Mémoire présenté par

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Concentration and quantitative regularity in homogenization and hydrodynamics

Soutenue le 4 octobre 2022 à la Maison Internationale de la Recherche, CY Cergy Paris Université, devant le jury composé de :

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Abstract

This habilitation thesis is about a selection of my works concerned with the study of regularity for Partial Differential Equations, with a focus on equations stemming from fluid dynamics. In a broad sense, regularity theory is the study of the local behavior of solutions. In our view the main objectives are: (i) to identify a range of scales where there is a certain self-similar behavior, (ii) to find basic objects i.e. building blocks, that represent well the solutions at these scales, (iii) to prove scaling laws for excess quantities, i.e. local error estimates, at these scales. Our analysis is motivated by physics: concentration effects in composite materials, fluids slipping over rough surfaces, generation of turbulence near boundaries in fluids. Our aim is to contribute to analyzing such phenomena from the perspective of regularity theory.

There are two main parts in this thesis. The first part is concerned with large-scale regularity and quantitative homogenization of three-dimensional stationary Navier-Stokes and elliptic equations. With Higaki (former postdoctoral researcher, now at Kobe University) and Zhuge (The University of Chicago), we prove large-scale regularity results in bumpy regions possibly as rough as fractals. With Armstrong (NYU), Kuusi (University of Helsinki) and Mourrat (ENS de Lyon), we obtain near-optimal error estimates for the homogenization of boundary layer correctors. The second part is concerned with the three-dimensional non stationary Navier-Stokes equations. With Maekawa (Kyoto University) and Miura (Tokyo Institute of Technology), we find new pressure estimates that enable us to control the strong nonlocality in the half-space. With Albritton (IAS, Princeton) and Barker (University of Bath), we investigate norm and geometric concentration near potential singularities. We also establish a connection between concentration and quantitative regularity in the critical case that leads to a slight breaking of the criticality barrier.

Résumé

Concentration et régularité quantitative en homogénéisation et en hydrodynamique

Cette thèse d'habilitation à diriger des recherches porte sur une sélection de mes travaux sur la régularité des solutions d'Équations aux Dérivées Partielles, provenant en particulier de la mécanique des fluides. En un sens, l'étude de la régularité est l'analyse du comportement local des solutions. Dans cette optique, les objectifs principaux sont : (i) d'identifier une gamme d'échelles avec une certaine auto-similarité, (ii) de trouver des briques de base qui représentent les solutions à ces échelles, (iii) d'établir des lois pour l'erreur locale à ces échelles. Notre travail est motivé par la physique : concentration dans les composites, glissement de fluides sur des surfaces rugueuses, génération de la turbulence près de bords. Notre objectif est de contribuer à l'analyse de ces phénomènes à travers le prisme de la régularité.

Il y a deux parties principales dans cette thèse. La première est dédiée à la régularité dite améliorée ou aux grandes échelles et à l'homogénéisation quantitative pour les équations de Navier-Stokes tri-dimensionnelles stationnaires et elliptiques. Avec Higaki (ancien postdoctorant, à présent à Kobe University) et Zhuge (The University of Chicago), nous démontrons des estimations de régularité à grande échelle au-dessus de bords rugueux, possiblement aussi irréguliers que certaines fractales, mais asymptotiquement plats. Avec Armstrong (NYU), Kuusi (University of Helsinki) et Mourrat (ENS de Lyon), nous obtenons des estimations quantitatives presque optimales pour l'homogénéisation de couches limites. La seconde partie est consacrée à l'étude des équations de Navier-Stokes tri-dimensionnelles non stationnaires. En collaboration avec Maekawa (Kyoto University) et Miura (Tokyo Institute of Technology), nous trouvons de nouvelles estimations pour la pression qui nous permettent de contrôler la forte nonlocalité due à l'incompressibilité et à la condition de non-glissement dans le demi-espace. Avec Albritton (IAS, Princeton) et Barker (University of Bath) nous étudions les phénomènes de concentration en norme et de concentration géométrique au voisinage de singularités potentielles. Nous explorons aussi les liens entre concentration et régularité quantitative dans le cas critique, ce qui nous permet de faire sauter dans une certaine mesure le verrou de la criticalité.

List of publications after Ph.D.

Research articles and preprints presented in this habilitation thesis

The following papers are the main papers that are discussed in this habilitation thesis. The papers are ranked in the reverse chronological order taking into account the first release on arXiv. All the papers are available on arXiv and on my webpage.

— [10], with Dallas Albritton and Tobias Barker, Localized smoothing and concentration for the Navier-Stokes equations in the half space, *submitted* (2021).

— [184], with Mitsuo Higaki and Jinping Zhuge, Large-scale regularity for the stationary Navier-Stokes equations over non-Lipschitz boundaries, to appear in *Analysis & PDE* (2021).

— [37], with Tobias Barker, Mild criticality breaking for the Navier-Stokes equations, *J. Math. Fluid Mech.* (2021).

— [38], with Tobias Barker, Quantitative regularity for the Navier-Stokes equations via spatial concentration, *Comm. Math. Phys.* (2021).

— [183], with Mitsuo Higaki, Regularity for the stationary Navier-Stokes equations over bumpy boundaries and a local wall law, *Calc. Var. Partial Differential Equations* (2020).

— [36], with Tobias Barker, Scale-invariant estimates and vorticity alignment for Navier-Stokes in the half-space with no-slip boundary conditions, *Arch. Ration. Mech. Anal.* (2020).

— [35], with Tobias Barker, Localized Smoothing for the Navier–Stokes Equations and Concentration of Critical Norms Near Singularities, *Arch. Ration. Mech. Anal.* (2020).

— [247], with Yasunori Maekawa and Hideyuki Miura, Local energy weak solutions for the Navier-Stokes equations in the half-space, *Comm. Math. Phys.* (2019).

— [249], with Yasunori Maekawa and Hideyuki Miura, Estimates for the Navier-Stokes equations in the half-space for non localized data, *Analysis & PDE* (2020).

— [23], with Scott Armstrong, Tuomo Kuusi and Jean-Christophe Mourrat, Quantitative analysis of boundary layers in periodic homogenization, *Arch. Ration. Mech. Anal.* (2017).

Other research articles and preprints posterior to Ph.D. thesis

The following papers are posterior to my Ph.D. thesis, but are not presented in this habilitation thesis. They are ranked in the reverse chronological order taking into account the first release on arXiv. All the papers are available on arXiv and on my webpage.

— [187], with Richard Höfer and Franck Sueur, Motion of several slender rigid filaments in a Stokes flow, *Journal de l'École polytechnique – Mathématiques* (2022).

— [54], with Edoardo Bocchi and Francesco Fanelli, Anisotropy and stratification effects in the dynamics of fast rotating compressible fluids, to appear in *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire* (2022).

— [248], with Yasunori Maekawa and Hideyuki Miura, On stability of blow-up solutions of the Burgers vortex type for the Navier-Stokes equations with a linear strain, *J. Math. Fluid Mech.* (2019).

— [214], with Carlos Kenig, Improved regularity in bumpy Lipschitz domains, *J. Math. Pures Appl.* (2018).

— [213], with Carlos Kenig, Uniform Lipschitz estimates in bumpy half-spaces, *Arch. Ration. Mech. Anal.* (2015).

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Chapter 1

Introduction

Contents

This habilitation thesis is about a selection of my works carried out since my recruitment as a CNRS researcher in the fall of 2015. These works are concerned with the study of regularity for Partial Differential Equations, with a focus on equations stemming from fluid dynamics. In a broad sense, regularity theory is the study of the local behavior of solutions. We promote an extended point of view of regularity, that goes beyond the one of classical regularity in two aspects. First regularity in not necessarily measured in the sense of C^k or H^s function spaces. Second regularity may hold at certain scales but not all, which is emphasized by the terminology 'improved', 'mesoscopic' or 'large-scale' regularity. Therefore, in our view the main objectives are: (i) to identify a range of scales where there is a certain self-similar behavior, (ii) to find basic objects i.e. building blocks, that represent well the solutions at these scales (polynomials, piecewise polynomials, singular functions in corners, oscillating polynomials in homogenization…), (iii) to prove scaling laws for excess quantities, i.e. local error estimates, at these scales. Our analysis is motivated by physics (concentration effects in composite materials, fluids slipping over rough surfaces, generation of turbulence near boundaries in fluids). It is our hope that the tools that I developed together with my collaborators can further mature and contribute to analyzing such phenomena from the perspective of regularity theory (scales relevant for turbulence, shape of potential singularities, breaking of the criticality barrier, cascade of energy, dissipation near boundaries, effect of rough or flat walls on the generation of vorticity).

There are two main parts in this thesis. The first part (2 chapters) is concerned with large-scale regularity and quantitative homogenization of stationary Navier-Stokes and elliptic equations. The second part (3 chapters) is concerned with the three-dimensional non stationary Navier-Stokes equations. There is a gap in terms of difficulty between the stationary and the non stationary Navier-Stokes equations. In the latter case, the nonlocality due to the incompressibility constraint is stronger especially near boundaries, and the only known controlled quantity, the energy, is supercritical. This explains that two major questions remain essentially open for the 'finite-energy weak' or 'turbulent' solutions constructed by Leray in 1934 [238]: (a) the uniqueness of finite-energy solutions relevant to our ability to predict the behavior of fluids, see the paper 'The real butterfly effect' by Palmer, Döring and Seregin [278], (b) the question of the regularity of solutions, which is believed to be important to the understanding of turbulence, see the description of the Millenium problem by Fefferman [131]. As a result, for stationary fluids our program is much more advanced than for non stationary fluids.

As for the first part of the thesis, our main results are on the following topics. First, in collaboration with Higaki (former postdoctoral researcher, now at Kobe University) and Zhuge (The University of Chicago), we prove large-scale Lipschitz and higher-order $C^{1,\mu}$, $C^{2,\mu}$ regularity in bumpy (i.e. asymptotically flat) Lipschitz and John domains, that are very rough in two regards: the small-scale oscillations of the boundary are arbitrary and the boundary may be as rough as that of certain fractals. Second, in collaboration with Armstrong (NYU), Kuusi (University of Helsinki) and Mourrat (ENS de Lyon), we prove quantitative error estimates for the homogenization of boundary layer correctors and obtain near-optimal error estimates that outperform previous bounds.

As for the second part of the thesis, let us point to our main contributions. First, we investigate the impact of unbounded boundaries on the nonlocal effects due to the incompressibility constraint. In collaboration with Maekawa (Kyoto University) and Miura (Tokyo Institute of Technology), we find new pressure formulas that on the one hand emphasize the strong nonlocality of the Navier-Stokes equations in the half-space, but on the other hand enable to find controls that are good enough for a number of applications. With Maekawa and Miura, we prove the existence of local energy solutions in the half-space, which was an open question since similar work in the whole-space. In collaboration with Barker (University of Bath), we prove fractional pressure estimates in the half-space, that enable us to unify Type I blow-up notions and hence to develop a new strategy for the proof of regularity under local vorticity alignement, 'geometric concentration' being the other side of the coin. Second, we investigate the strength of nonlocal effects in view of local smoothing properties. With Barker, we prove local (in time and space) smoothing in the critical case and 'norm concentration' in the whole space. We extend these results to the half-space, where the nonlocality is much stronger, in collaboration with Albritton (IAS, Princeton) and Barker. Third, with Barker, we develop a new method for the proof of quantitative regularity that grounds on the analysis of concentration phenomena. That line of research enables us to give a partial answer of a conjecture by Tao about the blow-up of slightly supercritical Orlicz norms.

The main contributions are summarized in Section 1.2 below, mainly via tables or graphical representations. We now take a closer look at a few transversal themes.

1.1 A few transversal themes

1.1.1 Steady vs. unsteady incompressible viscous fluids

In this thesis we study the three-dimensional stationary Stokes and Navier-Stokes equations

$$-\Delta U + U\cdot\nabla U + \nabla P = 0, \quad \nabla\cdot U = 0, \tag{1.1}$$

for velocity $U = U(x) \in \mathbb{R}^3$ and pressure $P = P(x) \in \mathbb{R}$, in bumpy half-spaces, see Chapter 2. We also study the non stationary three-dimensional Navier-Stokes equations

$$\partial_t U - \Delta U + U\cdot\nabla U + \nabla P = 0, \quad \nabla\cdot U = 0, \tag{1.2}$$

for velocity $U = U(x,t) \in \mathbb{R}^3$ and pressure $P = P(x,t) \in \mathbb{R}$, in the whole-space and in the flat half-space. We recall that the Stokes system is the linearized version of the Navier-Stokes system around 0.

The stationary Navier-Stokes equations (1.1) and the non stationary Navier-Stokes equations (1.2) share similar scaling invariance: for $\lambda \in (0,\infty)$

$$U \quad \text{solves (1.1) implies} \quad U_\lambda = \lambda U(\lambda\cdot) \quad \text{solves (1.1)}, \tag{1.3}$$

$$U \quad \text{solves (1.2) implies} \quad U_\lambda = \lambda U(\lambda\cdot,\lambda^2\cdot) \quad \text{solves (1.2)}. \tag{1.4}$$

These scaling properties are fundamental for the study of the regularity or singularity of solutions, since they enable to zoom-in or zoom-out. These scaling properties enable to discriminate between norms/quantities that are invariant under the scaling, subcritical or supercritical. Scale-critical quantities play a key role for the regularity theory: ε-regularity results [75, 243, 342] among many other works, quantitative regularity in Chapter 6 (see also Subsection 1.1.8), etc.

One key difficulty of the non stationary three-dimensional Navier-Stokes equations is the fact that the only known controlled quantity, the global energy

$$\mathscr{E}(U,T) = \sup_{t\in(0,T)} \tfrac12 \int_{\mathbb{R}^3} |U(\cdot,t)|^2 + \int_0^T\int_{\mathbb{R}^3} |\nabla U|^2$$

is supercritical with respect to the scaling (1.1) i.e.

$$\mathscr{E}(U_\lambda, T/\lambda^2) = \frac{1}{\lambda}\mathscr{E}(U,T).$$

Hence the energy does not control well the nonlinearity, or in other words the smoothing from the parabolic part of the equation is not strong enough to bootstrap the regularity starting from the energy class. As a consequence, most of the regularity results in this thesis, and in the mathematical literature concerned with (1.2), are under critical or slightly supercritical a priori bounds, in scale-critical regimes (self-similar solutions), or assume certain special structure (axisymmetry without swirl, local vorticity alignement).

A second key difficulty is related to the incompressibility that introduces non local effects via the pressure. These effects can already be seen at the level of the linear Stokes system. They are stronger in the non stationary Stokes equations than in the stationary Stokes equations. In the latter case, one can directly estimate the pressure in terms of the velocity

$$\left\|P - (P)_{B(1)}\right\|_{L^2(B(1))} \leq C\|\nabla U\|_{L^2(B(1))}$$

via the Bogovskii estimate (see (2.14)), while in the unsteady Stokes system we have

$$\|P - (P)_{B(1)}\|_{L^2(Q(1))} \leq C\|\partial_t U - \Delta U\|_{L^2(-1,0;H^{-1}(B(1)))}.$$

We see that the pressure competes with the time derivative of the velocity. This is emphasized by the Serrin examples, see (5.1) and [311]. The nonlocal effects are also stronger in the half-space with no-slip boundary condition than in the whole-space or the half-space with perfect slip boundary condition; see (1.1.2). Finally, these effects are obstacles to local smoothing properties for the non stationary systems, see [347, 225] for the interior regularity and [295, 208, 301] for the boundary regularity.

We finish this very concise overview by defining 'regular' and 'singular' space-time points in the whole-space (resp. the half-space). We say that $(\bar{x}, t) \in \mathbb{R}^3 \times (0, \infty)$ (resp. $(\bar{x}, t) \in \overline{\mathbb{R}^3_+} \times (0, \infty)$) is a 'regular point' of U, if there exists $r \in (0, \infty)$ such that $U \in L^\infty(B_{\bar{x}}(r) \times (t - r^2, t))$ (resp. $U \in L^\infty(B_{\bar{x},+}(r) \times (t - r^2, t))$). A contrario, a point $(\bar{x}, t) \in \mathbb{R}^3 \times (0, \infty)$ (resp. $(\bar{x}, t) \in \overline{\mathbb{R}^3_+} \times (0, \infty)$) is a 'singular point', or a 'blow-up point' if it is not regular. A time $T^* \in (0, \infty)$ is called a blow-up time if there exists $\bar{x} \in \mathbb{R}^3$ such that $(\bar{x}, T^*)$ is a singular point.

1.1.2 Interplay of incompressibility and no-slip

Boundaries break the structure of free flows by introducing additional constraints. In the case of the no-slip boundary condition the fluid sticks to the boundary $U = 0$, which results in the creation of small scales in the vicinity of the boundary. These small scales can be further amplified by the nonlinearity of the fluid equation. The nonlocal effects due to the incompressibility are stronger near the boundary, which is reflected by the fact that the pressure, its harmonic component, depends on the history of the flow and involves the initial data, see Subsection 5.3.2. This makes the analysis of the regularity near the boundary particularly challenging. From the mathematical point of view, some of the difficulties are:

(1) the nonlinear and nonlocal boundary condition for the vorticity, see [246, 159] and (4.1);

(2) the lack of local smoothing in spatial variables, see the examples in [208, 301, 210, 91] that demonstrate that local boundedness of the solution near the flat boundary does not imply boundedness of its gradient, see Subsection 4.1.4;

(3) the fact that one cannot estimate the pressure in terms of the nonlinearity $U \otimes U$ contrary to the whole-space, see [220] and Subsection 4.1.2.

We investigate these effects thoroughly in Section 4.1.2 and Subsection 4.1.4. Some of the open problems in the half-space are also mentioned in Subsection 4.1.6. Let us mention that we manage to obtain certain results in a scale-critical (so-called Type I) regime where the nonlinear effects compete directly with the diffusion. Finally, the relationship between boundary effects and potential singularity formation is discussed in [232].

1.1.3 An enlarged point of view of regularity theory: building blocks

The goal of the regularity theory is to describe the local behavior of solutions, which means finding functions that represent well the solutions locally, at certain scales, and measuring quantitatively the local error, or excess decay. This goal can be very different from proving regularity in the sense of C^k or H^s function spaces. Let us list several examples:

(i) **Homogenization**
The idea is already present in the work of Avellaneda and Lin [29], where '$\mathbf{a}(\frac{\cdot}{\varepsilon})$-harmonic linear functions', i.e. linear functions perturbed by oscillating cell correctors of the form

$$x + \varepsilon\chi(\tfrac{x}{\varepsilon})$$

are the building blocks of the $C^{1,\mu}$ regularity theory of elliptic operators $-\nabla\cdot\mathbf{a}(\frac{x}{\varepsilon})\nabla\cdot$ with highly oscillating ε-periodic coefficients. These building blocks retain some of the oscillations of the solutions. Notice the highly singular behavior of derivatives in ε, which allows to say that in that sense these building blocks are not smooth. Further developments include the following works: Armstrong, Kuusi and Smart [24] higher-order $\mathbf{a}$-harmonic polynomials in periodic homogenization, Armstrong, Gloria and Kuusi [20] for almost-periodic homogenization, Armstrong, Kuusi and Mourrat [22, 21] and Gloria, Otto and Neukamm [167] for stochastic homogenization.

(ii) **Transmission problems**
In the paper [355], Zhuge introduces 'piecewise linear functions' that play the role of building blocks for the $C^{1,\mu}$ regularity theory for transmission problems between periodic structures separated by a flat interface.

(iii) **Corners**
In the paper [205], Josien, Raithel and Schäffner introduce '$\mathbf{a}$-harmonic singular functions' that play the role of building blocks for the higher-order boundary regularity theory of elliptic equations with highly-oscillating random coefficients near corners. These building blocks contain the singularities of the function due to the corner.

(iv) **Wall laws**
In our work with Higaki [183], see Theorem 2.7 and Remark 2.8, we introduce 'Navier polynomials' that describe a local wall law for fluids above a periodic Lipschitz boundary. Higher-order correctors were constructed in a similar context in collaboration with Higaki and Zhuge [184], see Theorem 2.9 and Theorem 2.10, and by Higaki and Zhuge in [185].

The building blocks are 'blow-up limit' or 'blow-down limit' solutions depending on the context. Their role for the regularity is emphasized by Liouville-type results such as Theorem 2.11.

1.1.4 An enlarged point of view of regularity theory: improved regularity

The terminology 'improved' regularity refers to the fact that we are able to prove, in different contexts, regularity estimates at certain scales that are false at other scales. This is typically the case in homogenization, where improved regularity is inherited at large-scales from the convergence to a homogenized, constant coefficient operator, with improved regularity. This improvement of flatness at larges scales was successfully exploited (by compactness or more quantitative arguments) in many works concerned with periodic homogenization by Avellaneda and Lin [29], almost-periodic homogenization by Armstrong and Shen [27], Zhuge [352] and stochastic homogenization by Armstrong and Smart [25], Armstrong, Kuusi and Mourrat [22, 21], Gloria, Otto and Neukamm [167] of elliptic operators $-\nabla\cdot\mathbf{a}(\frac{x}{\varepsilon})\nabla\cdot$ or variants. If one considers elliptic operators with barely bounded coefficients, the Lipschitz regularity that one proves at larges scales $\gtrsim\varepsilon$ may certainly fail at small scales $\lesssim\varepsilon$. Hence the terminology 'improved regularity'. We also sometimes refer to 'large-scale regularity' or 'mesoscopic regularity'.

Our leitmotiv is that of a scale-dependent notion of regularity. We pursue that objective in our works concerned with improved regularity above rough boundaries that are asymptotically flat. This program was initiated while I was a postdoctoral researcher at the University of Chicago working with Kenig [214], and continued first in collaboration with Higaki [183] and then with Higaki and Zhuge [184]; see also the related works of Zhuge [356], Gu and Zhuge [178]. A major objective is to develop tools that enable us to free ourselves from the lack of smoothness at small scales, see Subsection 1.2.1 below and Chapter 3.

1.1.5 Self-similarity and building blocks

Here we allude to the fact that the building blocks described in Subsection 1.1.3 above ensure a certain 'self-similarity' across the scales where one proves 'improved' regularity. They are essential to iteration arguments, where the control of excess quantities at scale θ^{k+1} by excess quantities at scale θ^k is similar, by rescaling, to the control of excess quantities at scale θ by excess quantities at scale 1. This idea is further developed in Remark 2.1 and Remark 2.12 below.

1.1.6 Scaling laws for excess quantities

By the Morrey-Campanato characterization of Hölder continuity [157, Theorem 5.5], Hölder norms $C^{0,\mu}$, $\mu \in (0,1]$, can be measured in terms of the decay of 'excess quantities' or 'oscillation'

$$\inf_{a\in\mathbb{R}} \fint_{B(r)} |U-a|^2 = \fint_{B(r)} |U-(U)_r|^2 \lesssim r^{2\mu}$$

where $(U)_r$ denotes the mean of U over $B(r)$ and $\fint = |B(r)|^{-1}\int$. This characterization is very useful since L^2 or L^p norms are easy to accessed for solutions of elliptic equations, Stokes or Navier-Stokes, by energy estimates. Caccioppoli's inequality for solutions of divergence-form elliptic equations and Poincaré-Wirtinger's inequality show that Lipschitz regularity is equivalently measured in terms of the boundedness of the 'energy density' [22]

$$\fint_{B(r)} |\nabla U|^2 \lesssim 1.$$

Let us give two further examples to illustrate the previous ideas. In order to measure classical $C^{1,\mu}$ regularity, $\mu \in (0,1]$, one considers the decay of the 'excess quantities' or 'oscillation'

$$\inf_{a\in\mathbb{R},\, b\in\mathbb{R}^d} \fint_{B(r)} |U-a-b\cdot x|^2 = \fint_{B(r)} |U-(U)_r-(\nabla U)_r\cdot x|^2 \lesssim r^{2\mu}.$$

The linear polynomials $a+b\cdot x$ are for instance the building blocks for the $C^{1,\mu}$ regularity theory of constant coefficients elliptic equations. The large-scale $C^{1,\mu}$ regularity for the operator $-\nabla\cdot\mathbf{a}(\frac{x}{\varepsilon})\nabla\cdot$, is measured in terms of the decay of the 'excess quantities' or 'oscillation'

$$\inf_{a\in\mathbb{R},\, b\in\mathbb{R}^d} \fint_{B(r)} |U^\varepsilon-a-b\cdot(x+\varepsilon\chi(\tfrac{x}{\varepsilon}))|^2 = \fint_{B(r)} |U^\varepsilon-(U^\varepsilon)_r-(\nabla U^\varepsilon)_r\cdot(x+\varepsilon\chi(\tfrac{x}{\varepsilon}))|^2 \lesssim r^{2\mu}.$$

Finally, let us note that there is also a version of this characterization for parabolic Hölder spaces, see [237, Lemma 13.2].

1.1.7 Concentration

In this thesis, concentration refers to three phenomena:

(1) **Concentration in boundary layers**
These boundary layers describe the strong gradients of solutions near bumpy boundaries in Chapter 2 or near the boundary of a domain with a singular, highly oscillating, boundary data in Chapter 3. In Chapter 2 the main difficulty is to construct first-order and second-order boundary layers in a context where there is no smoothness of the boundary. In Chapter 3 the main difficulty is to paste together boundary layers with very different sizes (due to resonance or non-resonance properties) in a curved geometry.

(2) **Norm concentration**
This refers to the concentration/accumulation of certain critical norms on concentrating sets near potential Type I singularities of the unsteady Navier-Stokes equations in Chapter 5 and the use of norm concentration to obtain quantitative estimates in Chapter 6.

(3) **Geometric concentration**
This refers to breaking of smoothness of the vorticity direction on concentrating sets near potential Type I singularities of the unsteady Navier-Stokes equations in Chapter 4. Such results were previously known only globally.

1.1.8 Concentration and quantitative regularity

In Chapter 6 we develop a scheme that enables us to obtain explicit quantitative estimates of the form

$$\text{a subcritical norm of the solution} \lesssim \mathscr{G}(\text{a critical norm of the solution}) \tag{1.5}$$

where $\mathscr{G}$ is an explicit function; see (6.1) for a more precise form. Our scheme is a physical space analog of the one developed by Tao in 2019 [334].

To illustrate the strategy we give a physical space version for $U \in L^5_{t,x}$ and a Fourier space version for $U \in L^\infty_t L^3_x$; see Figure 1.1 for a summary of the method. First the critical standing assumption, $U \in L^5_{t,x}$ or $U \in L^\infty_t L^3_x$ implies that certain scale-critical quantities are small (in terms of the size of U in $L^5_{t,x}$ or $L^\infty_t L^3_x$) near final time or for high frequencies. The threshold scales can be interpreted as certain Kolmogorov dissipation scales where diffusion starts to dominate the nonlinearity. One then, this is the second step, infers regularity via ε-regularity or tools from the mild solution theory. The outcome is the exponential bound (6.24) ($\mathscr{G} = \exp(\cdot^5)$) for the control of the $L^\infty_{t,x}$ norm in terms of the $L^5_{t,x}$ norm. For the other case handled by Tao [334], one gets the triple exponential bound (6.12), with $\mathscr{G} = \exp\exp\exp(\cdot^C)$, for the control of the $L^\infty_{t,x}$ norm in terms of the $L^\infty_t L^3_x$ norm.

1.1.9 Quantitative asymptotics and regularity

There is a close link between quantitative estimates and regularity. As far as we know, this link was first used by Caffarelli and Peral [76] and then brought to the field of stochastic homogenization by Armstrong and Smart [25]. The idea is simple. If one has a family of functions U^ε that is quantitatively close (at any scale) to a family of functions $\bar{U}$ that have

<table>
<tr><td colspan="3">the global scale-critical standing assumption</td></tr>
<tr><td>$U \in L^5_{t,x}$
see Subsection 6.3.1</td><td>$U \in L^\infty_t L^{3,\infty}_x$ and $U(\cdot,0) \in L^3$
see Subsection 6.3.2</td><td>$U \in L^\infty_t L^3_x$
see Subsection 6.3.3</td></tr>
<tr><td colspan="3">prevents the following scale-critical quantities</td></tr>
<tr><td>$(-t)^{\frac{1}{5}}\|U(\cdot,t)\|_{L^5}$</td><td>$\sqrt{-t}\int_{B_{O(\|U\|_{L^\infty_t L^{3,\infty}_x})\sqrt{-t}}} |\omega(\cdot,t)|^2$</td><td>$N^{-1}|P_N U(t',x')|$</td></tr>
<tr><td colspan="2">to concentrate to close to final time</td><td>to concentrate for large frequencies $N \geq N_0$</td></tr>
<tr><td colspan="3">no concentration, i.e. smallness, implies regularity</td></tr>
</table>

Figure 1.1 – Quantitative regularity via concentration: a summary

improvement of integrability or improvement of flatness, one can use that error estimate in a Schauder-type iteration to infer regularity of the first family of functions U^ε. It is enough to have suboptimal error estimates on $U^\varepsilon - \bar{U}$ to make this scheme work. Regularity in turn implies improved error estimates. We describe this 'quantitative' scheme in full details in Subsection 2.1.3 and compare it to a 'compactness' scheme, see Subsection 2.1.2, initiated by Avellaneda and Lin [29] in the context of periodic homogenization.

Such a quantitative scheme is particularly useful to us, see Subsection 2.3.2, for the regularity in bumpy John domains. By a careful bootstrap of the integrability of the gradient of the solution (Caccioppoli, Meyers, Calderón-Zygmund and Lipschitz), we manage to go around the lack of boundary layer correctors.

1.1.10 Regularity and spatial decay

By using scaling properties, large-scale regularity can be transferred to large-scale decay estimates for Green and Poisson kernels associated to elliptic or Stokes operators, see the pioneering work by Avellaneda and Lin [30], and the subsequent papers [216] by Kenig, Lin and Shen, the paper [282] from my Ph.D. work, [177] by Gu and Zhuge. This property also plays a role in our paper [184, Appendix B] with Higaki and Zhuge.

For the non stationary Navier-Stokes equations, scaling properties were used in combination with Hölder regularity to obtain a priori estimates for forward self-similar solutions in the work by Jia and Šverák that constructs forward self-similar solutions for arbitrarily large data [197].

1.2 Presentation of our main contributions

To facilitate the reading, we summarize our results in three different manners in the next subsections. We emphasize different aspects: new methods that we develop in Subsection 1.2.1, chronology of the results (main works that inspire us, chronology of our results and some further developments that followed our papers) in Subsection 1.2.2, location of our results on 'complexity of equation', 'criticality' and 'complexity of domain' axes in Subsection 1.2.3.

1.2.1 Overview of the main new methods

We list here the main new methods that I developed with my collaborators for **Part I** of the thesis:

(1) **Tools to decouple large-scale from small scale regularity I**
In the context of bumpy Lipschitz domains, in collaboration with Higaki we adapt a method that enables us to build boundary layer correctors over (rough) Lipschitz boundaries; see Subsection 2.3.1. The method is based on a domain decomposition and local energy estimates (so-called Saint-Venant estimates) near the boundary.
As a consequence we prove *improved regularity in bumpy Lipschitz domains* (Theorem 2.5, Theorem 2.7), i.e. large-scale regularity estimates (Lipschitz, $C^{1,\mu}$) that are false at the small scales.

(2) **Tools to decouple large-scale from small scale regularity II**
In the context of bumpy John domains (that include certain fractal boundaries such as Koch's snowflake), in collaboration with Higaki and Zhuge we develop a method that circumvents the construction of boundary layer correctors for the proof of large-scale Lipschitz estimates; see Subsection 2.3.2. Indeed the method of construction of boundary layers for bumpy John domains breaks down. Therefore, one has to perseveringly bootstrap the integrability of the gradient of the solution via an interplay between quantitative suboptimal error estimates, regularity estimates (Caccioppoli inequalities, regularity in smooth domains) and arguments oblivious to the equation (iterations or real variable arguments).
As a consequence we prove *improved Lipschitz regularity in bumpy John domains* (Theorem 2.6).

(3) **Construction of higher-order boundary layers**
In collaboration with Higaki and Zhuge, we pioneer a new method to build second-order boundary layers in bumpy John domains, i.e. with data growing linearly in the horizontal direction; see Subsection 2.3.2. Our construction uses the first-order boundary layer corrector in an Ansatz for the second-order boundary layer. The method can be made more systematic to address the existence of correctors of arbitrary high order.
As a consequence we prove *higher-order $C^{2,\mu}$ regularity* (Theorem 2.10) and *Liouville theorems for entire solutions of the stationary Stokes* in bumpy John domains (Theorem 2.11).

(4) **Calderón-Zygmund decomposition of a boundary layer**
In order to handle the homogenization of systems with highly oscillating boundary conditions, so-called boundary layers, that are strongly heterogeneous and anisotropic, in collaboration with Armstrong, Kuusi and Mourrat we initiate a new stopping time argument to decompose the boundary layer; see Subsection 3.3.2. This enables us to construct the boundary layer by a careful pasting of local boundary layers that are thick near bad almost resonant directions and thin near good non-resonant directions.
As a consequence we prove *near optimal error estimates* for the quantitative homogenization of elliptic systems with highly oscillating Dirichlet data in uniformly convex domains (Theorem 3.1). The method can be successfully extended to more general domains or different boundary conditions.

We list here the main new methods that I obtained with my collaborators for **Part II** of the thesis:

(5) **ε-regularity for a Navier-Stokes equation with scale-critical drifts**
In collaboration with Barker, we adapt the method of Caffarelli-Kohn-Nirenberg for the proof of ε-regularity for a perturbed Navier-Stokes system in the whole-space with scale-critical drift terms. Our method is flexible enough to handle drifts in critical (Lebesgue) and ultra-critical (Lorentz or Besov) spaces.
As a consequence we prove *local-in-space short-time smoothing* in the critical case (Theorem 5.1) and initiate the study of *'norm concentration'* for critical norms near potential singularities in the whole-space (Theorem 5.5).

(6) **New method for pressure estimates in the half-space**
In collaboration with Maekawa and Miura, we find a new decomposition of the pressure for the unsteady Navier-Stokes equations in the half-space. The interplay between the incompressibility condition and the no-slip boundary condition is responsible for strong non local effects, that are concentrated in a part of the pressure that we call the 'harmonic pressure'. We obtain bounds for the harmonic pressure near initial time that enable us to use local energy methods.
As a consequence we prove the existence of *global-in-time local energy solutions* in the half-space (Theorem 5.10) that was mentioned as an open question in several works. This provides then a good framework to study local smoothing properties and norm concentration near potential singularities in the half-space. Our formulas are also useful outside the uniformly locally finite energy framework, for instance for Morrey-type local energies and 'intermittent data'.

(7) **A new scheme for regularity under local coherence of the vorticity field** In collaboration with Barker, we introduce a new strategy for the proof of regularity for the three-dimensional Navier-Stokes equations under continuous alignement of the vorticity; see Subsection 4.3.3. That strategy is based on compactness in a scale-critical regime and persistence of singularities. It avoids the use of Liouville theorems that are particularly complicated to prove in the half-space. Our method works in the whole-space and in the half-space.
As a consequence we prove *'geometric concentration' results* (Theorem 4.3), i.e. breaking of the continuity of the vorticity direction on concentrating sets. Such results are new even for the whole-space.

(8) **Quantitative regularity via concentration of scale-critical quantities**
In collaboration with Barker, we develop a method based on the concentration of scale-invariant quantities to prove quantitative regularity under the boundedness of a standing critical assumption (such as $L^5_{t,x}$, $L^\infty_t L^3_x$ or a Type I condition); see Subsection 6.3.1 for a toy model and Subsection 6.3.2 for the presentation of the method. Our method is the pendant in physical space of the Fourier space method pioneered by Tao in 2019; see Subsection 6.3.3. Working directly in physical space, with a scale-invariant local enstrophy, as we do, enables us to have localized results. Our method as well as Tao's rely on the quantitative estimate of a 'Kolmogorov scale' (either a sufficiently high frequency in Tao's work, or a sufficiently small space/time length in our work), at which diffusion overtakes the nonlinearity.
As a consequence we prove the *localized blow-up of the critical L^3 norm* in a Type I blow-up scenario (Theorem 6.1), and *quantify Seregin's 2012 regularity criteria* (Theorem 6.3).

(9) **Transferring subcriticality of the data forward in time**
In collaboration with Barker, we bring a method from the field of dispersive equations

to the field of fluid mechanics; see Subsection 6.3.5. The method is taken from a work by Bulut for nonlinear defocusing Schrödinger that was pointed out to us by Patrick Gérard. Subcritical energy estimates can be combined with quantitative regularity estimates as obtained by Tao in 2019. Hence the growth of the subcritical norm along the evolution can be estimated.
As a consequence we prove *mild criticality breaking* (Lemma 6.7) and the *blow-up of slightly supercritical Orlicz norms* (Theorem 6.5), which answers an open question mentioned by Tao in his 2019 work.

Let us emphasize that the schemes developed in points (7), (8) and (9) above are non perturbative.

1.2.2 Chronology of the results in this thesis

We refer to Figure 1.2 on page 23 that describes the chronology of the results in Part I of the thesis and to Figure 1.3 on page 24 that describes the chronology of the results in Part II of the thesis.

1.2.3 Two graphs to represent our main results

We refer to the Figure 1.4 on page 25 that orders our results according to 'complexity of the equation' against 'complexity of the domain' and to the Figure 1.4 on page 26 that order the results of Part II according to 'criticality'.

1.3 Outline of the thesis

There are two parts in the thesis. The first part (Chapter 2 and Chapter 3) is devoted to steady problems: elliptic systems and the steady three-dimensional Stokes and Navier-Stokes systems. The second part (Chapter 4, Chapter 5 and Chapter 6) is devoted to the unsteady three-dimensional Navier-Stokes equations. To increase the readability of the manuscrit, each chapter has the exact same structure with four sections: 'context: state of the art and obstacles', 'main results' (with a paragraph underlining the 'novelty of our results' and another 'further developments' if relevant) and 'new ideas and strategy for the proofs'.

In Chapter 2 we study the large-scale local boundary regularity of the steady Navier-Stokes system over very rough Lipschitz or John boundaries, that are asymptotically flat; we call such domains 'bumpy Lipschitz' or 'bumpy John domains'. This is work in collaboration with Higaki (Kobe University) and Zhuge (The University of Chicago).

In Chapter 3, we study the homogenization of elliptic systems with a highly oscillating Dirichlet boundary condition and obtain near optimal error estimates. This is work with Armstrong (NYU), Kuusi (University of Helsinki) and Mourrat (ENS de Lyon).

In Chapter 4, we study the regularity for the three-dimensional Navier-Stokes equations under continuous alignement of the vorticity. We introduce a new method in the scale-critical regime (Type I scenario) that enables us to obtain 'geometric concentration' results in the whole-space and the half-space. This is work in collaboration with Barker (University of Bath).

In Chapter 5, we study local smoothing properties for the three-dimensional Navier-Stokes equations in the whole-space and the half-space. As a corollary of 'local-in-space short-time smoothing' we get 'norm concentration' results near potential Type I singularities. This is work with Albritton (IAS, Princeton) and Barker (University of Bath). We also obtain new pressure estimates that enable us to study solutions with uniformly locally (but not necessarily globally) finite energy in the half-space. This is work with Maekawa (Kyoto University) and Miura (Tokyo Institute of Technology).

In Chapter 6, we study quantitative regularity for the three-dimensional Navier-Stokes equations in three different scenarios: (i) in a Type I blow-up scenario, (ii) in a slightly supercritical scenario (boundedness of an Orlicz norm), (iii) under boundedness of a critical norm along a sequence of times. This is work with Barker (University of Bath).

	quantitative large-scale regularity in bumpy domains	**concentration in homogenization**
main sources of inspiration	- Avellaneda, Lin [29] (compactness methods in homogenization) - Caffarelli, Peral [76] (quantitative method for regularity) - Kenig, Prange [213] (elliptic equations, smooth boundaries with no structure) - Kenig, Prange [214] (elliptic equations, Lipschitz boundaries with no structure) - Gérard-Varet, Masmoudi [154] (boundary layers for Stokes in rough domains)	- Gérard-Varet, Masmoudi [155] (boundary layers in polygonal domains) - Gérard-Varet, Masmoudi [156] (boundary layers in convex domains) - Kenig, Lin, Shen [216] (homogenization of Poisson kernels)
main results of the thesis	*for the stationary Navier-Stokes equations in bumpy domains* - Theorem 2.7 with Higaki in 2019 (local wall law in Lipschitz domains) - Theorem 2.6 with Higaki and Zhuge in 2021 (improved regularity in bumpy John domains) - Theorem 2.10 with Higaki and Zhuge in 2021 (higher-order regularity in John domains)	*for boundary layers in periodic homogenization* - Theorem 3.1 with Armstrong, Kuusi and Mourrat in 2016 (near-optimal error estimates)
some further developments	- Gu, Zhuge [178] (system of elasticity in large-scale $C^{1,\alpha}$ domains) - Zhuge [356] (elliptic equations in large-scale Reiffenberg flat domains) - Higaki, Zhuge [185] (arbitrary order regularity in bumpy John domains)	- Shen, Zhuge [316] (optimal estimates in low dimensions) - Shen, Zhuge [317] (almost Lipschitz regularity of the homogenized boundary data) - Shen, Zhuge [353] (relaxation of uniform convexity)

Figure 1.2 – Chronology of results, Part I: boundary layers and regularity for steady problems in disordered environments; *the dates of the theorems are those of first release on arXiv*

	boundary regularity and pressure estimates	**local smoothing and norm concentration**	**concentration and quantitative regularity**
main sources of inspiration	- Desch, Hieber, Prüss [124] (Stokes resolvent problem in the half-space) - Lemarié-Rieusset [235] (local energy solutions) - Chang, Kang [89] (fractional pressure estimates) - Constantin, Fefferman [107] (regularity under vorticity alignement) - Giga, Miura [162] (regularity under vorticity alignement in Type I case)	- Caffarelli, Kohn, Nirenberg [75] (ε-regularity) - Lin [243] (ε-regularity via compactness) - Jia, Šverák [197] (local-in-space smoothing for subcritical data) - Li, Ozawa, Wang [240] (concentration)	- Escauriaza, Seregin, Šverák [129] (qualitative regularity for $L^\infty_t L^3_x$) - Seregin [294] (blow-up of the L^3 norm) - Tao [334] (quantitative regularity for $L^\infty_t L^3_x$) - Bulut [73] (mild criticality breaking for nonlinear Schrödinger)
main results of the thesis	*for the unsteady 3D Navier-Stokes equations*		
	in the half-space	*in the whole-space or the half-space*	*in the whole-space*
	- Theorem 5.10 with Maekawa and Miura in 2017 (global-in-time local energy solutions) - Theorem 4.1 with Barker in 2019 (fractional pressure estimates) - Theorem 4.2 with Barker in 2019 (unification of Type I blow-ups) - Theorem 4.3 with Barker in 2019 (geometric concentration under Type I)	- Theorem 5.1 with Barker in 2018 (local-in-space smoothing in $\mathbb{R}^3$ for critical data) - Theorem 5.5 with Barker in 2018 (critical norm concentration in $\mathbb{R}^3$) - Theorem 5.11 with Albritton and Barker in 2021 (local-in-space smoothing in $\mathbb{R}^3_+$ for critical and subcritical data) - Theorem 5.13 with Albritton and Barker in 2021 (critical norm concentration in $\mathbb{R}^3_+$)	- Theorem 6.1 with Barker in 2020 (localized quantitative blow-up of the L^3 norm near a Type I singularity) - Theorem 6.3 with Barker in 2020 (quantification of Seregin's 2012 result about the blow-up of the L^3 norm) - Theorem 6.5 with Barker in 2021 (blow-up of a slightly supercritical Orlicz norm)
some further developments	- Bradshaw, Kukavika, Ożański [63] (global-in-time existence for 'intermittent' data)	- Kang, Miura, Tsai [211] (local version of local-in-space smoothing) - Kang, Miura, Tsai [212] (concentration for the supercritical L^2 norm)	- Palasek [276] (quantitative regularity axisymmetric) - Palasek [277] (quantitative regularity 4D Navier-Stokes) - Feng, He, Wang [133] (quantitative regularity for critical Lorentz norms)

Figure 1.3 – Chronology of results, Part II: regularity and concentration for unsteady fluids; *the dates of the theorems are those of first release on arXiv*

increasing complexity of the domain →

increasing complexity of the equation ↓

	interior or whole-space	**flat boundary or half-space**	**bumpy Lipschitz**	**bumpy John** (possibly fractal)	**curved**
linear divergence-form elliptic					*quantitative homogenization boundary layers* Theorem 3.1
steady 3D Stokes and Navier-Stokes			*improved Lipschitz regularity* Theorem 2.5	*improved Lipschitz regularity* Theorem 2.6	
			local wall law Theorem 2.7	*higher-order regularity* Theorem 2.10	
unsteady 3D Stokes and Navier-Stokes		*existence of global-in-time local energy solutions* Theorem 5.10			
	local-in-space smoothing Theorem 5.1	*local-in-space smoothing* Theorem 5.11			
	norm concentration Theorem 5.5	*norm concentration* Theorem 5.13			
		fractional pressure Theorem 4.1			
		unification of Type I Theorem 4.2			
		geometric concentration under Type I Theorem 4.3			
	localized quantitative blow-up under Type I Theorem 6.1				
	quantitative blow-up of the L^3 norm Theorem 6.3				
	quantitative blow-up of a slightly supercritical Orlicz norm Theorem 6.5				

Figure 1.4 – Main results presented in this thesis ordered according to ‘complexity of the equation’ against ‘complexity of the domain’

increasing level of criticality →

borderline critical	**borderline endpoint critical**	**slightly supercritical**	**supercritical**		
$L_t^\infty L_x^3$	Type I	'log' breaking	$\sup_k \lVert U(\cdot, t_k)\rVert_{L^3} < \infty$	geometric breaking	energy
Escauriaza, Seregin, Šverák [129] (qualitative) Tao [334] (quantitative)		Pan [279], Seregin [299], Chen, Tsai, Zhang [95] (axisymmetric)	Seregin [294] (qualitative regularity)	Constantin, Fefferman [107] (nonlinearity depletion by vorticity alignement)	?
	- Theorem 4.3 (geometric concentration under Type I) - Theorem 5.5 (norm concentration in the whole-space) - Theorem 5.13 (norm concentration in the half-space) - Theorem 6.1 (quantitative blow-up of the L^3 norm on concentrating sets)	- Theorem 6.5 (quantitative blow-up of a slightly supercritical Orlicz norm)	- Theorem 6.3 (quantitative regularity)		

Figure 1.5 – Selected results of Part II and from the literature ordered according to 'criticality'

Part I

Boundary layers and regularity for steady problems in disordered environments

Chapter 2

Compactness and quantitative methods for regularity

This chapter relies mainly on the papers:

— [184], with Mitsuo Higaki and Jinping Zhuge, Large-scale regularity for the stationary Navier-Stokes equations over non-Lipschitz boundaries, to appear in *Analysis & PDE* (2021).

— [183], with Mitsuo Higaki, Regularity for the stationary Navier-Stokes equations over bumpy boundaries and a local wall law, *Calc. Var. Partial Differential Equations* (2020).

Contents

2.1 Context: state of the art and obstacles

The study of fluids over rough boundaries plays a prominent role in the field of hydrodynamics, at least for three reasons.

First, rough, bumpy or corrugated surfaces are ubiquitous in nature and engineering. They appear at any scales from geophysics to zoology and microfluidics. At the microstructure, the geometry may be anything from fractal to periodic and crenellated. No surface is perfectly smooth, and the lack of smoothness may actually enable us to resolve certain oddities, such as the no-collision paradox for a sphere dropped in a viscous fluid under the action of gravity [318, 202, 121, 152, 191]. Moreover, certain roughness patterns are either selected by biological processes and environmental pressure such as scales of sharks for their drag reduction properties, or designed for industrial applications especially in aeronautics, microfluidics and for the transport of fluids in pipes.

Second, the study of roughness is strongly tied to the derivation of boundary conditions in fluid mechanics. The question whether fluids slip or not over surfaces is still a matter of active debate. Experiments show that there is no universal answer and that the slip behavior depends a lot on the geometry and microstructure of the surface [55, 233]. A widespread idea is that roughness favors slip. To give one specific example where finding the most accurate boundary condition is critical, let us cite the field of glaciology. The assessment of various friction laws for the flow of a glacier over a rough bedrock is crucial in order to understand the speed of glacier discharge and eventually estimate the sea level rise as a result of global warming [206, 262].

Third, the study of the impact of roughness on the behavior of fluids accompanied the development of turbulence research, as underlined by Jiménez in [199]:

> Turbulent flows over rough walls have been studied since the early works of Hagen (1854) and Darcy (1857), who were concerned with pressure losses in water conduits. They have been important in the history of turbulence. Had those conduits not been fully rough, turbulence theory would probably have developed more slowly. The pressure loss in pipes only becomes independent of viscosity in the fully rough limit, and this independence was the original indication that something was amiss with laminar theory. Flows over smooth walls never become fully turbulent, and their theory is correspondingly harder.

Investigations of the effect of roughness on fluid flows span three distinct regimes. In the laminar regime, studies focus on the drag reducing properties of roughness elements [46, 146]. As for the onset of turbulence [292, 326], there are some indications that roughness lowers the critical Reynolds number for the transition from the laminar to the turbulent regime [341]. In the fully turbulent regime, a similarity hypothesis for the flow over flat surfaces and for the flow over rough surfaces was put forward [337]. The extent to which such a universal law holds is still being disputed [199, 82, 137, 291].

Our main (ambitious) objective can be vaguely stated as follows:

> **Our goal:** Study the impact of roughness on the behavior of fluids and the possible generation of vorticity from the point of view of the regularity theory.

The results in this chapter should be seen as first steps in this longterm research program.

2.1.1 Asymptotic analysis in bumpy domains

First, there is an extensive body of works that deal with wall (or friction) laws, or in other words, effective or homogenized boundary conditions. One aims at replacing rough boundaries by fictitious, smooth or flat boundaries. The wall law catches an averaged effect from the $O(\varepsilon)$-scale on large-scale flows of order $O(1)$ through homogenization. In that line of research, it is well-known that Navier-slip boundary conditions provide refined approximations for fluids above bumpy boundaries. This effective boundary condition reads for instance in two dimensions

$$u_1 = \varepsilon\alpha\partial_2 u_1\,, \quad u_2 = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+ \tag{2.1}$$

with a constant α depending only on the boundary function γ. Under some quantitative ergodicity assumptions, one can get error estimates. Historically, periodic roughness profiles were first looked at [18, 4, 192, 193]. Analysis of almost-periodic [154] or random stationary ergodic [150, 45] boundary oscillations was done more recently. Let us also mention a few works that address non-stationary fluids [71, 182], for which the analysis is less developed due to its inherent difficulties. We also point out that some authors attempted to justify boundary conditions arising in fluid mechanics starting from boundary conditions at the microscopic scale; see for instance [81, 72, 56] for the derivation of the no-slip boundary condition from a perfect slip boundary condition at the microscale, or [114] for the computation of the homogenized effect starting from Navier-slip boundary conditions at the microscale.

A second topic is the study of the effect of roughness on singular limits. The topics of rotating fluids and of the homogenized effect of bumpiness on Ekman pumping was studied in numerous papers [149, 151, 116, 115]. The paper [153] carries out an analysis of the vanishing viscosity limit in a specific scaling regime. There are also studies concerned with equations in singularly perturbed domains such as the Stokes equations in rough thin films [101] or water waves above a rough topography in the shallow regime [110].

Third, rough domains pose considerable numerical challenge. This aspect has certainly driven the development of wall laws in a model reduction perspective; see for instance [4, 123]. Other approaches are being elaborated, such as Direct Numerical Simulations [80], Lattice Boltzmann Methods that are adapted to intricate geometries [341] and Large Eddy Simulations [19, 57] that in this context cause important parametrization issues of the small scales.

2.1.2 Compactness methods for regularity

Compactness arguments in the regularity theory originate from the works of De Giorgi [122] and Almgren [17] in the calculus of variations, and were adapted to various contexts, notably in homogenization by Avellaneda and Lin [29], for the study of partial regularity for five-dimensional stationary fluids by Struwe [327] and for three-dimensional unsteady fluids by Lin [243] or Ladyženskaja and Seregin [230]. There are far more works relating to this topic. Let us just point that our works with Kenig [213, 214], with Higaki [183] presented here, and the work with Albritton and Barker [10] presented in Chapter 5 are based on a compactness argument. Schematically the compactness argument is a two-steps process:

(Step-1) **Improvement of flatness** in certain certain small-scale (regular coefficients), large-scale (homogenization), or linear regime (ε-regularity) limit;
one gets an improved estimate at a fixed scale using regularity for the limit system and compactness of the family of solutions (via Caccioppoli-type inequalities, local energy estimates, Aubin-Lions-type lemmas);

(Step-2) **Iteration** of the one-scale improvement of flatness estimate;
this step relies on the reiterated use of the first step, which requires a sort of self-similarity of the approximation.

Remark 2.1 (About the self-similarity property). Let us illustrate the self-similarity that is needed to iterate the one-scale improvement of flatness estimate by looking at just three different situations:

(1) In the case of the local interior regularity for the Laplace operator, harmonic polynomials are the right objects to measure the regularity of solutions; see the standard regularity for harmonic functions [157].

(2) In the case of the local interior regularity for divergence-form elliptic equations with oscillating coefficients, such as $-\nabla \cdot \mathbf{a}(x/\varepsilon)\nabla\cdot$, harmonic polynomials are not solutions of the equations. The right objects to consider are hence corrected polynomials that we dub $\mathbf{a}$-harmonic polynomials. If the coefficients $\mathbf{a}$ are periodic, the $\mathbf{a}$-harmonic polynomials of degree one are the functions $y + \chi(y)$, where χ is the usual cell corrector; see [29] and [24] for the higher-order $\mathbf{a}$-harmonic polynomials.

(3) In the case of the local boundary regularity for the Laplace equation near a bumpy boundary $x_3 = \varepsilon\gamma(x'/\varepsilon)$, the trace of harmonic polynomials does not vanish on the bumpy boundary. The right objects are hence harmonic polynomials corrected with boundary layer correctors, that vanish on the boundary; see [213, 214] and Subsection 2.3.1 below.

Some other recent works rely on the compactness method to prove uniform estimates in homogenization. In the work of Niu and Zhuge [272], the compactness method is used to prove uniform Hölder estimates for elliptic systems oscillating with multiple non separated scales. Notice that in the case of Hölder estimates, no correctors are needed because the right objects to measure the regularity are just constants, which are trivial solutions to the equation with oscillating coefficients. The Lipschitz estimate is stated there as an open problem. In [314], Shen studies the large-scale regularity for the Stokes equation in periodically perforated domains. The compactness method is effective in this setting because the size of the holes is comparable to the size of the periodicity cell. In certain situations, the compactness method reaches its limits and a quantitative method is more successful.

2.1.3 Quantitative methods for regularity

In some situations (stochastic homogenization, homogenization of high-contrast materials, homogenization of porous media with dilute holes) there is a need for quantitative versions of the compactness method of Subsection 2.1.2. The main idea is that the original problem with variable coefficients or small scales is quantitatively close to a homogeneous limit problem, which has improved regularity. The scheme is reminiscent of the perturbation argument in Schauder's theory for solutions of equations with smooth coefficients at the small scale. The general framework was laid down by Caffarelli and Peral [76] for the proof of $W^{1,p}$ estimates, p finite, for divergence-form elliptic equations. There the method is dubbed '$W^{1,p}$ estimates by approximation' and encompasses the case of regular coefficients, where the perturbation is at the small scales around a fixed regular point, as well as the case of homogenization, where the perturbation is at the large scales around the homogenized equation. The scheme was summarized in [76] as follows:

(1) **Improvement of flatness** for a fixed operator $-\nabla \cdot \mathbf{a}_0\nabla\cdot$ (either the operator frozen at a fixed point when the coefficients are smooth, or the homogenized operator);

(2) **A local (sub-optimal) error estimate at any scale** to approximate the solutions of the variable coefficient operator $-\nabla\cdot\mathbf{a}\nabla\cdot$ by solutions of the fixed operator $-\nabla\cdot\mathbf{a}_0\nabla\cdot$;

(3) **A real variable argument** which is a version of the Calderón-Zygmund lemma; see the original paper by Caffarelli and Peral [76, Lemma 1.3] and the book by Shen [313, Theorem 4.2.3] for a refined version.

Lipschitz and higher-order $C^{k,\mu}$ estimates can be obtained following a similar strategy. The last step, which requires a Calderón-Zygmund type argument to get on top of the L^p integrability in the case of $W^{1,p}$ estimate, is less subtle in the case of higher-order estimates because one can afford to loose a bit on the exponent μ.

Such a scheme is effectively implemented in the context of homogenization. In periodic homogenization, quantitative arguments may be needed to treat:

(i) borderline cases where there is no improvement of regularity as in the proof of boundary Rellich estimates by Shen [312],

(ii) multiple-scales problems as in the paper by Geng and Shen [147] for the homogenization of parabolic equations $\partial_t \cdot -\nabla \cdot \mathbf{a}(x/\varepsilon, t/\varepsilon^k)\nabla\cdot$ with $k \neq 2$ i.e. time and space scaling in a non self-similar way, or in the paper [271] by Niu, Shen and Xu for divergence-form elliptic equations with multiple separated scales,

(iii) singularly perturbed problems, such as the fourth-order elliptic system studied by Niu and Shen [270], or elliptic equations with high-contrast coefficients studied by Russel [288] (contrast parameter $\delta = 0$), [289] (contrast parameter $\delta \in (0,1)$) and Shen [315] (contrast parameter $\delta \in (0,\infty)$).

In stochastic and almost-periodic homogenization, the existence of correctors with certain growth at most sub-linear at space infinity, or even bounded as in the periodic case, is subtle and related to the large-scale regularity theory, see the book by Armstrong, Kuusi and Mourrat [22, Chapter 3]. A starting point for the regularity theory of such operators is therefore to consider approximate correctors, that enable to get an approximation of solutions at any scales and sub-optimal error estimates; see [22, page (viii) and (iv)] for a general description of the method. These quantitative approximations, along with improvement of flatness for the homogenized problem, can then be used in a classical Schauder $C^{1,\mu}$ regularity scheme similarly as explained above for the $W^{1,p}$ estimate. The method was pioneered by Armstrong and Smart [25] and developed in the papers by Gloria, Neukamm and Otto [167] and by Armstrong, Kuusi and Mourrat [21] for the random case. The method was also successfully applied in almost-periodic homogenization to the large-scale interior regularity by Armstrong and Shen [27] and boundary regularity by Zhuge [352].

2.2 Main results

Our results are concerned with the local large-scale quantitative regularity for the steady incompressible three-dimensional Navier-Stokes equations in bumpy domains

$$\begin{cases} -\Delta U^\varepsilon + \nabla P^\varepsilon = -U^\varepsilon \cdot \nabla U^\varepsilon & \text{in} \quad B^\varepsilon_+(1), \\ \nabla \cdot U^\varepsilon = 0 & \text{in} \quad B^\varepsilon_+(1), \\ U^\varepsilon = 0 & \text{on} \quad \Gamma^\varepsilon(1), \end{cases} \tag{NS$^\varepsilon$}$$

where the functions $U^\varepsilon = U^\varepsilon(x) \in \mathbb{R}^3$ and $P^\varepsilon = P^\varepsilon(x) \in \mathbb{R}$ denote respectively the velocity and the pressure fields of the fluid and $\varepsilon \ll 1$ is the typical size of the microscopic roughness of the boundary, see below.

The following two general objectives in regularity theory motivate our results: (i) decouple the large-scale regularity question from the small-scale regularity (or lack of regularity) of the boundary, (ii) handle very rough or singular boundaries (Lipschitz, possibly fractals), (iii) remove structure assumptions on the microscopic oscillations of the boundary, (iv) identify building blocks that describe the local behavior of solutions, (iv) quantify local errors at mesoscopic scales, i.e. estimate the decay of certain excess quantities at various scales. Our longterm research program stated in the introduction of this chapter is to investigate the impact of roughness on regularity for steady or unsteady fluids.

Our research program was started with the works [213, 214] in collaboration with Kenig. These papers are concerned with uniform regularity estimates above highly oscillating boundaries for elliptic equations. The improved regularity results obtained at the large scales are generally false at the small scales due to the roughness of the boundary. Our results are in the spirit of large-scale regularity estimates pioneered in [29] for periodic homogenization, and later extended to stochastic homogenization; see for instance [25, 26, 167, 166] and [21] for the higher-order large-scale regularity theory.

The tools we develop in the papers [213, 214, 183, 184] in collaboration with Kenig, Higaki and Zhuge enable us to decouple the large-scale regularity from the small-scale properties of the boundary. We prove that fluids above bumpy boundaries, that are very rough at the microscopic scale but asymptotically flat, have improved regularity at large scales.

As for the regularity of the domain, we study two cases:

(1) in the work [183] with Higaki, we take a boundary given by the graph of a Lipschitz function;

(2) in the work [184] with Higaki and Zhuge, we consider John domains, which in some sense that we explain below are the most general domains in which we can handle the regularity for the Stokes system.

Lipschitz domains This is the setting of the paper [183]. We say that Ω is a 'bumpy Lipschitz half-space' if it is defined by

$$\Omega := \{y \in \mathbb{R}^3,\ y_3 > \gamma(y')\}, \tag{2.2}$$

where the boundary graph $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ is assumed to satisfy $\gamma(y') \in (-1, 0)$ for all $y' \in \mathbb{R}^2$. For $\varepsilon \in (0, 1]$, we set $\Omega^\varepsilon := \varepsilon\Omega$, which is the highly oscillating bumpy Lipschitz half-space. For $r \in (0, 1]$,

$$B^\varepsilon_+(r) := B(r) \cap \Omega^\varepsilon \qquad \text{and} \qquad \Gamma^\varepsilon(r) := B(r) \cap \partial\Omega^\varepsilon. \tag{2.3}$$

John domains In the work [184] we consider bumpy John domains that are very rough in two aspects: (i) the boundary may be fractal or have inward cusps, (ii) the boundary is highly oscillating. Hence, these boundaries get closer to the modeling of real surfaces found in nature (for instance porous media), that in particular do not need to be graphs, see Figure 2.1. John domains have in a broad sense the minimal properties for the analysis of incompressible fluids. Indeed, we rely on a Bogovskii operator to estimate the pressure. As far as we know, John domains are the most irregular ones for which we have the existence of such a tool.

These domains were introduced by John in [201] and named after John in [253].

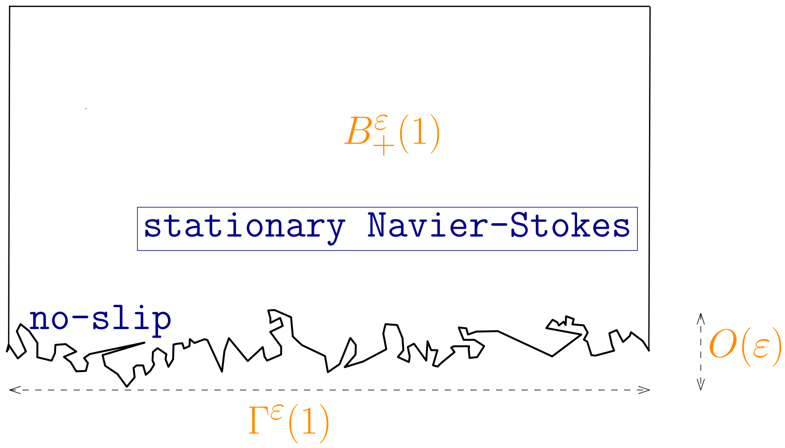

Figure 2.1 – A bumpy John domain

Definition 2.2 ([184, Definition 1.1] taken from [5]). Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set and $\bar{x} \in \Omega$. We say that Ω is a John domain (or a bounded John domain) with respect to $\bar{x}$ and with constant L if for any $y \in \Omega$, there exists a Lipschitz mapping $\rho : [0, |y - \bar{x}|] \to \Omega$ with Lipschitz constant $L \in (0, \infty)$, such that $\rho(0) = y, \rho(|y - \bar{x}|) = \bar{x}$ and $\mathrm{dist}(\rho(t), \partial\Omega) \geq t/L$ for all $t \in [0, |y - \bar{x}|]$.

Our analysis takes advantage of a key property of John domains, namely the existence of a right inverse of the divergence operator, see [5]. Such an operator is usually called a Bogovskii operator.

Examples of John domains are: Lipschitz domains, NTA domains, domains with inward cusps or certain fractals such as Koch's snowflake. Notice that domains with outward cusps are not John domains. For our work, we generalize the above definition from bounded domains to a class of unbounded domains.

Definition 2.3. Let Ω be a domain containing the upper half-space $\mathbb{R}^3_+$ and assume that $\partial\Omega \subseteq \{-1 < x_3 < 0\}$. We say that Ω is a 'bumpy John domain' with constant $L \in (0, \infty)$, if for any $x \in \partial\mathbb{R}^3_+$ and any $R \geq 1$, there exists a bounded John domain $\Omega_R(x)$ with respect to $\bar{x}_R = x + Re_d$ and with constant L according to Definition 2.2 such that

$$B_{x,+}(R) \subseteq \Omega_R(x) \subseteq B_{x,+}(2R). \tag{2.4}$$

The above definition guarantees that the constants of John domains are rescaling- and translation-invariant. This is a natural requirement as we are considering unbounded domains. Let $\Omega^\varepsilon := \varepsilon\Omega = \{x \in \mathbb{R}^3 |\ \varepsilon^{-1}x \in \Omega\}$. We refer to Ω^ε as a 'highly oscillating bumpy John domain'. Note that

$$\partial\Omega^\varepsilon \subseteq \{x \in \mathbb{R}^3,\ -\varepsilon < x_3 < 0\}. \tag{2.5}$$

A key fact about Ω^ε is that it is still a John domain with the same constants as in Definition 2.3, as these constants are scale-invariant.

Definition 2.4. We say that Ω is a 'periodic bumpy John domain' if the following holds:

(i) Ω is a bumpy John domain with constant L,

(ii) and Ω is $(2\pi\mathbb{Z})^2$-translation invariant, namely $2\pi\xi + \Omega = \Omega$ for any $\xi \in \mathbb{Z}^2 \times \{0\}$.

Periodicity is only needed at one place, to construct second-order boundary layer correctors in Theorem 2.10 below.

Finally, we define $B^\varepsilon_+(r)$ and $\Gamma^\varepsilon(r)$ in the same way as (2.3) above.

2.2.1 Large-scale Lipschitz regularity

We state first a large-scale (in other words 'mesoscopic', 'improved') Lipschitz estimate in the highly oscillating bumpy *Lipschitz* domain defined above, see Theorem 2.5. Second, we state a large-scale Lipschitz estimate in the highly oscillating bumpy *John* domain defined above, see Theorem 2.6. Notice that the John boundary case includes of course the case of Lipschitz boundaries. The statement of the theorems in both the Lipschitz and the John case may therefore seem a bit redundant. There are two main reasons for that: (i) historically we understood the case of boundaries given by Lipschitz graphs first and we were only able to handle John domains later, (ii) the proofs of these two theorems rely on completely different approaches, see Subsection 2.3.1 and Subsection 2.3.2 below, and therefore the estimates that we get are also a bit different.

Theorem 2.5 (improved Lipschitz regularity in Lipschitz domains; [183, Theorem 1], in collaboration with Higaki)**.** *For all $\varepsilon \in (0,1/2)$ and for all $M \in (0,\infty)$ the following statement holds. Let γ be a* Lipschitz graph *as defined above. If $(U^\varepsilon, P^\varepsilon) \in H^1(B^\varepsilon_+(1))^3 \times L^2(B^\varepsilon_+(1))$ is a weak solution of* (NS$^\varepsilon$) *we have*

$$\left(\fint_{B^\varepsilon_+(1)} |\nabla U^\varepsilon|^2\right)^{\frac12} \leq M \qquad \textit{implies} \qquad \sup_{r\in(\varepsilon,1/2)} \left(\fint_{B^\varepsilon_+(r)} |\nabla U^\varepsilon|^2\right)^{\frac12} \leq C(M)\,, \quad (2.6)$$

where the constant $C(M)$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ and M. Moreover, $C(M)$ is a monotone increasing function of M and converges to zero as M goes to zero.

Notice that $C(M)$ grows polynomially in M. Of course, if one considers the linear Stokes equation, then $C(M)$ grows linearly in M.

Theorem 2.6 (improved Lipschitz regularity in John domains; [184, Theorem A], in collaboration with Higaki and Zhuge)**.** *For all $\varepsilon \in (0,1/2)$, $L \in (0,\infty)$, $M \in (0,\infty)$ and $\delta \in (0,1)$, the following statement holds. Let Ω be a* bumpy John domain *with constant L according to Definition 2.3. If $(U^\varepsilon, P^\varepsilon) \in H^1(B^\varepsilon_+(1))^3 \times L^2(B^\varepsilon_+(1))$ is a weak solution of* (NS$^\varepsilon$)*, we have*

$$\left(\fint_{B^\varepsilon_+(1)} |\nabla U^\varepsilon|^2\right)^{\frac12} \leq M \quad \textit{implies} \quad \sup_{r\in(\varepsilon,1/2)} \left(\fint_{B^\varepsilon_+(r)} |\nabla U^\varepsilon|^2\right)^{\frac12} \leq C(M + M^{4+\delta})\,, \tag{2.7}$$

where the constant C is independent of ε, M and r, and depends on L and δ.

Notice that in both results above, we also have the large-scale boundedness of the pressure

$$\left(\fint_{B^\varepsilon_+(r)} \left|P^\varepsilon - \fint_{B^\varepsilon_+(1/2)} P^\varepsilon\right|^2\right)^{1/2}.$$

This is explicitly stated in [184, Theorem A].

In the vein of the seminal works [29, 30] and of [216, 177], we provide pointwise estimates for the large-scale decay of the velocity and pressure parts of the Green function associated to the Stokes system in bumpy John half-spaces, [184, Appendix B]. These estimates are pivotal to construct the first-order boundary layers in Subsection 2.3.2.

Let us insist on the fact that we separate the small-scale regularity, i.e. at scales $\lesssim \varepsilon$, from the mescopic- or large-scale regularity, i.e. at scales $\varepsilon \lesssim r \leq 1$. Concerning the small scales, the classical Schauder regularity theory for the Stokes and the Navier-Stokes equations was started by Ladyženskaja [229] using potential theory and by Giaquinta and Modica [158] using Campanato spaces. These classical estimates require some smoothness of the boundary and typically depend on the modulus of continuity of $\nabla\gamma$ when the boundary is given by the graph $x_3 = \gamma(x')$. Therefore, these estimates degenerate for highly oscillating boundaries $x_3 = \varepsilon\gamma(x'/\varepsilon)$ with $\varepsilon \ll 1$. As for the large scales, on the contrary, the regularity is inherited from the limit system when $\varepsilon \to 0$ posed in a domain with a flat boundary. Here no regularity is needed for the original boundary, beyond the boundedness of γ and of its gradient. The mechanism for the regularity at small scales and at large scales is hence completely different.

As in the works [29, 150, 213] one can combine the mesoscopic regularity estimate with the classical regularity at small scales provided the boundary is regular enough, i.e. $C^{1,\kappa}$ with $\kappa \in (0,1)$. In that case, we can prove the full Lipschitz estimate on $\|\nabla U^\varepsilon\|_{L^\infty(B_+^\varepsilon(1/2))}$. However, one cannot expect such an estimate to hold in Lipschitz domains, not mentioning John domains, even for the Laplace equation with the Dirichlet boundary condition. This fact justifies the terminology of 'improved regularity'.

Novelty of our results

The common novelties of these two results are that:

(1) They do not rely on any smoothness of the boundaries at the microscopic scale. Works on this topic prior to the paper [214] in collaboration with Kenig all relied on microscopic smoothness of the boundary. Let us cite the paper [150], where a uniform Hölder estimate for weak solutions of the Stokes equations is obtained when the boundary graph $\gamma \in C^{1,\omega}(\mathbb{R}^2)$ for a fixed modulus of continuity ω. In the work [213] with Kenig we proved uniform Lipschitz regularity for elliptic equations above bumpy $C^{1,\kappa}$ boundaries, $\kappa \in (0,1)$. In the paper [214] Kenig and I managed to decouple the small-scale regularity from the large-scale regularity for elliptic equations. Our paper [183] with Higaki and Theorem 2.5 in particular builds upon this progress. Theorem 2.6 from our work [184] in collaboration with Higaki and Zhuge goes even one step further since the regularity of the boundary is reduced to that of a John domain, and moreover the boundary does not need to be a graph.

(2) They do not rely on any structure assumption on the microscopic oscillations of the boundary. In other words, we do not need any periodicity, almost-periodicity or stationary ergodicity assumptions. The only important point is that the bumpy boundary is asymptotically flat.

(3) They hold outside the perturbative regime, i.e. without a priori smallness on the size of the solutions, which is due to the energy subcritical nature of the three-dimensional stationary Navier-Stokes equations. This is in stark contrast with previous works

concerned with linear equations (elliptic or Stokes systems), see [29, 150, 175, 176, 213, 214].

2.2.2 Higher-order regularity and local wall laws

Beyond the large-scale Lipschitz estimate above we prove:

Higher-order $C^{1,\mu}$ and $C^{2,\mu}$ estimates for $\mu \in (0,1)$
For higher-order $C^{1,\mu}$ regularity results, we refer to Theorem 2.9, estimate (2.11). For higher-order $C^{2,\mu}$ regularity results, we refer to Theorem 2.10, estimate (2.12) in the John case. In these estimates we measure the oscillation of the solution with respect to modified polynomials that vanish on the bumpy boundary. These modified polynomials are polynomials of degree one and two that are corrected by the first-order and second-order boundary layers.

Local wall laws
Remark 2.8 establishes the connection between Theorem 2.7 and the Navier wall law. Again, as explained above, we state the case of Lipschitz domains separately, although it is mostly covered by the theorems handling John domains.

Theorem 2.7 (a local wall law in bumpy Lipschitz domains; [183, Theorem 2], in collaboration with Higaki)**.** *Fix $M \in (0,\infty)$ and $\mu \in (0,1)$. We assume in addition that $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ is 2π-periodic in each variable. Then for all weak solutions $(U^\varepsilon, P^\varepsilon) \in H^1(B^\varepsilon_+(1))^3 \times L^2(B^\varepsilon_+(1))$ to (NS$^\varepsilon$) satisfying the a priori bound*

$$\left(\fint_{B^\varepsilon_+(1)} |\nabla U^\varepsilon|^2\right)^{\frac12} \leq M, \tag{2.8}$$

the following statement holds. There exists a constant vector field $\alpha^{(j)} = (\alpha^{(j)}_1, \alpha^{(j)}_2, 0)^T \in \mathbb{R}^3$, $j \in \{1,2\}$, depending only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ such that for all $\varepsilon \in (0,1/2)$ and $r \in (\varepsilon, 1/2)$, we have

$$\left(\fint_{B^\varepsilon_+(r)} \Big|U^\varepsilon(x) - \sum_{j=1}^{2} c^\varepsilon_{r,j}(x_3\mathbf{e}_j + \varepsilon\alpha^{(j)})\Big|^2 \,\mathrm{d}x\right)^{\frac12} \leq C(M)(r^{1+\mu} + \varepsilon^{\frac32} r^{-\frac12}), \tag{2.9}$$

where the coefficient $c^\varepsilon_{r,j}$, $j \in \{1,2\}$, is a functional of U^ε depending on ε, r, M, μ and $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$. The constant $C(M) \in (0,\infty)$ is independent of ε and r, but polynomial in M and dependent on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ.

Remark 2.8 (relation with the wall law).

(1) Let us denote the polynomial in (2.9) by $\mathcal{Q}^\varepsilon_{N,j}$, $j \in \{1,2\}$:

$$\mathcal{Q}^\varepsilon_{N,j}(x) = x_3\mathbf{e}_j + \varepsilon\alpha^{(j)}. \tag{2.10}$$

These polynomials are the building blocks of the large-scale regularity theory of the Navier-Stokes equations above bumpy Lipschitz boundaries. Then each $\mathcal{Q}^\varepsilon_{N,j}$ is a shear flow in the half-space $\mathbb{R}^3_+$ and is an explicit solution to the following Navier-Stokes equations in the half-space with a Navier-slip boundary condition

$$\begin{cases} -\Delta\mathcal{U}^\varepsilon_N + \nabla\mathcal{P}^\varepsilon_N = -\mathcal{U}^\varepsilon_N\cdot\nabla\mathcal{U}^\varepsilon_N & \text{in } \mathbb{R}^3_+, \\ \nabla\cdot\mathcal{U}^\varepsilon_N = 0 & \text{in } \mathbb{R}^3_+, \\ \mathcal{U}^\varepsilon_{N,3} = 0 & \text{on } \partial\mathbb{R}^3_+, \\ (\mathcal{U}^\varepsilon_{N,1},\mathcal{U}^\varepsilon_{N,2})^T = \varepsilon\overline{M}(\partial_3\mathcal{U}^\varepsilon_{N,1},\partial_3\mathcal{U}^\varepsilon_{N,2})^T & \text{on } \partial\mathbb{R}^3_+, \end{cases} \tag{NS$^\varepsilon_N$}$$

with a trivial pressure $\mathcal{P}_N^\varepsilon = 0$. Here the 2×2 matrix $\overline{M} = (\alpha_i^{(j)})_{1\le i,j\le 2}$ can be proved to be positive definite. Thus the estimate (2.9) in Theorem 2.7 reads as follows: any weak solution U^ε to (NS$^\varepsilon$) can be approximated at any mesoscopic scale by a linear combination of the Navier polynomials $\mathcal{Q}_{N,1}^\varepsilon$ and $\mathcal{Q}_{N,2}^\varepsilon$. This is a local version of the Navier wall law, which has been widely studied in the global framework; see Subsection 2.1.1.

(2) Our result can be extended to the stationary ergodic or the almost-periodic setting. We also note that the wall law breaks down when the boundary does not have any structure at all, see [154] where it is showed that some ergodicity is needed to study the tails of boundary layer correctors that determine the Navier polynomials.

For the John domains, the analogous result reads as follows. We state the $C^{1,\mu}$ estimate with the boundary layer correctors. An asymptotic wall law in the spirit of (2.9) would also hold in periodic John domains, but we do not state it here.

Theorem 2.9 (large-scale $C^{1,\mu}$ regularity for John domains; [184, Theorem B], in collaboration with Higaki and Zhuge)**.** *For all $\mu \in [0,1)$, $\varepsilon \in (0,1/2)$, $L \in (0,\infty)$, $M \in (0,\infty)$ and $\delta \in (0,1)$, the following statement holds. Let Ω be a bumpy John domain with constant L according to Definition 2.3. If $(U^\varepsilon, P^\varepsilon) \in H^1(B_+^\varepsilon(1))^3 \times L^2(B_+^\varepsilon(1))$ is a weak solution of (NS$^\varepsilon$) satisfying the a priori bound (2.8), then, there exists a constant $\bar{P}_1$ depending on P^ε such that, for any $r \in (\varepsilon, \frac{1}{2})$,*

$$\inf_{(\mathcal{W},\mathcal{P})\in\mathscr{Q}_1(\Omega)} \left\{ \frac{1}{r}\left(\fint_{B_+^\varepsilon(r)} |U^\varepsilon - \varepsilon\mathcal{W}(x/\varepsilon)|^2\,\mathrm{d}x \right)^{1/2} + \left(\fint_{B_+^\varepsilon(r)} |P^\varepsilon - \mathcal{P}(x/\varepsilon) - \bar{P}_1|^2\,\mathrm{d}x \right)^{1/2} \right\} \le Cr^\mu(M + M^{4+2\mu+\delta}), \tag{2.11}$$

where $\mathscr{Q}_1(\Omega)$ is the class of all solutions to the Stokes equations in the bumpy John half-space Ω with linear growth at infinity that vanish on $\partial\Omega$, see [184, equation (5.1)]. The constant C is independent of ε, M and r, but depends on L, μ and δ.

While Theorem 2.9 holds for arbitrary bumpy John half-spaces, for the next result, we work in periodic John domains. As we outlined above, the extra periodicity assumption makes the analysis of the second-order boundary layers more manageable. However, we do not claim that this assumption is necessary. It is possible that a similar theorem can be proved in other frameworks with quantitative ergodicity properties, such as the quasiperiodic setting with some non-resonance condition; see Chapter 3 for related thoughts. The following result contains of course the case when the boundary is Lipschitz.

Theorem 2.10 (large-scale $C^{2,\mu}$ regularity for John domains; [184, Theorem C], in collaboration with Higaki and Zhuge)**.** *For all $\mu \in [0,1)$, $\varepsilon \in (0,\frac{1}{2})$, $L \in (0,\infty)$, $M \in (0,\infty)$ and $\delta \in (0,1)$, the following statement holds. Let Ω be a periodic bumpy John domain with constant L according to Definition 2.4. If $(U^\varepsilon, P^\varepsilon) \in H^1(B_+^\varepsilon(1))^3 \times L^2(B_+^\varepsilon(1))$ is a weak solution of (NS$^\varepsilon$) satisfying (2.8), then, there exists a constant $\bar{P}_2$ depending on P^ε such* *that, for any* $r \in (\varepsilon, 1/2)$,

$$
\begin{aligned}
\inf_{\substack{(\mathcal{W}_1,\mathcal{P}_1)\in\mathscr{Q}_1(\Omega)\\(\mathcal{W}_2,\mathcal{P}_2)\in\mathscr{Q}_2(\Omega)}} \Bigg\{ \frac{1}{r}\left(\fint_{B_+^\varepsilon(r)} |U^\varepsilon - \varepsilon\mathcal{W}_1(x/\varepsilon) - \varepsilon^2\mathcal{W}_2(x/\varepsilon)|^2\,\mathrm{d}x\right)^{1/2} &\\
+ \left(\fint_{B_+^\varepsilon(r)} |P^\varepsilon - \mathcal{P}_1(x/\varepsilon) - \varepsilon\mathcal{P}_2(x/\varepsilon) - \bar{P}_2|^2\,\mathrm{d}x\right)^{1/2}\Bigg\} &\\
\leq Cr^{1+\mu}(M + M^{6+2\mu+\delta}), &
\end{aligned}
\tag{2.12}
$$

where $\mathscr{Q}_1(\Omega)$ *is used in Theorem 2.9 and defined in [184, equation (5.1)] and* $\mathscr{Q}_2(\Omega)$ *is the class of all solutions to the Stokes equations in the periodic bumpy John half-space* Ω*, with quadratic growth at infinity, that vanish on* $\partial\Omega$*, see [184, equation (5.2)]. The constant* C *is independent of* ε*,* M *and* r*, but depends on* L*,* μ *and* δ*.*

We point out that the building blocks in $\mathscr{Q}_1(\Omega)$ and $\mathscr{Q}_2(\Omega)$ are defined through the first-order and second-order boundary layers. To emphasize the distinguished role of the classes $\mathscr{Q}_1(\Omega)$ and $\mathscr{Q}_2(\Omega)$ we state a Liouville-type theorem for the entire solutions of the Stokes system in Ω

$$
\begin{cases}
-\Delta\mathcal{W} + \nabla\mathcal{P} = 0 & \text{in} \quad \Omega,\\
\nabla\cdot\mathcal{W} = 0 & \text{in} \quad \Omega,\\
\mathcal{W} = 0 & \text{on} \quad \partial\Omega.
\end{cases}
\tag{2.13}
$$

Theorem 2.11 (Liouville theorems for sub-linear, sub-quadratic and sub-cubic growth; [184, Corollary 3.1] and [184, Theorem 5.8] in collaboration with Higaki and Zhuge)**.** *Let* Ω *be a bumpy John domain according to Definition 2.3. Let* (W, Π) *be a weak solution of* (2.13).

sub-linear *If*

$$
\lim_{R\to\infty}\frac{1}{R}\left(\fint_{B_R(0)\cap\Omega}|\mathcal{W}|^2\right)^{1/2} = 0,
$$

then $\mathcal{W}\equiv 0$*, hence* $\mathcal{P}$ *is constant.*

sub-quadratic *If for some* $\sigma \in (0, 1]$

$$
\liminf_{R\to\infty}\frac{1}{R^{1+\sigma}}\left(\fint_{B_+(R)}|\mathcal{W}|^2\right)^{1/2} = 0,
$$

then $(\mathcal{W},\mathcal{P}) \in \mathscr{Q}_1(\Omega)$*, up to a constant for* $\mathcal{P}$*.*

sub-cubic *In addition, assume* Ω *is periodic bumpy John domain according to Definition 2.4. If for some* $\sigma \in (0, 1)$*,*

$$
\liminf_{R\to\infty}\frac{1}{R^{2+\sigma}}\left(\fint_{B_{R,+}}|\mathcal{W}|^2\right)^{1/2} = 0,
$$

then $(\mathcal{W},\mathcal{P}) \in \mathscr{Q}_1(\Omega) + \mathscr{Q}_2(\Omega)$*, up to a constant for* $\mathcal{P}$*.*

Notice that in the case of sub-cubic growth, contrary to the sub-quadratic case, we cannot reach the exponent $\sigma = 1$ because we do not have a large-scale C^3 estimate.

Novelty of our results

As far as we know, the results stated above go far beyond the existing litterature. They are the first ones concerned with the large-scale high-order regularity above bumpy boundaries. They also do not have any counterpart for elliptic equations.

Theorem 2.7 from our work [183] with Higaki is the first local justification of a wall law. It is important physically as well as mathematically since we are interested in the effects of rough boundaries on viscous fluids. Our result is a step toward a better understanding of roughness effects on the Navier-Stokes flows in view of regularity improvement.

Moreover, our work [184] with Higaki and Zhuge is the first to construct the second-order boundary layers with a linear growth in the direction tangential to the boundary. To make the analysis more tractable, we assume that the boundary is periodic. We are aware of the papers [42, 70], where a refined second-order approximation is constructed for the Stokes equations in a two-dimensional rough channel. However, the boundary layers considered in [42, 70] only involve data spanned by linear and quadratic polynomials of the vertical variable, x_2 and x_2^2 in this two-dimensional case, which are bounded on the bumpy boundary. In our three-dimensional situation, the class of 'no-slip Stokes polynomials' needed for the $C^{2,\mu}$ regularity theory, see [184, Subsection 4.1], is much richer and involves boundary data with linear growth at spatial infinity.

Further developments

This line of research is currently being developed in the following directions:

(1) **Smoothness at large scales**
In the paper [178], Gu and Zhuge study the system of elasticity in domains which are not asymptotically flat, but smooth at the large scales; see [178, equation (1.22)] for the definition of the ε-scale $C^{1,\alpha}$ condition. Their domain is Lipschitz at the small scales, but this assumption can be relaxed to include John domains.

(2) **Rougher boundaries**
The paper [356] by Zhuge goes even beyond the framework of John domains for the study of the study of the large-scale Lipschitz regularity for linear elliptic equations. Indeed, no Bogovskii operator is needed for that analysis, which is the reason why we work in John domains for the Stokes system. In [356] the roughness of the boundary is arbitrary at small scales and satisfies a quantitative Reifenberg flatness condition at large scales.

(3) **Higher-order boundary layers**
In the paper [185], Higaki and Zhuge construct boundary layer correctors up to arbitrary order for the Stokes system in periodic bumpy John domains. They systematize the construction of the second-order boundary layers stated in Theorem 2.10 above. Such a study is in the vein of the paper of Armstrong, Kuusi and Mourrat [24] concerned with higher-order cell correctors for elliptic equations with periodic coefficients.

(4) **Building blocks**
The point of view adopted in Theorem 2.8, Remark 2.7 and Theorem 2.11 that certain functions play the role of 'building blocks' for the regularity theory is not new. The idea is already present in the work of Avellaneda and Lin [29], where $\mathbf{a}$-harmonic linear functions (linear functions perturbed by cell correctors) are the building blocks

of the $C^{1,\mu}$ regularity theory. Several recent original papers put forward the same idea in different contexts; see [355] (piecewise linear functions), [205] ($\mathbf{a}$-harmonic singular functions). These results are in line with the extended point of view of regularity theory that we aim at developping, see the introduction of this thesis, especially Subsection 1.1.3.

2.3 New ideas and strategy for the proofs

2.3.1 Boundary layers and large-scale boundary regularity in Lipschitz domains

In the work [183] with Higaki, we perform the analysis in bumpy Lipschitz domains. As in the papers with Kenig [213, 214], the proofs of Theorem 2.5 and of Theorem 2.7 are based on a compactness argument originating from the seminal work by Avellaneda and Lin [29] on uniform estimates in homogenization, see Subsection 2.1.2. The mains points are: (i) construction of a boundary layer corrector in the Lipschitz half-space, (ii) proof of the mesoscopic regularity by compactness and iteration. We focus on point (i) below.

Our analysis takes advantage of the progress made in [213] to remove structure assumptions on the oscillations of the boundary and in [214] to remove small-scale regularity assumptions. In the present case, there are additional difficulties related to:

(1) the incompressibility condition; a simplifying feature is the fact that in the case of a boundary given by a Lipschitz graph, the boundary condition in the boundary layer system (2.15) is a trace of a divergence-free function;

(2) the nonlocal pressure; since we consider the stationary Navier-Stokes equations, one can estimate the pressure directly in terms of the velocity using Bogovskii-type operators [5, 140]

$$\left\|P-(P)_{B(1)}\right\|_{L^2(B(1))} \leq C\|\nabla P\|_{H^{-1}(B(1))} \tag{2.14}$$

and the equation for ∇P; similar estimates hold close to the boundary and imply Caccioppoli-type estimates [183, Appendix B];

(3) the nonlinearity of the Navier-Stokes equations and the lack of smallness of the solutions; on the one hand, the stationary three-dimensional Navier-Stokes equations are energy subcritical, hence the solutions of the limit stationary Navier-Stokes equations in the flat domain are smooth, see [183, Appendix A]; on the other hand, the nonlinearity creates a difficulty when iterating the one-scale improvement of flatness estimate; we overcome this difficulty by choosing the free parameter θ in the compactness lemma in terms of the data $\|\gamma\|_{W^{1,\infty}}$ and M, see [183, Section 5].

Focus on the boundary layer corrector and a Saint-Venant estimate in a channel The use of boundary layer correctors is key to the compactness argument. Indeed, in order to iterate the one-scale improvement of flatness estimate, one needs some sort of self-similarity, see Subsection 2.1.2 above and in particular Remark 2.1. The boundary layers correct the highly oscillating trace on the bumpy boundary of the affine functions $(a, b, 0)^T x_3$, which are the building blocks for the $C^{1,\mu}$ regularity in the flat domain. The boundary layers correctors solve the following equations for $j \in \{1,2\}$

$$\begin{cases} -\Delta \mathcal{V} + \nabla \mathcal{P} = 0 & \text{in} \quad \Omega, \\ \nabla \cdot \mathcal{V} = 0 & \text{in} \quad \Omega, \\ \mathcal{V}(y', \gamma(y')) = -\gamma(y')\mathbf{e}_j, \end{cases} \tag{2.15}$$

where Ω is the bumpy Lipschitz half-space defined by (2.2). We emphasize that the boundary layer correctors solve the linear Stokes equations. This is expected from the following formal heuristics. Indeed, in the boundary layer $U^\varepsilon \simeq \varepsilon \mathcal{V}(x/\varepsilon)$, so that $\mathcal{V}$ solves $-\frac{1}{\varepsilon}\Delta \mathcal{V} + \varepsilon \mathcal{V} \cdot \nabla \mathcal{V} + \nabla \mathcal{Q} = 0, \quad \nabla \cdot \mathcal{V} = 0$.

We prove that there exists a unique weak solution $(\mathcal{V}, \mathcal{Q}) \in H^1_{loc}(\overline{\Omega})^3 \times L^2_{loc}(\overline{\Omega})$ to (2.15) satisfying the following energy bound, which is locally uniform in the tangential variable

$$\sup_{\eta \in \mathbb{Z}^2} \int_{\eta + (0,1)^2} \int_{\gamma(y')}^{\infty} |\nabla \mathcal{V}(y', y_3)|^2 \, dy_3 dy' \le C(\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}); \tag{2.16}$$

see [183, Proposition 8]

Our methodology for the analysis of the system (2.15) is close to the one developed in my work with Kenig [214] concerned with the boundary regularity of elliptic equations in bumpy Lipschitz domains. This strategy is inspired from the work of Gérard-Varet and Masmoudi [154] and was successfully implemented in other situations, such as the analysis of oceanic boundary layers [116, 115, 286] near a rough seabed (Ekman layer) or a rough coast (Munk layers), or the analysis of boundary layers for fluids slipping over a rough surface [114]. Their scheme enables us to: (i) compensate for the unavailability of certain tools (Fourier analysis, Green and Poisson kernel estimates) up to the boundary, (ii) circumvent the lack of structure of the boundary which implies a lack of compactness of the solution to the boundary layer problem. Concerning (ii), let us note for instance that contrary to the boundary layer system (3.5) analyzed in Chapter 3, there is no underlying quasiperiodic structure that allows to lift the system on a higher-dimensional cylinder, compact in the direction tangential to the boundary.

The basic idea of Gérard-Varet and Masmoudi [154] is inspired from domain decomposition methods in numerical analysis. We split the bumpy Lipschitz domain Ω in two parts: (i) a flat half-space $\mathbb{R}^3_+$ in which we can rely on explicit representation formulas, (ii) a bumpy channel $\Omega \cap \mathbb{R}^3_-$ in which we gain compactness in the vertical direction that allows the use of Poincaré-type inequalities. The half-space system (2.15) is then reduced (after lifting the Dirichlet boundary condition) to an equivalent system in the channel $\Omega \cap \mathbb{R}^3_-$,

$$\begin{cases} -\Delta \mathcal{W} + \nabla \mathcal{Q} = 0\,, & y \in \Omega \cap \mathbb{R}^3_-, \\ \nabla \cdot \mathcal{W} = 0\,, & y \in \Omega \cap \mathbb{R}^3_-, \\ \mathcal{W}(y', \gamma(y')) = 0, & \\ (-\partial_3 \mathcal{W} + \mathcal{Q}\mathbf{e}_3)|_{y_3=0} = \mathrm{DN}(\mathcal{W}|_{y_3=0}) - \mathbf{e}_j\,, & \end{cases} \tag{2.17}$$

with a transparent boundary condition on the upper boundary $\{y_3 = 0\}$ involving a Dirichlet-to-Neumann operator DN. That operator implicitly solves the Dirichlet-Stokes problem in the flat half-space.

We carry out local energy estimates in the channel $\Omega \cap \mathbb{R}^3_-$. Such local energy estimates were carried out by Ladyženskaja and Solonnikov [231] for the Navier-Stokes equations in an unbounded channel with Dirichlet boundary conditions. Here the nonlocality of the Dirichlet-to-Neumann operator DN makes the treatment of the large-scales particularly delicate. For an auxiliary parameter $m \in \mathbb{N}$, $m \geq 1$ and $k \in \mathbb{N}$ with $k \geq m$, one covers the two-dimensional space $\mathbb{R}^2$ by square tiles of area $(2m)^2$:

$$C_{k,m} := \left\{T = \eta + (-m,m)^2, \quad \eta \in \mathbb{Z}^2 \text{ and } T \subseteq \mathbb{R}^2 \setminus (-k-m+1, k+m-1)^2\right\},$$

We then estimate the local energy

$$E_k[\mathcal{W}] := \int_{(-k,k)^2} \int_{\gamma(y')}^{0} |\nabla \mathcal{W}|^2 \, dy_3 dy'.$$

This yields the following discrete differential inequality, so-called 'Saint-Venant' or 'Phragmén-Lindelöf' estimate,

$$E_k[\mathcal{W}] \leq C_* \Big(k^2 + E_{k+m}[\mathcal{W}] - E_k[\mathcal{W}] + \frac{k^4}{m^6} \sup_{T \in C_{k,m}} E_T[\mathcal{W}] \Big), \tag{SV}$$

where C_* is independent of m, and k, and depends only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$. Estimate (SV) yields the a priori bound

$$\sup_{\eta \in \mathbb{Z}^2} \int_{\eta+(0,1)^2} \int_{\gamma(y')}^{0} |\nabla \mathcal{W}(y', y_3)|^2 \, dy_3 dy' \leq C(\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}),$$

by backward induction and choosing m in terms of C_*. This bound in turn implies existence for (2.17), hence also for the original boundary layer system (2.15). Moreover, uniqueness for the linear system (2.17) is also deduced from (SV), without the contribution k^2 of the source term, by a backward induction.

2.3.2 Large-scale boundary regularity and boundary layers in John domains

In the paper [184] with Higaki and Zhuge, we perform the analysis of the large-scale regularity in John domains. As we mentioned above in Subsection 2.3.1, in the works [214] with Kenig and [183] with Higaki, several elementary tools were developed for the analysis of the first-order boundary layer correctors in bumpy Lipschitz domains without structure. Here, the analysis in bumpy John domains requires to push the techniques even further, to the limit of what seems technically possible.

Analysis in John domains The analysis in bumpy John domains relies on two elementary tools.

First, we can rely on the existence of a right inverse for the divergence operator with Dirichlet boundary conditions. The existence of such a Bogovskii operator in John domains is stated in [5]. This operator is required in order to estimate the pressure (as in the interior pressure estimate (2.14)) and hence prove a weak Caccioppoli inequality for the Stokes system [184, Lemma A.4], which then implies the reverse Hölder inequality as a starting point of the large-scale regularity theory.

Second, we have a Poincaré inequality for functions $H^1(B^\varepsilon(r))$ vanishing on $\Gamma^\varepsilon(r)$. We can for instance use [157, Proposition 3.15] to get that for all fixed bumpy John domain

Ω with constant $L \in (0, \infty)$ according to Definition 2.3, for all fixed $r \geq \varepsilon$, and for all $U \in H^1(B^\varepsilon_{r,+})$ such that $U = 0$ on Γ^ε_r,

$$\int_{B^\varepsilon_{r,+}} |U|^2 \leq Cr^2 \int_{B^\varepsilon_{r,+}} |\nabla U|^2, \tag{2.18}$$

where C is an absolute constant independent of ε and r. Notice that this estimate is only valid at scales $r \geq \varepsilon$. Indeed, below that scale the constant in (2.18) may degenerate due to possible inward cusps of highly oscillating bumpy John domains that lead to $|B(r) \setminus B^\varepsilon(r)| \ll r^3$, which is the measure of the region where we extend U by zero.

As a consequence of the fact that the Poincaré inequality fails at small scales, all the boundary estimates of our work are mesoscopic estimates in the sense that they involve averaged quantities such as

$$\mathcal{M}^p_t[\nabla U^\varepsilon](x) = \left(\fint_{Q_t(x)} |\nabla U^\varepsilon|^p \right)^{1/p}.$$

for $t \geq \varepsilon$ smoothing out the possibly rough microscales.

Quantitative method for the large-scale Lipschitz regularity There is one particular point, where we are completely unable to transfer the techniques used above Lipschitz graphs to the present context. Indeed, in [214, 183] we used a domain decomposition method pioneered in [154] to study the well-posedness of the Stokes system for the first-order boundary layer correctors, see Subsection 2.3.1. We do not manage to adapt this strategy to our current situation, because of difficulties to estimate the pressure in the local energy estimates in the bumpy channel; see below the paragraph 'Construction of boundary layers'. As a consequence, we cannot rely on boundary layer correctors to prove Lipschitz estimates directly as was done in [214, 183]. This requires to patiently bootstrap the regularity via quantitative approximation arguments:

(1) Caccioppoli's inequality in combination with the Poincaré-Sobolev inequality implies a **large-scale reverse Hölder inequality**.

(2) Combining the large-scale reverse Hölder inequality with Gehring's lemma, we obtain a **large-scale Meyers type estimate**, which is a first improvement of integrability for $\mathcal{M}^2_t[\nabla U^\varepsilon]$, see [184, Lemma 2.2].

(3) The Meyers estimate enables us to prove at any scale a sub-optimal error estimate between U^ε and an approximation $\bar{U}$ in the flat domain, see [184, Lemma 2.6]. Along with the real variable argument of [313, Theorem 4.2.3] and improved flatness of $\bar{U}$, this enables us to prove a **large-scale** $W^{1,p}$ **estimate** for finite p for the Stokes system, see [184, Theorem 2.4], following the scheme of Caffarelli and Peral [76] explained above in Subsection 2.1.3.

(4) In order to handle the nonlinear Navier-Stokes system, we consider the nonlinear term as a perturbation of the linear Stokes system. Hence we use the linear large-scale $W^{1,p}$ estimate along with a large-scale Sobolev embedding result [184, Theorem 2.7] to **bootstrap the integrability of the nonlinear source term** $-U^\varepsilon \otimes U^\varepsilon$; see [184, Theorem 2.8].

(5) Finally, we prove the **large-scale Lipschitz estimate** stated in Theorem 2.6 by an approximation argument of U^ε by $\bar{U}$ [184, Lemma 3.2] and improved flatness for $\bar{U}$.

Construction of boundary layers In the work [183] the well-posedness of the first-order boundary layer system (2.15) was proved over Lipschitz graphs by a domain decomposition method: coupling of the Stokes problem in a bumpy channel $\Omega \cap \mathbb{R}^3_-$ with the Stokes problem in the flat half-space $\mathbb{R}^3_+$ via a nonlocal Dirichlet to Neumann boundary condition at the interface $\partial\mathbb{R}^3_+$. We face considerable technical difficulties when trying to adapt this strategy to the case of bumpy John domains. Indeed, the local energy estimates in the bumpy channel require to estimate the pressure, or to work with divergence-free test functions. In either case, we need to construct a Bogovskii operator for a sequence of exhausting domains containing $\Omega \cap \{|x'| \leq k,\, x_3 < 0\}$ with a constant uniform in k. The construction of the Bogovskii operator of [5] relies on connecting any point in the bumpy John domain to a fixed neighborhood of a reference point $\bar{x}$. Such a procedure gives, for a slim domain such as $\Omega \cap \{|x'| \leq k,\, x_3 < 0\}$, a constant in the Bogovskii estimate that scales proportionately to the horizontal size k of the domain. We are unable to take advantage of the small vertical extent of the domain to provide a modified construction of the Bogovskii operator. This would be needed to carry out the downward iteration on the local energy estimates, also called Saint-Venant estimates, in [183].

This obstacle related to the existence of boundary layer correctors in bumpy John domains lead us to the new strategy explained above, that is inspired from almost-periodic and random homogenization where similar issues for the existence of correctors with good properties arise; see Subsection 2.1.3 above.

Now that the large-scale Lipschitz estimate of Theorem 2.6 is proved without any use of boundary layers, we develop a new strategy for the existence of solutions to the boundary layer system (2.15). In fact, the large-scale Lipschitz regularity in Theorem 2.6 makes it possible to construct the velocity and pressure parts of the Green function in bumpy John domains, and to estimate their decay at large scales, see [184, Appendix B]. Similar arguments were used for instance in [30, 216] for elliptic equations and in [177] for the Stokes equations. These estimates are the key for our new proof of the existence of the first-order boundary layer correctors; see [184, Theorem 4.1]. In this way we are able to completely by-pass the difficulties posed by the domain decomposition method used in [154, 116, 115, 214, 183].

To the best of our knowledge, our present work is also the first to carry out a thorough analysis of the second-order boundary layer correctors, allowing for linear growth of the boundary data in the tangential direction. Our key observation is an algebraic connection between the first-order and second-order boundary layers on the boundary, which allows us to use the first-order boundary layer correctors in an Ansatz for the second-order boundary layers, see [184, Section 4.3]. Unlike the first-order boundary layers (which form a two-dimensional vector space), the space of second-order boundary layers is six-dimensional and needs three different ways of construction, based on the structures of the associated Stokes polynomials. For our analysis to go through, we also need some good quantitative convergence/decay of the first-order boundary layers away from the boundary. Hence we work in a periodic framework, according to Definition 2.4, but this is by no means an optimal assumption. Other structures, such as almost-periodic structures with a non-resonance condition, or random ergodic with quantitative decorrelation properties at large scales, would certainly be manageable.

Quantitative method for the higher-order regularity Let us summarize the chain of results that we obtain in the paper [184]:

(1) proof of a large-scale Lipschitz estimate by a quantitative method, see Theorem 2.6,

(2) proof of large-scale Green kernel bounds, see [184, Proposition B.3-B.5],

(3) proof of the existence of first-order boundary layer correctors, see [184, Theorem 4.1],

(4) large-scale $C^{1,\mu}$ estimates, see Theorem 2.9,

(5) proof of the existence of second-order boundary layer correctors see [184, Theorem 4.3],

(6) large-scale $C^{2,\mu}$ estimates, see Theorem 2.10.

We emphasize that we rely on a quantitative perturbation method inspired from the classical Schauder regularity theory. In the vein of Caffarelli and Peral [76], the method follows the lines given in Subsection 2.1.3. In a nutshell, we use the improved regularity of the approximate problem, to get the scale-by-scale decay of excess quantities (measuring for instance, Hölder continuity, Lipschitz, $C^{1,\mu}$, $C^{2,\mu}$, or higher regularity) for the original rough problem, up to a small error. We then conclude by a real variable argument or an iteration lemma, which are in some sense black boxes oblivious to the equations.

In the context of homogenization, the homogenized limit problem with constant coefficients is the approximate problem. Here, the approximate problem is a Stokes problem in a domain with a flat boundary. Both problems have improved regularity, in the sense that the solutions are basically as smooth as one wishes.

Remark 2.12 (about the self-similarity). This remark is the pendant of Remark 2.1 above. For the $C^{1,\mu}$ estimate, a crucial fact is that $(\mathcal{W},\mathcal{P})$ belongs to $\mathscr{Q}_1(\Omega)$, where the class $\mathscr{Q}_1(\Omega)$ is defined in Theorem 2.9. In other words, the building blocks of the $C^{1,\mu}$ regularity, which are linear polynomials corrected by boundary layers, are solutions to the Stokes system that vanish on the bumpy boundary. Hence, by rescaling, $(U^\varepsilon,P^\varepsilon)-(\varepsilon\mathcal{W}(x/\varepsilon),\mathcal{P}(x/\varepsilon))$ is still a weak solution with a no-slip boundary condition. This observation allows us to capture the regularity beyond the Lipschitz estimate. A similar remark holds for the case of the $C^{2,\mu}$ regularity.

Chapter 3

Quantitative homogenization of boundary layers

This chapter relies mainly on the paper:

— [23], with Scott Armstrong, Tuomo Kuusi and Jean-Christophe Mourrat, Quantitative analysis of boundary layers in periodic homogenization, *Arch. Ration. Mech. Anal.* (2017).

Contents

This chapter is concerned with the quantitative analysis of a boundary layer problem that arises in the study of periodic homogenization. We consider the oscillating Dirichlet problem for uniformly elliptic systems with periodic coefficients, taking the form

$$\begin{cases} -\nabla\cdot\left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla U^\varepsilon(x)\right) = 0 & \text{in } \Omega, \\ U^\varepsilon(x) = g\left(x,\frac{x}{\varepsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{BL$^\varepsilon$}$$

Here $0<\varepsilon\ll 1$, the dimension $d\geq 2$ and $\Omega\subseteq\mathbb{R}^d$. We will require in addition that Ω is polygonal or smooth, convex or uniformly convex, although this last two assumptions can be relaxed. The coefficients are given by a tensor $\mathbf{a}=\left(\mathbf{a}_{ij}^{\alpha\beta}\right)_{i,j=1,\ldots,L}^{\alpha,\beta=1,\ldots,d}$ and the unknown function $U^\varepsilon=(U^\varepsilon_j)_{j=1,\ldots,L}$ takes values in $\mathbb{R}^L$. The coefficients are assumed to satisfy, for some fixed constant $\lambda\in(0,1)$, the uniformly elliptic condition, for all $\xi=(\xi_i^\alpha)\in\mathbb{R}^d\times\mathbb{R}^L$, for all $y\in\mathbb{R}^d$,

$$\lambda\left|\xi\right|^2\leq\mathbf{a}_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta\leq\lambda^{-1}\left|\xi\right|^2. \tag{3.1}$$

Both the coefficients $\mathbf{a}$ and the Dirichlet boundary condition $g : \partial\Omega \times \mathbb{R}^d \to \mathbb{R}$ are assumed to be smooth functions,

$$\mathbf{a} \in C^\infty\big(\mathbb{R}^d; \mathbb{R}^{L\times L\times d\times d}\big) \quad \text{and} \quad g \in C^\infty(\partial\Omega \times \mathbb{R}^d) \tag{3.2}$$

and periodic in the fast variable, that is,

$$\mathbf{a}(y) = \mathbf{a}(y+\xi) \quad \text{and} \quad g(x,y) = g(x,y+\xi) \quad \forall x \in \partial\Omega,\ y \in \mathbb{R}^d,\ \xi \in \mathbb{Z}^d. \tag{3.3}$$

The goal is to understand the asymptotic behavior of the system (BL^ε) as $\varepsilon \to 0$.

3.1 Context: state of the art and obstacles

The problem (BL^ε) arises naturally in the theory of elliptic homogenization when one attempts to obtain a two-scale expansion of solutions of the Dirichlet problem with non-oscillating boundary condition near the boundary

$$\begin{cases} -\nabla \cdot \mathbf{a}\big(\frac{x}{\varepsilon}\big)\nabla V^\varepsilon = 0 & \text{in } \Omega, \\ V^\varepsilon(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Indeed, the oscillating interior first-order corrector in the two-scale expansion induces a locally periodic perturbation of the boundary condition of order $O(\varepsilon)$:

$$V^\varepsilon(x) = \bar{V}(x) + \varepsilon\left(\chi\big(\tfrac{x}{\varepsilon}\big) \cdot \nabla\bar{V}(x) + U^\varepsilon_{bl}(x)\right) + \dots$$

where $\chi = \chi(y)$ is the periodic cell corrector and U^ε_{bl} solves (BL^ε) with $g\big(x, \frac{x}{\varepsilon}\big) = -\chi\big(\frac{x}{\varepsilon}\big) \cdot \nabla\bar{V}(x)$. For more details, we refer to [50, 313]. In other words, when examining the fine structure of solutions of the Dirichlet problem with oscillating coefficients, one expects to find a boundary layer in which the solutions behave qualitatively differently than they do in the interior of the domain, for which we have a complete understanding to arbitrary order [24]. The study of this boundary layer can be reduced to a problem of the form (BL^ε). This problem was identified at the start of the homogenization theory in the seventies by Bensoussan, Lions and Papanicolaou [50]. Despite this, major progress on this question was only achieved about ten years ago with the works of Gérard-Varet and Masmoudi [154, 155, 156]. As a matter of fact, the analysis of (BL^ε) is particularly challenging because of: (i) the high oscillations that create strong gradients in the vicinity of the boundary $\partial\Omega$, (ii) the boundary that breaks the periodic structure and (iii) the possible resonances between the periodic coefficients and the non-periodic oscillations on the boundary. This chapter is devoted to the quantitative analysis of these concentration phenomena.

3.1.1 Boundary layers in the half-space and polygonal domains

The analysis of the boundary layer in the domain Ω and the definition of the homogenized boundary condition $\bar{g}$ are based on an approximation procedure involving half-space boundary layer problems

$$\begin{cases} -\nabla \cdot \mathbf{a}(y)\nabla\mathcal{U} = 0, & y \cdot n(x) > a, \\ \mathcal{U} = g(x,y), & y \cdot n(x) = a, \end{cases} \tag{3.5}$$

where $n(x) \in \partial B_1, a \in \mathbb{R}, \mathcal{U} = \mathcal{U}(y; a, n(x))$. Full understanding of these boundary layers was achieved in the works [155, 156, 282], after earlier works [128, 267, 53, 266, 15, 268] concerned with rational half-spaces for which $n \in \mathbb{R}\mathbb{Z}^d$.

Convergence of $\mathcal{U}(y; a, n)$ when $y \cdot n \to \infty$ to a 'boundary layer tail' i.e. a constant vector field $\mathcal{U}^\infty$ holds whatever $n \in \partial B_1$, see my work [282] via a two-scale expansion for the Poisson kernel. Quantitative rates of convergence, though, are very sensitive to the Diophantine properties of the normal n. Let $\kappa > \frac{1}{d-1}$ be fixed. We say that $n \in \partial B_1$ is 'Diophantine' or 'non-resonant' with constant $A > 0$ if

$$|(\mathrm{I}_d - n \otimes n)\xi| \geq A|\xi|^{-\kappa}, \qquad \text{for all} \quad \xi \in \mathbb{Z}^d \setminus \{0\}, \tag{3.6}$$

where $(\mathrm{I}_d - n \otimes n)\xi$ denotes the projection of ξ on the hyperplane $n^\perp$.

Let M be an orthogonal matrix such that $Me_d = n$, and N be the matrix of the $d-1$ first columns of M. The matrices M and N essentially give us a convenient coordinate system to work in. They of course depend on $n \in \partial B_1$ and can be chosen in such a way as to be locally smooth functions of n in a neighborhood of any fixed $\bar{n} \in \partial B_1$. Thus, condition (3.6) is equivalent to

$$\left|N^T \xi\right| \geq A|\xi|^{-\kappa}, \qquad \text{for all} \quad \xi \in \mathbb{Z}^d \setminus \{0\}. \tag{3.7}$$

A key idea to study the system (3.5) is to take advantage of the quasiperiodic structure of the system along the boundary. One lifts the problem to a higher-dimensional cylinder $\mathbb{T}^d \times (a, \infty)$, where $\Upsilon = \Upsilon(\theta, t)$ solves a sub-elliptic problem

$$\begin{cases} -\begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \left(M^T \mathbf{a}(\theta + tn) M \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \Upsilon \right) = 0, & \theta \in \mathbb{T}^d,\ t > a, \\ \Upsilon = g(x, \theta + an), & \theta \in \mathbb{T}^d,\ t = a. \end{cases} \tag{3.8}$$

Notice that $\mathcal{U} = \Upsilon(x - (x \cdot n)n, x \cdot n)$. This enables to recover compactness properties in the horizontal direction. The price one pays is the degeneracy of the elliptic system (3.8). Under the non-resonance assumption (3.6) (see also (3.7)), one can show a higher-order Poincaré inequality: the degenerate derivative $N^T \nabla_\theta \Upsilon$ controls $\Upsilon - \fint_{\mathbb{T}^d} \Upsilon$, up to a finite loss of derivatives, see [155, Subsection 2.4]. Doing so it is possible to establish Saint-Venant type estimates for any $\alpha \in (\frac{1}{2}, 1)$, i.e. a differential inequality of the type

$$K(T) \leq C_p \big(- K'(T) \big)^\alpha$$

for the quantity

$$K(T) := \int_{\mathbb{T}^d} \int_T^\infty \left|N^T \nabla_\theta \Upsilon\right|^2 + \left|\partial_t \Upsilon\right|^2 d\theta dt,$$

which implies algebraic decay of $K(T)$ for $T \to \infty$. Hence, one obtains [156, Proposition 2.6], which roughly states the existence of a boundary layer tail $\mathcal{U}^\infty$ such that $\mathcal{U}$ and any derivatives converge to $\mathcal{U}^\infty$ faster than any polynomial of $Ay \cdot n$, where $A > 0$ is a constant such that (3.6).

Let us remark that the non-resonance condition was previously used to compensate for the degeneracy of $N^T \nabla_\theta \cdot$ in order to construct bounded correctors in the homogenization of quasiperiodic structures by Kozlov [222]. Moreover, the non-resonance condition enables to quantify ergodicity; see the quantitative ergodic theorem in our work [23, Proposition 2.1].

For further insights related to quantification of ergodicity for almost-periodic structures we refer to [20]. Furthermore, as is expected, the convergence in (3.5) can be arbitrarily slow for resonant directions, i.e. $n \in \partial B_1$ such that (3.6) does not hold. An explicit example is given in [282]; see also [12] for operators with oscillating coefficients. We finally emphasize that the existence of a boundary layer tail requires certain ergodicity properties, which here are implied by the quasiperiodicity of the boundary data along $y \cdot n = a$. Of course, in the case of a lack of ergodicity, no convergence at space infinity takes place, see for example [154, Proposition 11].

We conclude this overview of the literature, by a discussion of the homogenization of (BL^ε) in the case of convex polygonal domains

$$\Omega = \bigcap_{k=1}^{M} \{x,\ x \cdot n^k > a^k\} \subseteq \mathbb{R}^2.$$

If all normals n^k are rational, i.e. $n^k \in \mathbb{R}\mathbb{Z}^d \cap \partial B_1$, homogenization results were achieved by Moskow and Vogelius [266] and Allaire and Amar [15]. The work of Gérard-Varet and Masmoudi [155] handled the convergence in the case when $n^k \in \partial B_1$ satisfies the non-resonance condition (3.6). Let $\bar{V}$ be the homogenized limit in the two-scale expansion (3.1) and $g(x, y) = -\chi(y) \cdot \nabla \bar{V}$. For every edge $k \in \{1, \dots M\}$, we call $\mathcal{U}^{\infty,k}$ the boundary layer tail. Then, see [283, Theorem 3.3], U^ε solution to (BL^ε) converges to

$$\begin{cases} -\nabla \cdot \bar{\mathbf{a}} \nabla \bar{U} = & 0, \qquad\qquad\quad x \in \Omega \\ \qquad\quad \bar{U} = & -\mathcal{U}^{\infty,k} \cdot \nabla \bar{V}, \quad x \in \partial\Omega \cap \{x \cdot n^k = a^k\}, \text{ for all } 1 \le k \le M, \end{cases} \tag{3.9}$$

with a rate $O\left(\varepsilon^{\frac{1}{2}}\right)$ in $L^2(\Omega)$ on condition that $\bar{V}$ has enough regularity. Notice that the works [266, 283] are concerned with eigenvalue expansions for elliptic operators with highly oscillating coefficients, where the boundary layers determine the first-order corrections to the spectrum.

3.1.2 Boundary layers in convex domains

The first asymptotic convergence result for the homogenization of the system (BL^ε) in general uniformly convex domains was obtained by Gérard-Varet and Masmoudi [156]. They proved the existence of an homogenized boundary condition

$$\bar{g} \in L^\infty(\partial\Omega)$$

such that, for each $\delta > 0$ and $q \in [2, \infty)$,

$$\left\| U^\varepsilon - \bar{U} \right\|_{L^q(\Omega)}^q \le C \varepsilon^{\frac{2(d-1)}{3d+5} - \delta}, \tag{3.10}$$

where the constant $C(\delta, d, L, \lambda, \Omega, g, \mathbf{a}) \in (0, \infty)$ and $\bar{U}$ is the solution of the homogenized Dirichlet problem

$$\begin{cases} -\nabla \cdot \bar{\mathbf{a}} \nabla \bar{U} = 0 & \text{in } \Omega, \\ \bar{U}(x) = \bar{g}(x) & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

and $\bar{\mathbf{a}}$ is the usual homogenized tensor; see the similarity with the system (3.9). Notice that in [156], the estimate (3.10) is stated only for $q = 2$, but the statement for general q

can be recovered by interpolation since L^∞ bounds are available for both U^ε and $\bar{U}$. Besides giving the quantitative rate in (3.10), this result was the even first qualitative proof of homogenization of (BL^ε).

It is natural to approximate $\partial\Omega$ locally by hyperplanes and thus the boundary layer by solutions of Dirichlet problems in flat tilted half-spaces. As emphasized above in Subsection 3.1.1, these hyperplanes destroy the periodic structure of the problem. The geometry of the domain Ω thus enters in a nontrivial way. The local behavior of the boundary layer depends on whether or not the angle of the normal vector to $\partial\Omega$ is non-resonant with the periodic structure of $g(x,\cdot)$ and $\mathbf{a}$ i.e the lattice $\mathbb{Z}^d$. In addition to that difficulty, already present for polygonal domains, there is a further issue in smooth domains related to pasting the different half-spaces together. This is complicated due to the strength of singularities in the boundary layer and to the difficulty in obtaining any regularity of the homogenized boundary condition $\bar{g}$.

The lack of a periodic structure means the problem requires a quantitative approach as opposed to the softer arguments based on compactness that are more commonly used in periodic homogenization. Such a strategy was pursued in [156], based on gluing together the solutions of half-space problems with boundary hyperplanes having Diophantine (non-resonant) slopes, and it led to the estimate (3.10). As pointed out by the authors of [156], the exponent in (3.10) is not optimal and was obtained by balancing two sources of error. Roughly, if one approximates $\partial\Omega$ by too many hyperplanes, then the constant in the Diophantine condition for some of the hyperplanes is small, leading to a worse estimate. If one approximates with too few planes, the error in the local approximation caused by the difference between the local hyperplane and $\partial\Omega$ becomes large. In [156], the authors approximate Ω by polygons having edges of comparable size $O(\varepsilon^\alpha)$ for some $\alpha \in (0,1)$, which is a source of non-optimality of the rate (3.10). This leaves room for improvement that we exploited in collaboration with Armstrong, Kuusi and Mourrat, see Theorem 3.1 below.

Given the role of the problem (BL^ε) in quantifying asymptotic expansions in periodic homogenization, obtaining the optimal convergence rate of $\|U^\varepsilon - \bar{U}\|_{L^p(\Omega)}$ to zero is of fundamental importance. To make a guess for how far the upper bound for the rate in (3.10) is from being optimal, one can compare it to the known rate in the case that $\mathbf{a}$ is constant-coefficient (i.e., $\mathbf{a} = \bar{\mathbf{a}}$). In the latter case, the recent work of Aleksanyan, Shahgholian and Sjölin [14] gives, for every $q \in [1,\infty)$,

$$\left\|U^\varepsilon - \bar{U}\right\|_{L^q(\Omega)}^q \leq C \cdot \begin{cases} \varepsilon^{\frac{1}{2}} & \text{in } d=2, \\ \varepsilon\,|\log\varepsilon| & \text{in } d=3, \\ \varepsilon & \text{in } d\geq 4. \end{cases} \tag{3.12}$$

One should not expect a convergence rate better than $O\big(\varepsilon^{\frac{1}{q}}\big)$ for $\|U^\varepsilon - \bar{U}\|_{L^q(\Omega)}$. Indeed, observe that the difference in the boundary conditions is $O(1)$ and that we should expect this difference to persist at least in an $O(\varepsilon)$-thick neighborhood of $\partial\Omega$. Thus the solutions will be apart by at least $O(1)$ in a set of measure at least $O(\varepsilon)$, and this already contributes $O\big(\varepsilon^{\frac{1}{q}}\big)$ to the L^q norm of the difference, which is observed in [14, Theorem 1.6] and (3.12).

3.2 Main result

Our main result is the following quantitative homogenization result for the elliptic system with highly oscillating Dirichlet data (BL^{ε}). There are two main points in the theorem: (i) a statement about the regularity of the homogenized boundary data, (ii) a convergence rate.

Theorem 3.1 (nearly optimal quantitative estimates; [23, Theorem 1], in collaboration with Armstrong, Kuusi and Mourrat)**.** *Assume that $\Omega \subseteq \mathbb{R}^d$ is a smooth bounded uniformly convex domain, that $\mathbf{a}$ is uniformly elliptic and that $\mathbf{a}$ and g are smooth and $\mathbb{Z}^d$-periodic (see assumptions (3.1), (3.2) and (3.3)). Let $\bar{\mathbf{a}}$ denote the homogenized coefficients associated to $\mathbf{a}$ obtained in periodic homogenization. Then,*

(1) there exists a function $\bar{g} \in L^\infty(\partial\Omega)$ satisfying

$$\begin{cases} \bar{g} \in W^{s,1}(\partial\Omega) \quad \text{for all } s < \frac{2}{3} & \text{in } d = 2, \\ \nabla \bar{g} \in L^{\frac{2(d-1)}{3},\infty}(\partial\Omega) & \text{in } d > 2, \end{cases}$$

(2) for every $q \in [2, \infty)$ and $\delta > 0$, for every $\varepsilon \in (0, 1]$, the solutions U^ε and $\bar{U}$ of the problems (BL^{ε}) and (3.11) satisfy the estimate

$$\left\| U^\varepsilon - \bar{U} \right\|_{L^q(\Omega)}^q \leq C \cdot \begin{cases} \varepsilon^{\frac{1}{3}-\delta} & \text{in } d = 2, \\ \varepsilon^{\frac{2}{3}-\delta} & \text{in } d = 3, \\ \varepsilon^{1-\delta} & \text{in } d \geq 4, \end{cases} \tag{3.13}$$

where $C(q, \delta, d, \lambda, \mathbf{a}, g, \Omega) \in (0, \infty)$ is independent of ε.

In addition we obtain an explicit formula for the homogenized boundary data $\bar{g}$, see [23]. The formula for $\bar{g}(x)$, $x \in \partial\Omega$ fixed, is an average of g agains a kernel $\omega(x, \cdot)$ that depends on the normal $n(x)$:

$$\bar{g}(x) = \int_{\mathbb{T}^d} g(x, y)\omega(x, y)\, dy. \tag{3.14}$$

The kernel $\omega(x, \cdot)$ comes from the two-scale expansion of the Poisson kernel around x, see [23, Corollary 5.2]. It involves half-space boundary layer correctors around the point x.

Remark 3.2. The fundamental role of the uniform convexity assumption is to ensure that there are enough good non-resonant directions. We introduce the function $A = A(n) \in [0, 1]$ defined for $n \in \partial B_1$ in the following way

$$A(n) := \sup \{ A \geq 0 : \quad A \text{ satisfies (3.6)} \}. \tag{3.15}$$

In a bounded uniformly convex domain Ω, we have

$$A^{-1}(n(\cdot)) \in L^{d-1,\infty}(\partial\Omega). \tag{3.16}$$

The assumption of uniform convexity can be relaxed, see the paper of Zhuge [353] building upon our analysis. In a domain of finite type k, $A^{-1}(n(\cdot)) \in L^{\frac{1}{k-1},\infty}(\partial\Omega)$. The error estimates are worse as expected.

Remark 3.3 (about low dimensions, $d = 2, 3$). The reason that the rate is worse in low dimensions is because in some places near points of $\partial\Omega$ with good Diophantine normals, the boundary layer where $|U^\varepsilon - \bar{U}| \gtrsim 1$ will be $O(\varepsilon)$ thick, but in other places near points with rational normals with small denominator relative to $\varepsilon^{-\frac{1}{2}}$, the boundary layer will actually be worse, up to $O\big(\varepsilon^{\frac{1}{2}}\big)$ thick. In small dimensions, i.e. in $d = 2$ and with $d = 3$ being critical, the 'bad' points actually take a relatively large proportion of the surface area of the boundary (see the integrability of $A^{-1}(n(\cdot))$ given by (3.16)), leading to a worse error.

Remark 3.4 (on the smoothness assumption). We do not expect it to be possible to eliminate the small loss of exponent represented by $\delta > 0$ without upgrading the qualitative regularity assumption (3.2) on the smoothness of $\mathbf{a}$ and g to a quantitative one (for example, that these functions are analytic). Note that this regularity assumption plays an important role in the proof of Theorem 3.1 and is not a mere technical assumption or one used to control the small scales of the solutions. Rather, it is used to obtain control over the large-scale behavior of the solutions via the quasiperiodic structure of the problem since it gives us a quantitative version of the ergodic theorem, see [23, Proposition 2.1]. In other words, the norms of high derivatives of $\mathbf{a}$ and g control the ergodicity of the problem and thus the rate of homogenization.

Novelty of our result

Our result improves the best known results at that time in two directions:

(1) The rates of convergence that we obtain (3.13) are better in any dimension than the rates (3.10) obtained by Gérard-Varet and Masmoudi in [156]. In dimensions $d \geq 4$, we obtain the optimal convergence rate up to an arbitrarily small loss of exponent, since it agrees with (3.12). The difference in small dimensions between our rate and (3.12) is due to an error which arises only in the case of operators with oscillating coefficients: the largest source of error comes from the possible irregularity of the homogenized boundary condition $\bar{g}$. Reducing this source of error requires to improve the regularity of $\bar{g}$; see below the paragraph 'Further developments'.

(2) The statement asserting that $\nabla\bar{g} \in L^{\frac{2(d-1)}{3},\infty}$ in $d > 2$ and $\bar{g} \in W^{\frac{2}{3}-,1}$ in $d = 2$ is an improvement upon the one proved in [156, consequence of Corollary 2.9], where it was shown that $\nabla\bar{g} \in L^{\frac{d-1}{2},\infty}$ in $d > 2$ and $\bar{g} \in W^{\frac{1}{2}-,1}$ in $d = 2$. The limitation of this regularity comes from the lack of integrability of the inverse of the Diophantine constant, $A^{-1}(n(\cdot)) \in L^{d-1,\infty}$, see (3.16).

As we already mentioned and comment more on below, progress on the regularity of $\bar{g}$ enables us to improve the error estimates.

Further developments

The work of Gérard-Varet and Masmoudi [156] and our work with Armstrong, Kuusi and Mourrat [23] were followed by an intense research activity, that lead to significant progress in several directions:

(1) **Optimal estimates in low dimensions**
Several months after our paper [23] first appeared on the arXiv, Zhongwei Shen kindly pointed out to us that our method leads to optimal estimates for the boundary layer in dimensions $d = 2, 3$, up to an arbitrarily small loss of exponent. Indeed, Shen

and Zhuge [316] were able to upgrade the regularity statement for the homogenized boundary data in Theorem 3.1, reaching $\nabla \bar{g} \in L^q$ for any $q < d-1$ in dimension $d \geq 3$, and $\bar{g} \in W^{s,1}$ for any $s < 1$ in dimension $d = 2$. Their proof of the regularity of the homogenized boundary data follows ours, with a new ingredient, namely a weighted estimate for the boundary layer. Using this improvement of regularity for $\bar{g}$ and then following our argument for estimating boundary layers leads to the following improvement of the estimates of Theorem 3.1 in $d = 2, 3$, which is also proved in [316]: for every $q \in [2, \infty)$ and $\delta > 0$, there is a constant $C(q, \delta, d, \lambda, \mathbf{a}, g, \Omega) \in (0, \infty)$ such that, for every $\varepsilon \in (0, 1]$, the solutions U^ε and $\bar{U}$ of the problems (BL$^\varepsilon$) and (3.11) satisfy the estimate

$$\left\|U^\varepsilon - \bar{U}\right\|^q_{L^q(\Omega)} \leq C \cdot \begin{cases} \varepsilon^{\frac{1}{2}-\delta} & \text{in } d = 2, \\ \varepsilon^{1-\delta} & \text{in } d = 3. \end{cases} \tag{3.17}$$

This is optimal since it agrees with (3.12), up to an arbitrarily small loss of exponent.

(2) **Regularity of the homogenized boundary data**
As mentioned above, in the paper [316] Shen and Zhuge were able to upgrade the regularity of the homogenized boundary data by using a weighted estimate near the boundary for the boundary layers. In the subsequent paper [317], Shen and Zhuge manage a further step forward. They prove that $\bar{g} \in W^{1,p}(\partial\Omega)$ for any $p \in [1, \infty)$. This follows from combining: (i) bounds coming from the analysis of the half-space boundary layer problem in the higher-dimensional torus $\mathbb{T}^d \times (0, \infty)$, which involve the Diophantine constant A, see Subsection 3.1.1 above, (ii) bounds coming from the Poisson kernel in the 'physical' half-space and (iii) a formula [317, equation (2.18)] linking the higher-dimensional torus boundary layer to the 'physical' space boundary layer. With these simple ideas, they succeed in circumventing the lack of integrability of the inverse of the Diophantine constant $A^{-1}(n(\cdot))$. Along a different line of research, let us also mention the work of Feldman and Zhang [132]. They also manage to prove Hölder continuity of the homogenized boundary data using an intermediate scale homogenization problem related to the variation of the boundary layer tail in the vicinity of a given rational direction. We summarize progress on the regularity of the homogenized boundary data in Figure 3.1.

(3) **Relaxation of the convexity assumption**
In the work [353], Zhuge was able to extend the quantitative homogenization of the system (BL$^\varepsilon$) to domains that are not convex and are of finite type, which is more or less the most general framework in which one can carry out a quantitative analysis. Indeed, it is understood that in domains that have a flat part or infinite vanishing of derivatives at some point of the boundary, no algebraic rate of convergence can be expected, although some qualitative convergence may take place in certain conditions, see [148, 218]. The absence of an algebraic rate follows from results about convergence to the boundary layer tail in the half-space, see [282]. The qualitative convergence was proved in [218] by Kim, Lee and Shahgholian under the assumption that the domain satisfies an Irrational Direction Dense Condition, see [218, Theorem 1.5]. It is noted, see [218, Example 4.10], that the homogenized boundary data may be discontinuous if the boundary contains a flat part with a rational direction.

(4) **Other types of oscillating boundary conditions**
Progress on the quantitative homogenization of problems with highly oscillating Dirichlet data also opened the way to investigations of other types of oscillating boundary

Gérard-Varet and Masmoudi (2012) [156]	$\bar{g} \in L^{\infty}(\partial\Omega) \cap L^{\frac{d-1}{2},\infty}(\partial\Omega)$ for $d > 2$
Armstrong, Kuusi, Mourrat and Prange (2017) [23]	$\nabla\bar{g} \in L^{\frac{2(d-1)}{3},\infty}(\partial\Omega)$ for $d > 2$
Feldman and Zhang (2019) [132] *via* intermediate scale boundary layer problem	$\bar{g} \in C^{0,\frac{1}{d}-}(\partial\Omega)$
Shen and Zhuge (2020) [317] *via* weighted estimates	$\nabla\bar{g} \in L^{\infty-}(\partial\Omega)$

Figure 3.1 – Regularity of the homogenized boundary data

conditions. Shen and Zhuge studied the Neumann problem in [316]. Geng and Zhuge studied the Robin problem $n(x) \cdot \mathbf{a}(\frac{x}{\varepsilon})\nabla U^\varepsilon + b(\frac{x}{\varepsilon})U^\varepsilon = h$ in [148]. Notice that for Robin boundary conditions, the rates are obtained using in particular a duality argument from the Neumann problem.

(5) **Eigenvalue expansions**
Progress on the rates of convergence for the homogenization of the boundary layer problem opens the way to improved error estimates for the first-order expansions of eigenvalues and eigenvectors of the elliptic operator $-\nabla \cdot \mathbf{a}(\frac{x}{\varepsilon})\nabla\cdot$ in a bounded domain Ω. Such an analysis was done in my paper [283] for polygonal domains building upon [155], and by Zhuge in [354] for smooth uniformly convex domains building upon [23, 316]; for an applied view on wave propagation in bounded periodic media, see [108].

(6) **Random structures**
Half-space boundary layer correctors are constructed by Fischer and Raithel [136, Theorem 1] for random stationary ergodic structures. In this setting contrary to periodic structures, the position of the boundary of the half-space does not matter. The homogenization result for the system (BL^ε) in bounded domains with a random structure was announced at the '9th GAMM-Workshop on Analysis of Partial Differential Equations' at the University of Freiburg in October 2021.

(7) **Interfaces and corners**
The homogenization of elliptic systems with oscillating coefficients with different structures in two half-spaces separated with a flat interface is a very similar problem to the analysis of the half-space boundary layer problem (3.5). Progress on this problem was achieved by Josien [203] for periodic structures with commensurable periods and by Josien and Raithel [204] for general periodic or random structures. Such an analysis was further extended to the case of a domain with a corner by Josien, Raithel and Schäffner [205].

3.3 New ideas and strategy for the proofs

We first underline the novelty of our approach, before focusing on two specific aspects: the non uniform size of the boundary layer which is a discovery of [23] and the continuity estimates for the homogenized boundary data.

3.3.1 High level strategy

The proof of Theorem 3.1 blends techniques from previous works on the problem [155, 156, 13, 14] with some original estimates and then combines them using a new strategy. Like the approach of [156], we cut the boundary of $\partial\Omega$ into pieces and approximate each piece by a hyperplane. However, rather than gluing approximations of the solution together, we approximate, for a fixed x_0, the contribution of each piece of the boundary in the Poisson formula

$$U^\varepsilon(x_0) = \int_{\partial\Omega} P^\varepsilon(x_0, x) g\big(x, \tfrac{x}{\varepsilon}\big)\, dx, \tag{3.18}$$

where P^ε is the Poisson kernel for the heterogeneous operator $-\nabla \cdot \mathbf{a}\big(\tfrac{\cdot}{\varepsilon}\big)\nabla\cdot$.

The first step in our argument is to replace the Poisson kernel $P^\varepsilon(x_0, x)$ by its two scale expansion, using a result of Kenig, Lin and Shen [216] (based on the classical regularity theory of Avellaneda and Lin [29, 30]), which states that

$$P^\varepsilon(x_0, x) = \overline{P}(x_0, x)\omega^\varepsilon(x) \; + \; \text{small error},$$

where $\overline{P}(x_0, x)$ is the Poisson kernel for the homogenized operator $-\nabla \cdot \bar{\mathbf{a}}\nabla\cdot$ and $\omega^\varepsilon(x)$ is a highly oscillating function which is given explicitly in [216] and which depends mostly on the coefficients in an $O(\varepsilon)$-sized neighborhood of the point $x \in \partial\Omega$. We then show that this function $\omega^\varepsilon(x)$ can be approximated by the restriction of a smooth, $\mathbb{Z}^d$-periodic function on $\mathbb{R}^d$ which depends only on the outer unit normal to $\partial\Omega$ at x:

$$\omega^\varepsilon(x) = \omega\big(x, \tfrac{x}{\varepsilon}\big) \; + \; \text{small error}, \tag{3.19}$$

for a smooth $\mathbb{Z}^d$–periodic function $\omega(x, \cdot) \in C^\infty(\mathbb{R}^d)$. This is true because the boundary of $\partial\Omega$ is locally close to a hyperplane which is then invariant under $\mathbb{Z}^d$–translations. To bound the error in this approximation we rely in a crucial way on the $C^{1,1-}$ regularity theory of Avellaneda and Lin [29, 30] up to the boundary for periodic homogenization.

We can therefore approximate the Poisson formula (3.18) by

$$u^\varepsilon(x_0) = \int_{\partial\Omega} \overline{P}(x_0, x)\, \omega\big(x, \tfrac{x}{\varepsilon}\big) g\big(x, \tfrac{x}{\varepsilon}\big)\, dx \; + \; \text{small error}. \tag{3.20}$$

Finally, we cut up the boundary of $\partial\Omega$ into small pieces which are typically of size $O(\varepsilon^{1-})$ but sometimes as large as $O\big(\varepsilon^{\frac{1}{2}-}\big)$, depending on whether the local outer unit normal to $\partial\Omega$ resonates with the lattice structure of $\mathbb{Z}^d$ on scales smaller than $O(\varepsilon^{-1})$. This chopping has to be done in a careful way, which we handle by performing a Calderón-Zygmund-type cube decomposition; see Subsection 3.3.2. In each piece, we freeze the macroscopic variable $x = \bar{x}$ on both ω and g and approximate the boundary by a piece of a hyperplane, making another small error. The integral on the right of (3.20) is then replaced by a sum of integrals, each of which is a slowly varying smooth function $\overline{P}(x_0, \cdot)$ times the restriction of a smooth, $\varepsilon\mathbb{Z}^d$–periodic function $\omega\big(\bar{x}, \tfrac{\cdot}{\varepsilon}\big) g\big(\bar{x}, \tfrac{\cdot}{\varepsilon}\big)$ to a hyperplane. This is precisely the situation in which an appropriate quantitative form of the ergodic theorem for quasiperiodic functions allows us to compute the integral of each piece, up to a tiny error, which turns out to be close to the integral of $\overline{P}(x_0, \cdot)$ times $\int_{\mathbb{T}^d} g(\bar{x}, y)\omega(\bar{x}, y)\, dy$, the mean of the local periodic function. Therefore we deduce that

$$u^\varepsilon(x_0) = \int_{\partial\Omega} \overline{P}(x_0, x) \left(\int_{\mathbb{T}^d} g(x, y)\omega(x, y)\, dy \right) dx \; + \; \text{small error}. \tag{3.21}$$

The right side is now $\bar{u}(x_0)$ plus errors, since now we can see that the homogenized boundary condition should be defined by (3.14).

There is an important subtlety in the final step, since the function $\bar{g}$ is not known to be very regular. As a consequence, we have to be careful in estimating the error made in approximating the homogenized Poisson formula with the sum of the integrals over the flat pieces. This is resolved by showing that $\bar{g}$ is continuous at every $x \in \partial\Omega$ with Diophantine normal $n(x)$, with a quantitative bound for the modulus which leads to the conclusion that $\nabla\bar{g} \in L^{\frac{2(d-1)}{3}-}$. This estimate is a refinement of those of [156] and also uses ideas from [282]; see the discussion in Subsection 3.3.3.

We conclude this section by remarking that, while many of the arguments in the proof of Theorem 3.1 are rather specific to the problem, the high-level strategy based on two-scale expansion of the Poisson kernel, a suitable regularity theory (like that of Avellaneda and Lin) and the careful selection of approximating half-spaces (done here using a Calderón-Zygmund cube decomposition of the boundary based on the local Diophantine quality) is quite flexible and useful in other situations; see the paragraph 'Further developments' in Section 3.2.

3.3.2 The size of the boundary layer is not uniform

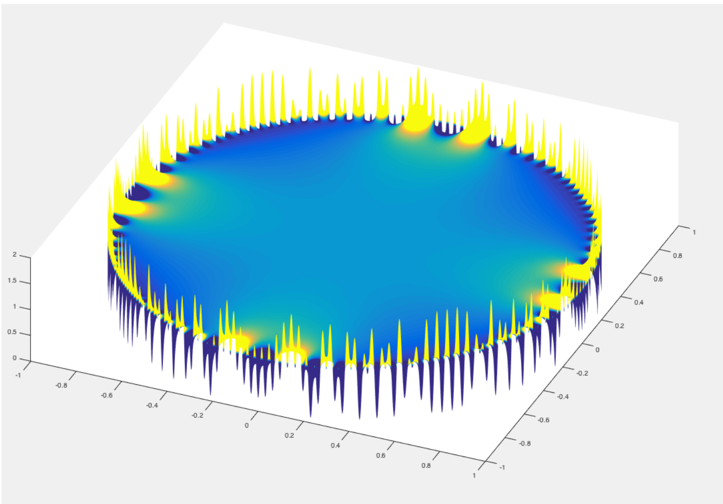

Figure 3.2 – Non-uniform boundary layer size; courtesy of Antti Hannukainen (Aalto University, Finland)

The main discovery of our paper [23] is that the boundary layer, i.e. the zone where U^ε and $\bar{U}$ are $O(1)$ apart has a variable size that depends on the geometry of the boundary $\partial\Omega$ and the resonance/non-resonance with the lattice $\mathbb{Z}^d$ of the underlying periodic structure. The function $A(n(\cdot))$ defined by (3.15) determines the local speed of convergence to the boundary layer tail, see the ergodic theorem [23, Proposition 2.1]. Roughly speaking the boundary layer will be thick where $A(n(\cdot))$ is small and thin where $A(n(\cdot))$ is close to 1. It turns out that: (i) near rational directions $\frac{\xi}{|\xi|}$ with $\gcd(\xi) = 1$ and $|\xi|$ small, the irrational non-resonant directions n have small Diophantine constant $A(n)$, while (ii) near rational directions $\frac{\xi}{|\xi|}$ with $\gcd(\xi) = 1$ and $|\xi|$ large, the irrational non-resonant directions n have large Diophantine constant $A(n)$. Hence, close to the rational directions $\frac{\xi}{|\xi|}$ with $|\xi|$ small, the boundary layer will be locally thick. In the vicinity of rational directions $\frac{\xi}{|\xi|}$ with $|\xi|$ large, the boundary layer will be locally thin. This can be seen on Figure 3.2, which is an unpublished simulation done by Antti Hannukainen (Aalto University, Finland) of the Laplace equation in a disk with oscillating Dirichlet boundary data.

These observations are fundamental when designing the chopping of the boundary. Indeed, one needs to balance the error between approximating by too many tangent hyperplanes, which results in capturing resonant normals, and approximating by too few tangent hyperplanes, which results in a large local error.

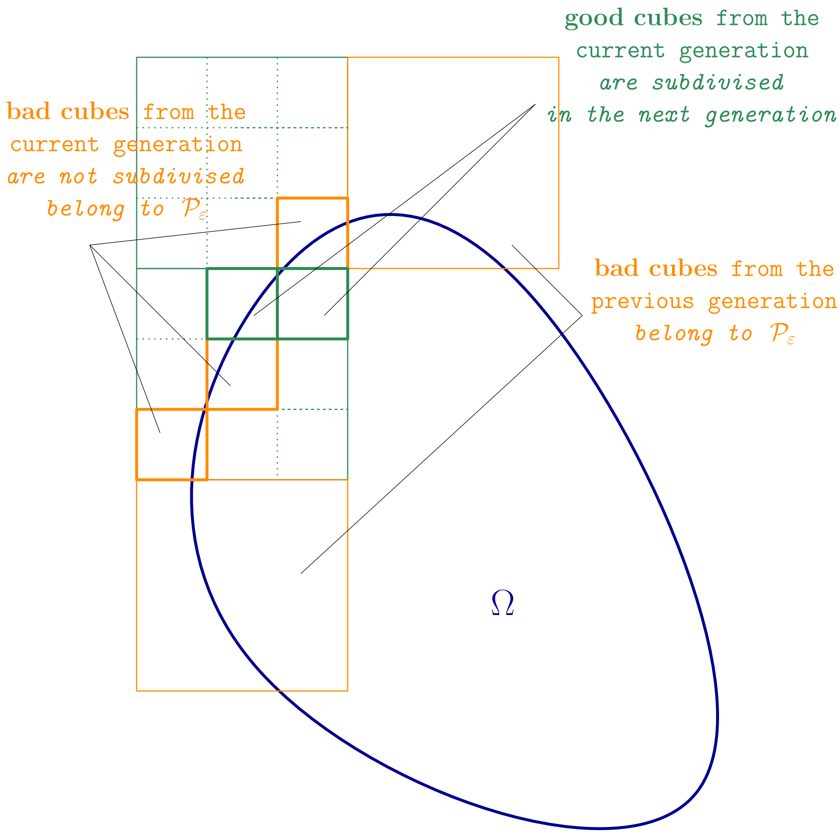

Figure 3.3 – Stopping time argument for the decomposition of the boundary layer

We perform a Calderón-Zygmund-type decomposition of the domain near the boundary [23, Proposition 3.1] which, when applied to the Diophantine constant of the normal to the boundary, will help us construct the approximation of the boundary layer. The decomposition is proved by a stopping time argument; see Figure 3.3. The abstract result is applied to to the function

$$F(x) := \varepsilon^{1-\delta} A^{-1}(n(x)), \quad x \in \partial\Omega,$$

where $A(n(\cdot))$ is defined by (3.15) and yields $\mathcal{P}_\varepsilon$ a the collection of triadic cubes; see Figure 3.4. The boundary layer is the union of the cubes in the decomposition. The decomposition has the following properties:

(i) for each cube $\square \in \mathcal{P}_\varepsilon$, there exists $\bar{x}(\square) \in 3\square \cap \partial\Omega$ such that

$$A(n(\bar{x}(\square))) \geq \frac{\varepsilon^{1-\delta}}{\text{size}(\square)}, \tag{3.22}$$

which means that each cube has at least a 'good' non-resonant direction;

(ii) we have an estimate for the number of cubes bigger than a certain threshold

$$\#\{\square \in \mathcal{P}_\varepsilon \,:\, \text{size}(\square) \geq 3^n\} \leq C 3^{-2n(d-1)} \varepsilon^{(1-\delta)(d-1)}; \tag{3.23}$$

notice that (3.23) implies in particular that the largest cube in P_ε has size at most $C\varepsilon^{\frac{1-\delta}{2}}$; since F is bounded below by $\varepsilon^{1-\delta}$, we have

$$\text{for all} \quad \square \in \mathcal{P}_\varepsilon, \quad c\varepsilon^{1-\delta} \leq \text{size}(\square) \leq C\varepsilon^{\frac{1-\delta}{2}}. \tag{3.24}$$

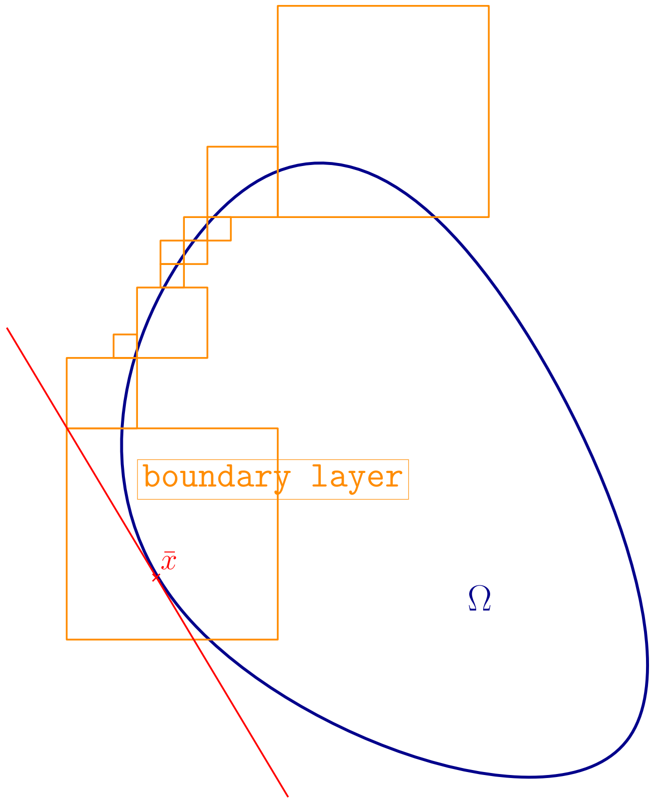

Figure 3.4 – Calderón-Zygmund decomposition of the boundary layer and good non-resonant direction

3.3.3 Regularity estimates for the homogenized boundary data

We explain here how to investigate the variation of $\mathcal{U}(\cdot; a, n(x))$ the half-space boundary layer corrector solving (3.5) in terms of the normal $n(x)$ near a non-resonant normal. This analysis results in 'continuity' estimates with respect to x for the kernel $\omega(x, \cdot)$ defined in (3.19), see [23, Proposition 5.4], and as a consequence for the boundary layer tail and for the homogenized boundary data $\bar{g}$ defined by (3.14), see [23, Proposition 6.3].

Gérard-Varet and Masmoudi were able to prove a Lipschitz estimate for the boundary layer tails for two directions n_1 and n_2, see [156, Corollary 2.9]. The Lipschitz constant is proportional to $\max(A(n_1)^{-2}, A(n_2)^{-2})$, which yields an estimate of the gradient in n like $A^{-2}(n(\cdot)) \in L^{\frac{d-1}{2}}(\partial\Omega)$ for $d > 2$, see Figure 3.1.

The method developed in [23] relies on ideas of [156]. A central idea is to compare $\mathcal{U}(\cdot; a, n_1)$ and $\mathcal{U}(\cdot; a, n_2)$ for $|n_1 - n_2| \ll 1$ at the level of the lifted boundary layers Υ_1 and Υ_2 solving ad hoc variants of (3.8). The advantage of this is that the boundary data for the difference vanishes. We only assume that n_2 satisfies the Diophantine condition (3.6). The source term in the system for the difference $\Upsilon_1 - \Upsilon_2$ is expressed in terms of derivatives of Υ_2 only. The estimates of [156, Proposition 2.6], see also Subsection 3.1.1 above, then allow to estimate this source term. Thus,

$$\int_{\mathbb{T}^d} \int_T^\infty \left|N^T \nabla_\theta (\Upsilon_1 - \Upsilon_2)\right|^2 + \left|\partial_t (\Upsilon_1 - \Upsilon_2)\right|^2 d\theta dt \leq \frac{C|n_1 - n_2|^2}{A^3} \left(1 + \frac{|n_1 - n_2|^2}{A^2}\right),$$

as well as for derivatives of any order. These estimates turn into $\bar{g} \in L^{\frac{2(d-1)}{3},\infty}$ for $d > 2$, see Figure 3.1.

Remark 3.5 (On the improvement by Shen and Zhuge). Let us comment on the upgrade of the regularity of $\nabla\bar{g}$ by Shen and Zhuge in [316]. Using a weighted estimate, they are able to refine the bounds on $\Upsilon_1 - \Upsilon_2$ in a layer close the boundary in the following way: for all $\sigma \in (0, 1)$, there exists a constant $C(d, L, \lambda, \mathbf{a}, \mathcal{V}_0, \sigma) \in (0, \infty)$ such that

$$\int_{\mathbb{T}^d} \left|N^T \nabla_\theta \left(\Upsilon_1(\theta, 0) - \Upsilon_2(\theta, 0)\right)\right| d\theta \leq \frac{C|n_1 - n_2|}{A^{1+\sigma}} \left(1 + \frac{|n_1 - n_2|}{A}\right).$$

Consequently, they can prove (see [316, Theorem 6.1]) that for any $\sigma \in (0, 1)$, there exists a constant $C(d, L, \lambda, \mathbf{a}, g, \sigma) < \infty$ such that for any $x_1, x_2 \in \partial\Omega$, if n_2 Diophantine with constant A and $|n_1 - n_2| \leq \nu_0$, then

$$|\bar{g}(x_1) - \bar{g}(x_2)| \leq \frac{C|n(x_1) - n(x_2)|}{A^{1+\sigma}} \left(1 + \frac{|n(x_1) - n(x_2)|}{A}\right).$$

Following our proof of Proposition [23, Proposition 6.3], this yields the improved regularity $\nabla\bar{g} \in L^q(\partial\Omega)$ for any $q < d - 1$ in dimension $d \geq 3$, and $\bar{g} \in W^{s,1}(\partial\Omega)$ for any $s < 1$ in dimension $d = 2$. We emphasize that these results were further strengthened by Shen and Zhuge in [317], see paragraph ‘Further developments’ in Section 3.2.

Part II

Regularity and concentration for unsteady fluids

Chapter 4

Geometric concentration

This chapter relies mainly on the papers:

— [36], with Tobias Barker, Scale-invariant estimates and vorticity alignment for Navier-Stokes in the half-space with no-slip boundary conditions, *Arch. Ration. Mech. Anal.* (2020).

— [247], with Yasunori Maekawa and Hideyuki Miura, Local energy weak solutions for the Navier-Stokes equations in the half-space, *Comm. Math. Phys.* (2019).

— [249], with Yasunori Maekawa and Hideyuki Miura, Estimates for the Navier-Stokes equations in the half-space for non localized data, *Analysis & PDE* (2020).

Contents

This chapter is devoted to nonlinearity depletion mechanisms under local coherence of the vorticity direction. This line of research was pioneered by Constantin and Fefferman in the early 1990s [107]. There is a clear physical motivation behind these studies. Concentrated vortices, in particular vortex filaments that have a constant vorticity direction, play a crucial role in the self-organization and spatial intermittency of turbulent fluids: see [223, 224] for experimental, [343, 344] for numerical and [264, 284, 285] for theoretical evidence. Concentrated vortices were also used to model the turbulent three-dimensional cascade of energy toward the small scales, see [244, 164, 274], in which a time-dependent linear strain, as in the Burgers vortex or Lundgren's spiral, represents the stretching of other vortices. Mathematically, models for concentrated vortices, notably the Burgers vortex, were analyzed for instance in works by Gallay and Wayne [145] for low Reynolds numbers and Gallay and Maekawa [143] for high Reynolds numbers.

In this chapter, we investigate vorticity depletion mechanisms in the whole-space and in the half-space. The combination of the incompressibility condition and the no-slip boundary condition for the velocity create strong nonlocal effects. This creates important difficulties and explains why certain results known in the whole-space are open questions for the half-space. We list such problems in Subsection 4.1.6 below. Some difficulties of the regularity in the half-space can be linked to the following facts:

(1) In the (flat) half-space one can freely differentiate in the direction tangential to the boundary, but not in the vertical direction. Therefore, to estimate the vertical derivatives $\partial_3^2 U$ one uses the equation and needs information about $\partial_t U$ and ∇P, which gives more interactions than in the whole-space. This can be seen in the example (4.2) of a parasitic flow in the half-space. Such flows are basic flows for finding counter-examples to the local regularity theory, see Subsection 4.1.4.

(2) In addition to the Helmholtz pressure, in the half-space there is also a non trivial harmonic pressure that is driven by the flow on the boundary, see for instance (4.5). This makes the pressure depend in a very nonlocal way on the pressure. We investigate this nonlocality in Subsection 4.3.1and Subsection 5.3.2.

(3) The vorticity satisfies a nonlocal nonlinear boundary condition as showed by Maekawa [246]: on the half-plane

$$\partial_2 \omega + (-\partial_1^2)^{\frac{1}{2}} \omega = N(\omega, \omega) \tag{4.1}$$

with N a bilinear operator involving the Biot-Savart kernel; similar formulas hold in higher dimensions. This makes the use of the vorticity unwieldy in the half-space, while in the whole space the vorticity is a very convenient tool, see for instance Subsection 4.1.5 below. Note that in the case of perfect slip for the velocity, the vorticity satisfies a homogeneous Dirichlet boundary condition, and hence the vorticity can be used successfully as in the whole-space. The complexity of the boundary condition (4.1) reflects the intricate vorticity generation mechanisms near the boundary that are understood, up to now, to a very limited extent.

In Section 4.1, we review some known results and obstacles for solutions in the half-space. Our fundamental contribution to the analysis of half-space solution is the derivation of pressure formulas that enable to circumvent the lack of certain estimates in the half-space. Results in these directions play a key role in this chapter and the next one. We refer in particular to the fractional pressure estimates stated in Theorem 4.1 and to their proof in Subsection 4.3.1, and to the local pressure estimates in Subsection 5.3.2 in the next chapter. The implications of these results are important, see: in this chapter Theorem 4.2 about the unification of notions of Type I blow-up in the half-space, which enables us to clear a significant hurdle, Theorem 4.3 about geometric concentration in the half-space, which is an entirely new result even in the whole-space; and in the next chapter Theorem 5.10 about the global-in-time existence of local energy solutions, which answers an open question asked in [40], Theorem 5.11 about local-in-space short-time smoothing in the half-space and Theorem 5.13 about norm concentration in the half-space.

4.1 Context: state of the art and obstacles

4.1.1 Global regularity in the half-space: some positive results

Several explicit representation formulas are derived for the unsteady Stokes system in the half-space $\mathbb{R}^3_+$. Golovkin [168] derived the formula for the Poisson kernel of the unsteady homogeneous Stokes system with a Dirichlet boundary condition. Using this Golovkin kernel and the Green kernel for the Stokes system, Solonnikov derived in a series of works [320, 321, 322], see also [323, 324, 252], the representation formulas for the unsteady nonhomogeneous Stokes system with a Dirichlet boundary condition. Note that these formulas are for divergence-free source terms, and hence dubbed 'restricted Green Stokes kernel'. A major drawback of this assumption is that it requires a tedious computation of the Helmholtz-Leray projection in the half-space when one uses the kernel to solve the Navier-Stokes equations, because the nonlinear term $\nabla\cdot(U\otimes U)$ is not divergence-free in general. The restriction on the source term was recently removed by Kang, Lai, Lai and Tsai [209], where bounds are proved for the 'unrestricted Green Stokes kernel'.

Other approaches include the one of Ukai [340] that derives an explicit solution formula, in terms of Riesz transforms and the fundamental solutions of the heat and Laplace equations, in the case of the unsteady homogeneous Stokes system with a Dirichlet boundary condition. The formula was extended to the nonhomogeneous case by Cannone, Planchon and Schonbek [79] and Danchin and Zhang [117]. These formulas work well in the case of data in nonendpoint Lebesgue spaces, but not for L^1 or L^∞ due to the unboundedness of the Riesz transform there.

The paper by Desch, Hieber and Prüss [124] pioneers a novel approach that avoids the use of the Helmholtz-Leray projection to compute the Stokes semigroup. The authors prove the analyticity of the Stokes semigroup in $L^\infty_\sigma(\mathbb{R}^d_+)$. Their approach is based on the study of the Stokes resolvent problem. In order to circumvent the use of the Helmholtz-Leray projection, one of the key ideas is to decompose the resolvent operator into a part corresponding to the Dirichlet-Laplace part and another part associated with the nonlocal pressure term. Our work [249] with Maekawa and Miura, that underlies our work [247] and the pressure formulas in Subsection 4.3.1 and Subsection 5.3.2, uses the same idea to handle non-decaying data that is L^p_{uloc} for $p\in(1,\infty]$.

On a tangential aspect, let us mention the works of Abe and Giga [1, 2] that proves the analyticity of the Stokes semigroup in L^∞ on admissible domains, such as for instance the half-space or a bounded C^3 domain. The authors rely on a compactness proof rather than on explicit formulas. A weighted (that degenerates near the boundary) scale-invariant estimate for the pressure is used to get a Liouville theorem for the limit blown-up problem; see also [9, Appendix C].

As is expected, the bounds obtained in these works for the potentials make it possible to prove space-time smoothness of solutions to the Stokes system [300, Theorem 2.1] and of bounded mild solutions to the Navier-Stokes equations in the half-space, see for instance [302, 39]. Moreover Solonnikov proved maximal regularity estimates for the Stokes system in the half-space, for isotropic [321] and anisotropic [323] space-time Lebesgue spaces. The pressure is estimated in these works. Let us also mention the maximal regularity result of Sohr and von Wahl [319]. These estimates though are not global in time in the sense that the constant depends on the time interval. This point was improved by Giga and Sohr in [163], where the estimates are global. The pressure estimates were however not given in [163]. The Schauder-type estimates, i.e. maximal regularity in space-time parabolic Hölder spaces, were proved by Solonnikov in [325].

4.1.2 Global regularity in the half-space: some negative results

On the endpoints L^1 and L^∞ The Stokes semigroup on the half-space fails to be analytic on $L^1(\mathbb{R}^d_+)$. Indeed, Desch, Hieber and Prüss [124, Theorem 5.1] proved that there exists data in $L^1(\mathbb{R}^d_+)$ such that the solution to the Stokes resolvent problem does not belong to $L^1(\mathbb{R}^d_+)$. Han [181] showed that a correction term involving the net force exerted by the fluid on the boundary must be substracted to recover the L^1 summability. Note though that the expected semigroup estimate holds for the gradient of the velocity, which belongs to L^1 for initial data in L^1 [161, Theorem 0.1] and [124, Proposition 5.3]; see also [251].

On the other end of the Lebesgue spaces scale, certain mild decay conditions have to be imposed in order to ensure a representation formula for the pressure. Indeed there are examples of non-decaying flows, solutions to the Stokes system in the half-space, that are sometimes dubbed 'parasitic solutions' because they are driven by the pressure. For instance in $\mathbb{R}^3_+$,

$$\mathcal{U}(x,t) := (V_1(x_3,t), V_2(x_3,t), 0) \quad \text{and} \quad \mathcal{P}(x,t) := -f(t)\cdot x', \tag{4.2}$$

where $f \in C^\infty_0((0,\infty);\mathbb{R}^2)$ and $V = V(x_3,t)$ solves the heat equation $\partial_t V - \partial_3^2 V = f$ with $V(0,t) = 0$. These Poiseuille-type flows generalize the 'Serrin-type' flows known in the whole-space [311]. Such flows also give counter-examples to the local regularity theory, see Subsection 4.1.4 below. Notice that (4.2) also solves the Navier-Stokes equations in the half-space with no-slip boundary condition. A Liouville theorem for infinite energy solutions of the Stokes system in the half-space was worked out by Jia, Seregin and Šverák [194, 195]. Their result holds for weak solutions in $L^\infty(\mathbb{R}^3_+ \times (-\infty, 0))$, so-called bounded ancient solutions. The proof in [194] is based on the use of the Fourier transform in the tangential variable, while the proof in [195] uses a duality argument and the kernel bounds by Solonnikov [320, 321, 322, 324]. Notice that in collaboration with Maekawa and Miura [249] we extended these results under the weaker integrability assumption that the velocity belongs to $L^\infty(0,T; L^1_{uloc,\sigma}(\mathbb{R}^3_+))$. For further discussions of Liouville theorems for the Stokes system in the half-space, we refer to [1, Section 4], [159, Section 4]

On difficulties related to the pressure There is a fundamental difference between the pressure in the half-space and the pressure in the whole-space. In the whole-space, if parasitic solutions are excluded for instance thanks to some mild decay assumption at space infinity, the pressure is equal to the Helmholtz pressure which comes from the Helmholtz-Leray decomposition of the source term; see [135, 69] for related discussions. In the half-space, the non-locality of the pressure is much stronger due to the interaction of the incompressibility condition and the no-slip boundary conditions. For the Stokes system in the half-space

$$\partial_t U - \Delta U + \nabla P = \nabla\cdot F, \quad \nabla\cdot U = 0 \ \text{ in } \mathbb{R}^3_+ \times (0,T) \tag{4.3}$$

with the no-slip boundary condition

$$U(\cdot,t) = 0 \quad \text{on} \quad \partial\mathbb{R}^3_+ \tag{4.4}$$

and the initial data $U(\cdot,0)=0$, one can first carry out the Helmholtz-Leray decomposition of the source term $\nabla\cdot F = \nabla P^F_{Helm} + \mathbb{P}(\nabla\cdot F)$, where P^F_{Helm} is the Helmholtz pressure and $\mathbb{P}$ is the Leray projector on divergence-free fields. Then taking formally the divergence of (4.3), one ends up with the following elliptic system

$$\begin{cases} -\Delta P^F_{harm}(\cdot,t) = 0 & \text{in } \mathbb{R}^3_+, \\ \partial_3 P^F_{harm}(\cdot,t) = \gamma|_{x_3=0}\Delta U_3(\cdot,t) & \text{on } \partial\mathbb{R}^3_+, \end{cases} \tag{4.5}$$

for a pressure $P^F_{harm}(\cdot,t)$ that is called the 'harmonic' part of the pressure, since it is harmonic. Note that in the whole-space the harmonic pressure is harmonic in the whole of $\mathbb{R}^3$, hence constant if the pressure is assumed to grow sub-linearly for instance.

Koch and Solonnikov [220, Theorem 1.3] showed that there are examples of Stokes systems in the half-space with divergence-form source term $\nabla\cdot F$ for which the harmonic pressure is not integrable in time. Hence in short, $P \not\precsim U\otimes U$ for the half-space, contrary to the whole-space. Indeed the example constructed in [220] is of the form $P = P_{Helm} + \partial_t Q$ with:

(i) the following integrability for the Helmholtz pressure $P_{Helm} \in L^q_{t,x}$, which is expected when the source term $F \in L^q_{t,x}$,

(ii) and $Q \in L^q_t W^{2,q}_x$, which means that the harmonic pressure $\partial_t Q$ is not integrable in time.

This fact is a source of considerable difficulties for our proof in Subsection 4.3.2 of Theorem 4.2. We go around this obstacle thanks to the fractional pressure estimates stated in Theorem 4.1. These two theorems are among the main contributions of our paper [36].

Tolksdorf [335] points to another difficulty can be seen at the level of the resolvent problem. Assuming that an estimate of the following type holds for the pressure of the resolvent problem

$$\|P\|_{L^2(\Omega)} \leq C(\lambda)\|f\|_{L^2(\Omega)}, \tag{4.6}$$

the question is to find an explicit quantitative dependence of $C(\lambda)$ in terms of the resolvent parameter λ for various bounded or unbounded domains $\Omega \subseteq \mathbb{R}^3$. Via Dunford's formula such estimates would imply short-time estimates for the pressure of the evolution problem. In the whole-space or the half-space, if an estimate such as (4.6) was true (which is unclear), scaling arguments would imply that $C(\lambda) \simeq |\lambda|^{-\frac{1}{2}}$. Tolksdorf [335, Proposition 3.1] shows that on bounded C^4 domains, $C(\lambda) \simeq |\lambda|^{-\frac{1}{2}}$ scales as is expected in view of scaling for homogeneous Neumann-type boundary conditions. Somewhat surprisingly, Tolksdorf [335, Proposition 3.3] shows that for no-slip boundary conditions on bounded Lipschitz domains, the following estimate is optimal: for any $\alpha \in [0,\frac{1}{4})$, $C_\alpha(\lambda) \lesssim_\alpha |\lambda|^{-\alpha}$ for λ large; see also [273]. Hence the scaling is broken for homogeneous Dirichlet boundary conditions. We further discuss this topic in Subsection 5.3.2 in connection with the pressure estimates of our work with Maekawa and Miura [247] for uniformly locally integrable data.

4.1.3 Local boundary regularity: some positive results

Despite the difficulties related to the strong nonlocality of the Navier-Stokes equations in the half-space, see Subsection 4.1.2 above and Subsection 4.1.4 below, many regularity results can be localized. For a general survey of the local regularity in the half-space, we refer to the paper by Seregin and Shilkin [300].

First, it is possible to localize maximal regularity results in the half-space. This was done in several steps by Seregin in the 2000s. The most accomplished result is in the paper [308]. There Seregin is able to remove extra integrability assumptions that were needed in previous works, such as [306, 305].

Second, we define the following scale-invariant quantities: for $r \in (0, \infty)$,

$$A(U,r) := \sup_{-r^2<s<0} \frac{1}{r} \int\limits_{B_+(r)} |U(y,s)|^2 dy, \tag{4.7}$$

$$C(U,r) := \frac{1}{r^2} \int\limits_{-r^2}^{0} \int\limits_{B_+(r)} |U|^3 dyds, \tag{4.8}$$

$$E(U,r) := \frac{1}{r} \int\limits_{-r^2}^{0} \int\limits_{B_+(r)} |\nabla U|^2 dyds, \tag{4.9}$$

$$D_{\frac{3}{2}}(P,r) := \frac{1}{r^2} \int\limits_{-r^2}^{0} \int\limits_{B_+(r)} |P - (P(\cdot,s))_{B_+(r)}|^{\frac{3}{2}} dyds. \tag{4.10}$$

It was proved by Mikhaylov in [258] that if U is a suitable solution in $Q_+(1)$, boundedness of one of the quantities $\sup_{r\in(0,1)} A(U,r)$, $\sup_{r\in(0,1)} C(U,r)$ or $\sup_{r\in(0,1)} E(U,r)$ implies boundedness of all quantities $\sup_{r\in(0,1)} A(U,r)$, $\sup_{r\in(0,1)} C(U,r)$, $\sup_{r\in(0,1)} E(U,r)$ and $\sup_{r\in(0,1)} D_{\frac{3}{2}}(P,r)$.

Third, we have partial regularity results. Indeed, Seregin [306, Theorem 2.3] proved that for a suitable solution U in $Q_+(1)$ smallness of one of the quantities $\sup_{r\in(0,1)} A(U,r)$, $\sup_{r\in(0,1)} C(U,r)$ or $\sup_{r\in(0,1)} E(U,r)$ implies parabolic Hölder regularity in a neighborhood of $(0,0)$; see also variations in [180] and [259]. Seregin, Shilkin and Solonnikov [309, Theorem 1.1] prove a one-scale ε-regularity criteria in the half-space with a compactness argument à la Lin [243]. In [306] and [309] the definition of suitability includes bounds that can be removed using maximal regularity estimates that are obtained in [308]. In [295], Seregin shows a one-scale regularity criteria based on the sole smallness of the velocity. It is reminiscent of Wolf's criteria [347] in the whole-space, see also the recent paper by Kwon [225].

Fourth, there are local Ladyženskaja-Prodi-Serrin type criteria under boundedness of non borderline critical norms. This is proved by Takahashi [329], see also Seregin's result [307], for finite energy weak solutions and extra integrability assumptions on the velocity gradient and the pressure, that can be removed using the maximal regularity result of [308]. For suitable solutions, the regularity follows from the ε-regularity criteria by Gustafson, Kang and Tsai [180].

Finally, we have versions of the Escauriaza-Seregin-Šverák [129] result. Regularity under $L^\infty_t L^3_x$ local control of a finite-energy weak solution is obtained by Seregin in [293] near a flat boundary and by Mikhailov and Shilkin in [257] near a curved boundary. Seregin's result [294] about the blow-up of the L^3 norm near a potential singularity was generalized to the half-space by Barker and Seregin [40] near a flat boundary. Note that the papers [294] and [40] are global results. A local version of these results near a curved boundary was obtained by Albritton and Barker in [9].

4.1.4 Local boundary regularity: some negative results

Contrary to local solutions in the whole-space, there are obstructions to spatial smoothing in the half-space for suitable solutions: in the local setting, there are bounded flows with unbounded derivatives. This was demonstrated in two bodies of works:

(i) **Finite-energy**
Using a compactly supported Hölder continuous flux term on the boundary, Kang [208] constructed an example of a bounded Stokes flow U in the half-space such that ∇U is unbounded away from the support of the flux. Kang, Lai, Lai and Tsai [210] recently showed that Kang's example has global finite-energy. They also find a flow with similar properties solving the Navier-Stokes equations. In the same vein, the optimality of the local maximal regularity estimates of Seregin [306, 305, 308] is investigated in the work by Chang and Kang [91].

(ii) **Shear flow**
Using the shear flow solution (4.2) for a well-chosen singular time-dependent source term f, Seregin and Šverák show the existence of a bounded flow with unbounded first-order derivatives. We emphasize that this flow is not decaying at space infinity, contrary to the flows constructed in [208, 210].

Finally, let us remark that Chang and Kang [90] proved the failure of Caccioppoli-type inequalities for suitable solutions of the Navier-Stokes equations in the half-space. Such inequalities, that do not involve the pressure, are known for suitable solutions in the whole-space [200, 348].

4.1.5 Geometric regularity criteria

For a general overview of geometric regularity criteria for the Navier-Stokes equations, we refer to the recent survey paper of Miller [261]. These criteria encapsulate some geometric information about the flow, either in the fact that certain quantities, such as the helicity, are small, or that the direction of the velocity or the vorticity is constant or varies smoothly… The observed structure of three-dimensional turbulent flows indicates that there is a high level of spatial intermittency, or sparseness, and anisotropy for the vorticity, with high vorticity regions organized as filaments; see for instance [130].

In [107], Constantin and Fefferman provided a geometric regularity criteria for the vorticity $\omega = \nabla \times U$ of solutions on the whole-space, which remarkably does not depend on scale-invariant quantities. Specifically, they showed that if

$$
\begin{aligned}
&|\sin(\angle(\omega(x+y,t),\omega(x,t))| \leq C|y| \\
&\qquad \text{for} \quad (x,t) \in \Omega_d := \{(y,s) \in \mathbb{R}^3 \times (0,T) : |\Omega(y,s)| > d\}
\end{aligned}
\tag{4.11}
$$

(here $\angle(a,b)$ denotes the angle between the vectors a and b), then U is smooth on $\mathbb{R}^3 \times (0,T]$. Notice that since the angle $\angle(\omega(x+y,t),\omega(x,t))$ appears in (4.11), antiparallel vortex alignment is covered by their result. The proof uses energy estimates for the vorticity equation

$$\partial_t \omega - \Delta\omega + U\cdot\nabla\omega - \omega\cdot\nabla U = 0 \tag{4.12}$$

to get

$$\|\omega(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_0^t\int_{\mathbb{R}^3}|\nabla w|^2 dxdt \leq 2\int_0^t\int_{\mathbb{R}^3}(\omega\cdot\nabla U)\cdot\omega dxdt \tag{4.13}$$

Then Constantin and Fefferman proceed with a careful analysis of the stretching term, i.e. the right hand side of (4.13). In particular, the Biot-Savart law, integration by parts and linear algebra identities are used to show that the most singular contribution of the stretching term can be controlled by

$$\int_0^t \iint_{\Omega_d\times\Omega_d} \frac{|\omega(x,s)|^2|\omega(y,s)||\sin(\angle(\omega(y,s),\omega(x,s))|}{|x-y|^3}dydxds. \tag{4.14}$$

Crucially (4.11) depletes the singularity in the integral of this stretching term. For certain extensions of this geometric regularity criterion, we refer (non-exhaustively) to [48, 172, 351, 174, 171, 241].

In the paper by Giga and Miura [162], a new strategy for regularity under a continuous alignment condition for the vorticity was introduced for the Navier-Stokes in the whole-space with a scale-invariant a priori control (a Type I condition). The Type I assumption gives compactness and a Liouville theorem for a two-dimensional flow leads to the conclusion. This scheme was adapted successfully to the Navier-Stokes equations in the half-space by Giga, Hsu and Maekawa [159]. In this case, the proof of the Liouville theorem is convoluted due to the intricate boundary condition (4.1) for the vorticity. One of our contribution, see Theorem 4.3 and Subsection 4.3.3 below, is to develop an approach that bypasses the use of Liouville theorems.

4.1.6 Regularity in the half-space: some open problems

The issue of vorticity creation at the boundary means that certain other situations that are understood for the Navier-Stokes equations in the whole-space, remain open, as far as we know, for the case of the fluid occupying the half-space with no-slip boundary condition.

On backward self-similar singularities in the half-space It is not known whether there exists backward self-similar solutions

$$U(x,t) = \frac{1}{\sqrt{T-t}}\mathbf{U}\Big(\frac{x}{\sqrt{T-t}}\Big), \quad (x,t)\in\mathbb{R}^3_+\times(0,T)$$

with finite local energy and no-slip boundary condition. For the whole-space such solutions were shown not to exist in [338]. Notice that the case when the profile $\mathbf{U}$ has more decay i.e. belongs to $L^3(\mathbb{R}^3_+)$ is ruled out by the blow-up result of [293].

On axisymmetric flows in the half-space The smoothness of solutions to the Navier-Stokes equations in $\mathbb{R}^3_+\times(0,\infty)$ with no-slip boundary condition that are axisymmetric without swirl is not known. Such solutions have the following form in cylindrical coordinates

$$U(r,z,t) = U_r(r,z)\vec{e_r} + U_z(r,z)\vec{e_z},$$

where $\vec{e_r} = (\cos(\theta), \sin(\theta), 0)$, $\theta \in (-\pi, \pi)$ and $\vec{e_z} = (0,0,1)$. For the whole-space, such solutions were shown to be smooth by Ladyženskaja [228]. For the half-space Kang proved in [207] that such solutions blow-up at maximum one point located at the intersection of the axis of symmetry and of the boundary.

Moreover, the regularity of axisymmetric flows in the half-space but under a Type I assumption is still open, as mentioned in [96]. We recall that such results are known for the whole-space since the works of Chen, Strain, Tsai and Yau [94, 93] on the one hand, and of Koch, Nadirashvili, Seregin and Šverák [219] on the other hand.

On vorticity depletion mechanisms in the half-space For the case of the Navier-Stokes equations in $\mathbb{R}^3_+$ with no-slip boundary conditions, vorticity is generated at the boundary. Specifically, ω has a non-zero trace on $\partial\mathbb{R}^3_+$ and satisfies a nonlocal boundary condition as showed by Maekawa [246], see (4.1) (written in two space dimensions). This provides an obstacle to applying energy methods to the vorticity equation (4.12). Consequently, the following statement analogous to the result of Constantin and Fefferman for the whole-space [107] remains open for the half-space

> **(Q)**: For finite energy solutions of the Navier-Stokes equations in $\mathbb{R}^3_+ \times (0,T)$, with no-slip boundary condition and divergence-free initial data $U_0 \in C^\infty_0(\mathbb{R}^3_+)$, does the Lipschitz continuity of the vorticity direction (4.11) imply that U is smooth on $\mathbb{R}^3_+ \times (0,T]$?

Note that with perfect slip boundary conditions

$$U_3 = \partial_3 U_2 = \partial_3 U_1 \quad \text{on} \quad \partial\mathbb{R}^3_+, \qquad U = (U_1, U_2, U_3), \tag{4.15}$$

vorticity is not created at the boundary and the geometric regularity condition is known to hold [111]. Li [242] was able to generalize this result for Navier-slip boundary conditions.

Theorem 4.3 below obtained in collaboration with Barker [36] investigates regularity under local vorticity alignment under a Type I assumption. The criticality enables to handle the strong nonlocal effects in the half-space due to the pressure or to the non-zero vorticity on the boundary.

4.2 Main results

4.2.1 Fractional pressure estimates in the half-space

The result below is a reformulation of global pressure estimates in the half-space that first appeared in the work of Chang and Kang [89]. These estimates are stated for the Stokes system in the half-space (4.3) with the no-slip boundary condition (4.4). We dub these estimates 'fractional pressure estimates' since they involve fractional derivatives of the source term F.

Theorem 4.1 (fractional pressure estimates; [36, Proposition 7], in collaboration with Barker)**.** *Let $q \in [1,\infty]$, $p \in (1,\infty)$, $F = (F_{\alpha\beta})_{1\leq\alpha,\beta\leq 3} \in L^q(0,T; W^{1,p}_0(\mathbb{R}^3_+))$. We decompose the pressure of the system* (4.3) *with the no-slip boundary condition* (4.4) *and the zero*

initial data $U(\cdot,0)=0$ *into* $P=P^F_{Helm}+P^F_{harm}$*, where* P^F_{Helm} *is the Helmholtz part of the pressure and* P^F_{harm} *is the harmonic part of the pressure.*
Then the following estimates hold: for all $F\in L^q(0,T;W^{1,p}_0(\mathbb{R}^3_+))$*, for all* $\kappa\in(\frac{1}{p},1]$*,*

$$\|P^F_{Helm}\|_{L^q(0,T;L^p(\mathbb{R}^3_+))}\leq C\|F\|_{L^q(0,T;L^p(\mathbb{R}^3_+))},$$
$$\|P^F_{harm}\|_{L^q(0,T;L^\infty_{x_3}(0,\infty;L^p_{x'}(\mathbb{R}^2)))}\leq CT^{\frac{1}{2}(\kappa-\frac{1}{p})}\Big\|\|F\|^{1-\kappa}_{L^p(\mathbb{R}^3_+)}\|\nabla F\|^{\kappa}_{L^p(\mathbb{R}^3_+)}\Big\|_{L^q(0,T)},$$

with a constant $C(\kappa,p)\in(0,\infty)$*.*

Novelty of our result

This result is not new. A similar estimate is contained in the work of Chang and Kang [89, Theorem 1.2, (1.14)]. Notice that we gain boundedness in the vertical direction for p^F_{harm}, but we are not able to recover the $L^p_{x_3}$ integrability as in Chang and Kang. Notice that due to the example of Koch and Solonnikov [220] it is not possible to take $\kappa=0$; see Subsection 4.1.2.

Our point in [36] is to revisit the proof of the estimate of Chang and Kang. We provide a completely new and elementary proof of these estimates, which takes advantage of new pressure formulas for the half-space discovered in [249, 247]. In particular, we avoid the use of space-time fractional Sobolev norms. A key point in the proof is that F vanishes on the boundary of the half-space, which is not a restriction in view of applications to the Navier-Stokes system where $F=-U\otimes U$. Indeed, the proof relies on the use of the one-dimensional Hardy inequality in the vertical direction, see Subsection 4.3.1 below.

We also provide a completely new and elementary proof of these estimates based on formulas for the harmonic pressure that I obtained in collaboration with Maekawa and Miura [247].

4.2.2 Unification of Type I blow-up notions in the half-space

The following result is a local result for solutions of the Navier-Stokes equations near the boundary of the half-space. It enables to infer the boundedness of scale-invariant energies under an 'ODE blow-up rate' notion of Type I singularities, which is the most classical notion of Type I singularity corresponding to the ODE blow-up rate in semilinear heat equations (see for instance Giga and Kohn [160]).

Theorem 4.2 (boundedness of scale-invariant energies; [36, Theorem 2], in collaboration with Barker)**.** *Let* $M\in(0,\infty)$*. Suppose that* (U,P) *is a suitable solution of the Navier-Stokes equations in* $Q_+(1)$ *(see [308, Definition 1.3] and [300, Definition 1.2] for a definition of suitable solutions in the half-space) that satisfies the 'ODE blow-up rate' Type I bound*

$$|U(x,t)|\leq\frac{M}{\sqrt{-t}}\quad in\quad Q_+(1). \tag{4.16}$$

Then,

$$\sup_{0<r<1}\big\{A(U,r)+E(U,r)+D_{\frac{3}{2}}(P,r)\big\}\leq\mathcal{M}\big(M,A(U,1),E(U,1),D_{\frac{3}{2}}(P,1)\big), \tag{4.17}$$

where A *(resp.* E*,* D*) are defined by* (4.7) *(resp.* (4.9)*,* (4.10)*).*

The importance of Theorem 4.2 lies in the fact that it links the natural notion of Type I blow-up (4.16) to a workable notion of Type I for solutions to the Navier-Stokes equations in the half-space with no-slip boundary condition. Indeed it turns out that the scale-invariant condition (4.17) is exactly what is needed to prove a number of results. This can be seen in particular in three situations:

(1) The generalized Type I bound (4.17) ensures that a mild bounded ancient solution originating from (U, P) has some sort of decay at space infinity.

(2) In the paper [35] on concentration phenomena for Type I blow-up solutions of the Navier-Stokes equations, the boundedness of scale-invariant energies was used to provide a control in $L^2_{uloc}(\mathbb{R}^3)$ of a rescaled solution, see Chapter 5.

(3) The scale-invariant bound (4.17) ensures that solutions rescaled according to the natural scaling of the Navier-Stokes equations satisfy the uniform bound in the energy norm required for applying the persistence of singularities. This fact is crucial for our new strategy to prove regularity under vorticity alignment, see Theorem 4.3 below and the description of the strategy in Subsection 4.3.3.

Novelty of our result

The implication 'ODE blow-up rate' Type I implies boundedness of scale-invariant energies is well-known in the whole-space. It is known that for a suitable weak solution (U, P) of the Navier-Stokes equations in $Q(1)$, the Type I condition

$$|U(x,t)| \leq \frac{M}{\sqrt{-t}} \quad \text{in} \quad Q(1) \tag{4.18}$$

implies

$$\sup_{0<r<1} \left\{ A(U,r) + E(U,r) + D_{\frac{3}{2}}(P,r) \right\} < \infty. \tag{4.19}$$

This was established by Seregin and Zajaczkowski in [310]; see also Seregin and Šverák [303] (statement and proof). It is discussed in [8] that the definition of Type I singularities given by (4.19) is very natural and includes most other popular notions of Type I used in the literature. The proofs in the whole-space crucially use the fact that $P \simeq U \otimes U$. For more on the whole-space we refer to Remark 5.6 and to the sketch of the proof in Subsection 4.3.2 below.

For the case of a suitable solution (U, P) in the half-space with no-slip boundary condition, it was shown by Mikhaylov [258] and by Seregin [296] that the definition of Type I singularity (4.17) is implied by $|U(x,t)| \leq \frac{M}{|x|}$; see also a similar result of Chernobay [96] that applies to axisymmetric Type I solutions. However, it was previously not understood if, for solutions in the half-space with no-slip boundary conditions, (4.17) was consistent with the 'ODE blow-up rate' notion of Type I singularities (4.16). Our Theorem 4.2 fills this gap and demonstrates that (4.17) is a reasonable notion of Type I singularity for suitable solutions in the half-space with no-slip boundary condition.

The fractional pressure estimates of Theorem 4.1 above are the main tools enabling our breakthrough as explained below in Subsection 4.3.2. Indeed, these estimates for the pressure enable to circumvent the fact that in the half-space $P \not\simeq U \otimes U$ (see [220] and Subsection 4.1.2 above).

4.2.3 Localized vorticity alignement with no slip boundary condition

The following result shows that for a potential Type I singularity the direction of the vorticity is very rough in the vicinity of the blow-up point.

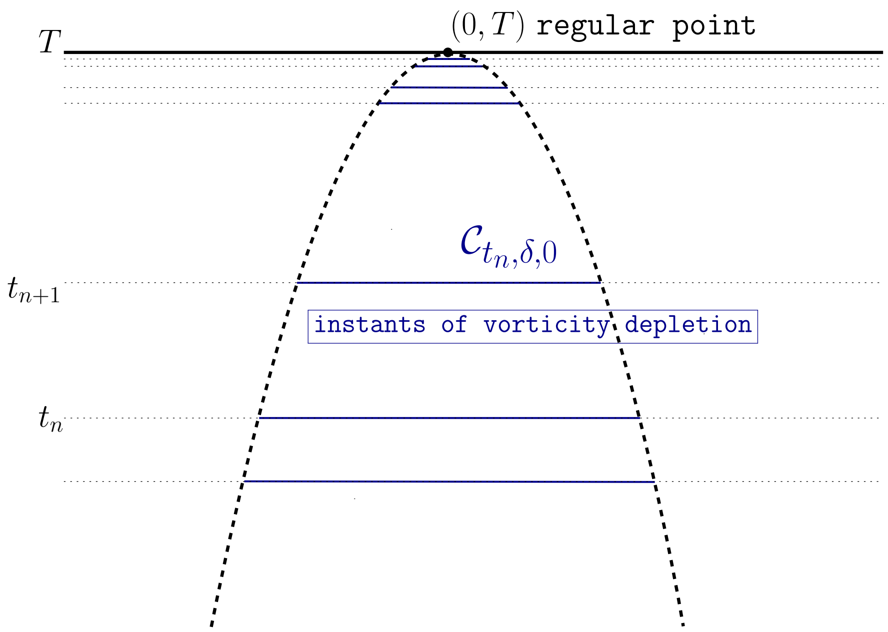

Figure 4.1 – Continuous alignement on concentrating sets

Theorem 4.3 (geometric concentration in the half-space; [36, Theorem 1], in collaboration with Barker)**.** *Let $M \in (0, \infty)$. Suppose U is a mild solution to the Navier-Stokes equations in $\mathbb{R}^3_+ \times (0, T)$, with the no-slip boundary condition (4.4) and initial data $U_0 \in C^\infty_{0,\sigma}(\mathbb{R}^3_+)$. Furthermore, suppose that for $(x, t) \in \mathbb{R}^3_+ \times (0, T)$:*

$$|U(x, t)| \leq \frac{M}{\sqrt{T - t}}. \tag{4.20}$$

Let $t_n \uparrow T$, $\bar{x} \in \overline{\mathbb{R}^3_+}$, $d > 0$ and $\delta > 0$. Define $\omega = \nabla \times U$, $\xi := \frac{\omega}{|\omega|}$, the set of high vorticity,

$$\Omega_d := \{(x, t) \in \mathbb{R}^3_+ \times (0, T) : |\omega(x, t)| > d\},$$

the cone

$$\mathcal{C}_{\delta,\bar{x}} := \bigcup_{t \in (T-1,T)} \{x \in \mathbb{R}^3_+ : |x - \bar{x}| < \delta\sqrt{T - t}\} \tag{4.21}$$

and the time-sliced cone

$$\mathcal{C}_{t_n,\delta,\bar{x}} := \bigcup_n \{(x, t_n) : x \in \mathbb{R}^3_+ \ \text{and} \ |x - \bar{x}| < \delta\sqrt{T - t_n}\}. \tag{4.22}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\eta(0) = 0$.
Under the above assumptions, there exists $\delta(M, U_0) > 0$ such that the following holds true: on the one hand

$$\sup_n \sup_{(x,t) \in \mathcal{C}_{t_n,\delta,\bar{x}}} |\omega(x, t)| < \infty \Rightarrow \quad (x_0, T) \quad \text{is a regular point of} \quad U, \tag{4.23}$$

and on the other hand

$$|\xi(x,t)-\xi(y,t)|\leq\eta(|x-y|)\qquad \textit{in}\quad \Omega_d\cap\mathcal{C}_{\delta,\bar{x}} \tag{CA}$$
$$\Downarrow$$
$$(\bar{x},T)\quad \textit{is a regular point of}\quad U.$$

The condition (CA) is dubbed the 'continuous alignment condition on concentrating balls'. We can read this result as a 'geometric concentration' result. Indeed, the regularity criteria based on continuous alignment of the vorticity direction involves the direction of the vorticity field. It is hence of geometric nature. Here we demonstrate that vorticity alignment is sufficient on concentrating sets. The contrapositive means that the direction of the vorticity breaks any modulus of continuity on sets concentrating on the singularity following the parabolic scaling, see Figure 4.1. In Chapter 5 we investigate 'norm concentration' near potential singularities.

Remark 4.4 (on localized criteria). In [36] we also obtain local variations of Theorem 4.3. To the best of our knowledge those are the first local results regarding regularity under the vorticity alignment condition, for a Navier-Stokes solution having no-slip on the flat part of the boundary. Previously all localisations were known for only interior cases [174, 171, 162]. For precise statements of our local results, we refer to [36, Theorem 4 in Section 6].

Remark 4.5 (on the converse statement). When $\bar{x}$ is a regular point of U belonging to the interior of the half-space, it can be seen that U has Hölder continuous spatial derivatives near $\bar{x}$ and the converse statement to Theorem 4.3 is true, i.e. the vorticity alignment condition (CA) holds. It is not clear to us whether or not that remains to be the case when $\bar{x}$ lies on $\partial\mathbb{R}^3_+$. The difficulty is that there are examples of Navier-Stokes solutions, see Subsection 4.1.4, with no-slip boundary condition on the flat part of the boundary that demonstrate that boundedness of U near the flat boundary does not imply boundedness of ∇U. It is interesting to note that these examples do not provide a counterexample to the converse statement of Theorem 4.3. In fact in [301] the construction is based upon a monotone shear flow, whose vorticity direction is constant.

Novelty of our result

To the best of our knowledge, the only vorticity alignment regularity criteria known for the Navier-Stokes equations in $\mathbb{R}^3_+\times(0,T)$ with no-slip boundary condition, was proved by Giga, Hsu and Maekawa in [159] for mild solutions under the additional assumption that U satisfies the Type I condition (4.20). In particular, in [159] it was shown that if $\eta:[0,\infty)\to\mathbb{R}$ is a nondecreasing continuous function with $\eta(0)=0$ then the following holds true. Namely, the assumption

$$|\xi(x,t)-\xi(y,t)|\leq\eta(|x-y|)\quad \text{for all } (x,t),(y,t)\in\Omega_d \tag{4.24}$$

implies that U is bounded up to $t=T$. Notice that (4.24) on the contrary to (CA) states vorticity alignment on the whole of Ω_d rather than on the concentrating sets $\Omega_d\cap\mathcal{C}_{\delta,\bar{x}}$.

As far as we know, the global or local statements with vorticity alignment on concentrating sets are new even for the case of the whole-space $\mathbb{R}^3$.

Given the strong nonlocal effects of the pressure in the half-space, see subsections 4.1.6, 4.1.4 and 4.1.2, and Subsection 5.3.2 in the next chapter, it is a priori far from clear that such a result is true for the half-space, although its equivalent for the whole-space is well known. Roughly speaking, thanks to the Type I assumption we manage to tame a bit the nonlocality of the pressure. Beyond that regime, no analog of Constantin and Fefferman's result [107] (see Subsection 4.1.5 above) is known for solutions of the Navier-Stokes equations in the half-space with no-slip boundary conditions. This was actually mentioned as an open problem in Subsection 4.1.6 above.

4.3 New ideas and strategy for the proofs

4.3.1 Resolvent pressure formulas in the half-space and pressure estimates

We outline the four main ingredients that we use to give an elementary proof of the fractional pressure estimates of Theorem 4.1. Our proof builds upon the formulas obtained in collaboration with Maekawa and Miura [249].

First ingredient: formulas for the harmonic pressure We decompose the pressure of the Stokes system (4.3) with the no-slip boundary condition (4.4) on the boundary of the half-space into a Helmholtz pressure P^F_{Helm} and a harmonic pressure P^F_{harm}. The treatment of the Helmholtz pressure is standard. We hence focus on the harmonic pressure that formally solves the elliptic Neumann boundary boundary value problem (4.5). We rely on a resolvent pressure formula derived in our work in collaboration with Maekawa and Miura [247, Section 2]; see also Subsection 5.3.2 below. We use resolvent kernel estimates established in the companion paper [249] joint with Maekawa and Miura, which is based on an earlier decomposition of the pressure for the Stokes resolvent problem carried out by Desch, Hieber and Prüss in [124]. We have the following formula for the harmonic pressure:

$$P^F_{harm}(x', x_3, t) = \frac{1}{2\pi i} \int_0^t \int_\Gamma e^{\lambda(t-s)} \int_{\mathbb{R}^3_+} \tilde{q}_\lambda(x' - z', x_3, z_3)(\mathbb{P}\nabla \cdot F(z', z_3, s))_3 dz' dz_3 d\lambda ds, \tag{4.25}$$

for all $(x', x_3) \in \mathbb{R}^3_+$. Here $\mathbb{P}$ denotes the Helmholtz-Leray projection and $\Gamma = \Gamma_\rho$ with $\rho \in (0, 1)$ is the curve

$$\{\lambda \in \mathbb{C} \mid |\arg \lambda| = \eta,\ |\lambda| \geq \rho\} \cup \{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \eta,\ |\lambda| = \rho\} \tag{4.26}$$

for some $\eta \in (\frac{\pi}{2}, \pi)$. The kernel $\tilde{q}_\lambda$ is defined by for all $x' \in \mathbb{R}^2$ and $x_3,\ z_3 > 0$,

$$\tilde{q}_\lambda(x', x_3, z_3) := -\int_{\mathbb{R}^2} e^{ix'\cdot\xi} e^{-|\xi|x_3} e^{-\sqrt{\lambda+|\xi|^2}(\xi) z_3} \left(\frac{\sqrt{\lambda + |\xi|^2}(\xi)}{|\xi|} + 1 \right) d\xi. \tag{4.27}$$

We will see below that the use of formula (4.25), involving the vertical component of the source term rather than the tangential one (on that subject see also Remark 5.14) is more adapted to the study of P^F_{harm}. Indeed the vertical component of $\mathbb{P}\nabla \cdot F$ vanishes on the boundary of $\mathbb{R}^3_+$, which is not necessarily the case for the tangential component. Notice that the estimates for $\tilde{q}_\lambda$ are derived along the exact same lines as [249, Proposition 3.7].

Second ingredient: a formula for the source term We compute the Helmholtz-Leray projection. From the work of Koch and Solonnikov [220, Proposition 3.1], there exists $G \in L^q(0,T;W^{1,p}(\mathbb{R}^3_+))$ such that

$$\mathbb{P}\nabla \cdot F = \nabla \cdot G. \tag{4.28}$$

Moreover, we have the following formula for G: for all $1 \leq \alpha,\ \beta \leq 3$,

$$\begin{aligned} G_{\alpha\beta} = F_{\alpha\beta} - \delta_{\alpha\beta}F_{33} + (1-\delta_{3\alpha})\partial_{x_\beta}\Bigg(&\int_{\mathbb{R}^3_+} \partial_{z_\gamma}N(x,z)F_{\alpha\gamma}(z)dz \\ &+ \int_{\mathbb{R}^3_+} \big(\partial_{z_3}N(x,z)F_{3\alpha}(z) - \partial_{z_\alpha}N(x,z)F_{33}(z)\big)dz\Bigg), \end{aligned} \tag{4.29}$$

where N is the Neumann kernel for the half-space. This formula implies by the Calderón-Zygmund theory that for all $p \in (1,\infty)$ the $W^{1,p}_x$ norm of G can be estimated in terms of the $W^{1,p}_x$ norm of F. Furthermore, notice that

$$(\mathbb{P}\nabla \cdot F)_3 = \sum_{\alpha=1}^{2} \partial_\alpha G_{\alpha 3} \tag{4.30}$$

because $G_{33} = 0$ by the formula (4.29). This cancellation property is essential since it enables to avoid the derivative of $\tilde{q}_\lambda$ in the vertical direction, which has less decay in x'; see [249, Proposition 3.7]. Finally, the fact that the trace of $F(\cdot,t)$ vanishes on $\partial\mathbb{R}^3_+$ implies that the trace of $G_{\alpha 3}(\cdot,t)$ also vanishes on $\partial\mathbb{R}^3_+$.

Third ingredient: a Calderón-Zygmund estimate for the harmonic pressure Our starting point to estimate the harmonic part of the pressure is formula (4.25), which we combine with the formula (4.30) for the source term. We have for almost all $x_3 \in (0,\infty)$, $x' \in \mathbb{R}^2$, $s \in (0,T)$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^2} \tilde{q}_\lambda(x'-z',x_3,z_3)(\mathbb{P}\nabla \cdot F)_3(z',z_3,s)dz'dz_3 \\ &= \sum_{\alpha=1}^{2} \int_0^\infty \int_{\mathbb{R}^2} \partial_{x_\alpha}\tilde{q}_\lambda(x'-z',x_3,z_3)G_{\alpha 3}(z',z_3,s)dz'dz_3 \\ &=: I(x',x_3,s). \end{aligned}$$

In [36] we show that for all $\alpha \in \{1,2\}$, $x_3,\ z_3 > 0$, the kernel k_{x_3,z_3} defined by for all $x' \in \mathbb{R}^2$

$$k_{\alpha,x_3,z_3}(x') = \left(1 + \frac{1}{|\lambda|^{\frac{1}{2}}z_3}\right)^{-1} |\lambda|^{-\frac{1}{2}} e^{c^*|\lambda|^{\frac{1}{2}}z_3} \partial_{x_\alpha}\tilde{q}_\lambda(x',x_3,z_3)$$

is a Calderón-Zygmund kernel.

Fourth ingredient: Hardy's inequality We now turn to estimating the harmonic pressure. We have for almost all $s \in (0,T)$,

$$\|I(\cdot,s)\|_{L_\infty(0,\infty;L_p(\mathbb{R}^2))} \leq C \sum_{\alpha=1}^{2} \left\| \int_0^\infty e^{-c|\lambda|^{\frac{1}{2}} z_3} \frac{\|G_{\alpha 3}(\cdot, z_3, s)\|_{L_p(\mathbb{R}^2)}}{z_3} dz_3 \right\|_{L_\infty(0,\infty)},$$

for some $c \in (0, c^*)$. Thanks to the fact that $G_{\alpha 3}$ has zero trace on the boundary, we can use Hardy's inequality in the vertical direction, see [36, Lemma 8], and get for almost every $s \in (0,T)$,

$$\|I(\cdot,s)\|_{L_\infty(0,\infty;L_p(\mathbb{R}^2))} \leq\ C|\lambda|^{-\frac{1}{2}(\kappa-\frac{1}{p})} \|G(\cdot,s)\|^{1-\kappa}_{L_p(\mathbb{R}^3_+)} \|\partial_{z_3} G(\cdot,s)\|^{\kappa}_{L_p(\mathbb{R}^3_+)},$$

with a constant $C(\kappa,p) \in (0,\infty)$. It remains to compute the integral in λ and estimate the convolution in time. This finishes the proof of the fractional pressure estimates of Theorem 4.1.

4.3.2 Boundedness of scale-invariant energies in Type I blow-up scenarios

In order to prove Theorem 4.2 a number of innovations are needed. Indeed, as we already mentioned at several places, the case near a flat boundary is considerably more intricate than the interior case. The proof relies on the local energy inequality; see below (5.20) in the context of local energy solutions, or [36, Definition 6] in the context of suitable solutions. The main difficulty lies in the estimate for the pressure term

$$I(U,P,r) := \int_{-r^2}^{0} \int_{B_+(r)} PU \cdot \nabla\Phi \, dxds \tag{4.31}$$

where $\Phi \in C_0^\infty(B(r) \times (-r^2,\infty))$ is a positive test function.

To highlight the difficulties concerned with the half-space, let us first discuss the proof of the simpler interior case [310, 303]. For the Navier-Stokes equations in the whole-space (with sufficient decay), the Calderón-Zygmund theory gives that the pressure P can be estimated directly in terms of $U \otimes U$, which we write in a formal way $P \simeq U \otimes U$. For the interior case of a suitable weak solution in $Q(1)$, this fact implies the decay estimate

$$D_{\frac{3}{2}}(P,\tau r) \leq C\big(\tau D_{\frac{3}{2}}(P,r) + \tau^{-2} C(U,r)\big), \tag{4.32}$$

for a given $\tau \in (0,1)$ and all $r \in (0,1]$. To prove the interior version of Theorem 4.2, it then suffices to bound the right hand side of (4.32). The game to play is to combine energy type quantities and the Type I bound (4.18) to obtain

$$C(U,r) \lesssim_M A^\mu(U,r) \left(\frac{1}{r^3} \int_{Q(r)} |U|^2 dxds + \frac{1}{r} \int_{Q(r)} |\nabla U|^2 dxds \right)^\theta \tag{4.33}$$

$$\Downarrow$$

$$C(U,r) + D_{\frac{3}{2}}(P,\tau r) \lesssim_M \varepsilon \left(\frac{1}{r^3} \int_{Q(r)} |U|^2 dxds + \frac{1}{r} \int_{Q(r)} |\nabla U|^2 dxds \right) + c(M,\varepsilon,\kappa,\tau) + C\tau D_{\frac{3}{2}}(P,r) \tag{4.34}$$

with

$$0 < \mu,\, \theta, \qquad \kappa := \mu + \theta < 1, \qquad \varepsilon \in (0,1) \quad \text{and} \quad c(M, \varepsilon, \kappa, \tau) > 0. \tag{4.35}$$

This estimate is based on interpolation. Figure 4.2 shows how to obtain it in a clear way. Estimate (4.33) is critical to get the decay estimate for the energy for ε and τ small enough; see [303, estimate (54)].

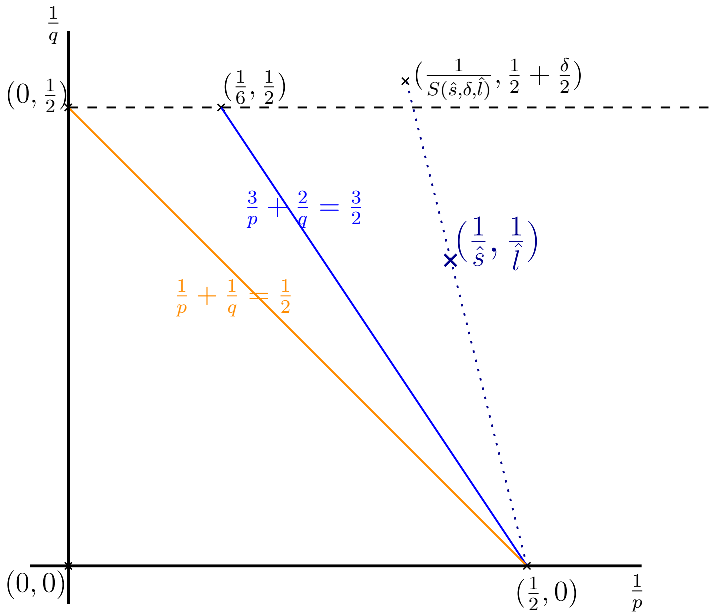

Figure 4.2 – Interpolation for $L^{\hat{l}}_t L^{\hat{s}}_x(Q_+(1))$

In the case of the half-space, on the contrary, we cannot estimate P in terms of $U \otimes U$; in short $P \not\simeq U \otimes U$. This issue is brought up in particular in the work of Koch and Solonnikov [220, Theorem 1.3]; see also Subsection 4.1.2 above. Considering the unsteady Stokes system in the half-space with no-slip boundary conditions, they prove that there are divergence-form source terms $\nabla \cdot F$, with $F \in L^{t,x}_q$, for which the pressure is not integrable in time. A possible alternative is to use maximal regularity for the Stokes system [163], i.e. $\nabla P \simeq U \cdot \nabla U$. Such an estimate was used for the local boundary regularity theory of the Navier-Stokes theory by Seregin in [306]. It remains a cornerstone for estimating the pressure locally near the boundary in particular given the localized version proved in [308]. Formally applying the maximal regularity estimate, together with the Poincaré-Sobolev and Hölder inequalities, the estimate of (4.31) (with $r = 1$) turns into

$$\begin{aligned} &\|U\|_{L^l_t L^s_x(Q_+(1))} \|P - (P)_{B_+(1)}(t)\|_{L^{l'}_t L^{s'}_x(Q_+(1))} \\ &\qquad\qquad \leq \|U\|_{L^l_t L^s_x(Q_+(1))} \|U\|_{L^{\hat{l}}_t L^{\hat{s}}_x(Q_+(1))} \|\nabla U\|_{L^2(Q_+(1))} \end{aligned} \tag{4.36}$$

with

$$\frac{1}{l} + \frac{1}{\hat{l}} = \frac{1}{2} \quad \text{and} \quad \frac{1}{s} + \frac{1}{\hat{s}} = \frac{5}{6}.$$

The presence of the gradient of U makes this quantity 'too supercritical'. Using sharp interpolation arguments with the Type I bound (4.16) gives

$$\|U\|_{L^l_t L^s_x(Q_+(1))} \|U\|_{L^{\hat{l}}_t L^{s}_x{\hat{s}}(Q_+(1))} \lesssim_M (A(U,1))^{\lambda} \tag{4.37}$$

with any λ slightly greater than $1 - (\frac{1}{l} + \frac{1}{\bar{l}}) = \frac{1}{2}$. Such a strategy will always produce a total power of energy quantities $\kappa = \lambda + \frac{1}{2} \geq 1$, as can be seen by combining (4.36) with (4.37). The analysis of Figure 4.2 enables to understand the limitation on the exponent κ. Interpolating the $L^4_{t,x}$ norm of U between the energy, i.e. the point $L^\infty_t L^2_x$, and the Type I condition, i.e. the region $L^q_t L^p_x$, $q < 2$, yields a power of the energy too large. Thus, any attempt to get an estimate similar to (4.34), with a small ε, in the half-space by using maximal regularity estimates for the pressure and the Type I condition (4.16) will not work.

We overcome this difficulty in the half-space thanks to the new fractional estimate on the pressure discovered by Chang and Kang [89, Theorem 1.2] and reproved by us in [36], see Theorem 4.1 above. The idea is to have an intermediate estimate between $P \simeq U \otimes U$ which is known to be false (see above the comment about the result of Koch and Solonnikov [220]), and the maximal regularity estimate $\nabla P \simeq U \cdot \nabla U$ which is too supercritical. The fractional pressure estimates can be formally written as $\nabla^\beta P \simeq \nabla^\beta (U \otimes U)$, for $1 \geq \beta$ sufficiently close to 1. With the help of the fractional pressure estimate, we are able to implement the strategy of [310, 303]. We believe that this is the first instance where the estimate of Chang and Kang [89] is used and is really pivotal.

4.3.3 A new strategy for regularity under vorticity alignement in a scale-invariant regime

The purpose of our work is to pave the way for a new method to prove regularity under the vorticity alignment and Type I condition. This new approach allows more flexibility in the rescaling procedure, and hence enables us to get geometric criteria on concentrating balls, as stated in Theorem 4.3 above. To reach a contradiction, we do not need to show that the blow-up profile U_∞ is zero as in [162, 159]. Instead it suffices to prove that it is bounded at a specific point, which is much easier. The vorticity alignment condition serves the purpose of showing that U_∞ is close to a two-dimensional flow in certain weak topologies. This is sufficient to infer that U_∞ is bounded at the desired point. We first recall the method of Giga, Hsu and Maekawa [159] for the half-space and then outline our new strategy.

A strategy based on a Liouville theorem

The proof by Giga, Hsu and Maekawa [159] for regularity under vorticity alignment in the half-space is inspired by the whole-space strategy of Giga and Miura [162]. The proof is by contradiction. One assumes vorticity alignment and the existence of a blow-up point. The proof relies on two steps:

(1) Use the assumption on vorticity alignment and a blow-up procedure to reduce to a non-zero two-dimensional nontrivial bounded limit solution with positive vorticity, which is defined on either the whole-space or the half-space. This limiting solution to the Navier-Stokes equations belongs to one of two special classes called 'whole-space mild bounded ancient solutions' or 'half-space mild bounded ancient solutions', see [36, Section 1.3].

(2) Obtain a contradiction by showing the limit function must be zero. This is done in [162] and [159] by proving a Liouville theorem for two-dimensional whole-space and half-space mild bounded ancient solutions having positive vorticity and satisfying an ODE blow-up Type I bound.

The rescaling used in the first step is very specific. Let $t_k \uparrow T$, y_k and $R_k \downarrow 0$ be selected such that

$$\frac{1}{R_{(k)}} - 1 \leq |U(y_k, t_k)| \leq \sup_{0 \leq s \leq t_k} \|U(\cdot, s)\|_{L_\infty(\mathbb{R}^3)} := \frac{1}{R_k}.$$

Then $U_k(y,t) = R_k U\big(R_k(y+y_k), (R_k)^2(t+t_k)\big)$ produces either a whole-space or half-space mild bounded ancient solution U_∞ in the limit. By [39, Theorem 1.3], U_∞ is smooth in space-time. To reach a contradiction, it is hence necessary to prove a strong fact about the limit, namely that the mild bounded ancient solution is zero. This is the purpose of the second step. For the case of the half-space, this step is nontrivial and involves a delicate analysis of the vorticity equation and its boundary condition.

Our new strategy based on persistence of singularities

Our strategy in [36] in collaboration with Barker allows to prove regularity under the Type I condition, assuming vorticity alignment on concentrating sets, as in Theorem 4.3. It also makes it possible to achieve local versions of the result of Giga, Hsu and Maekawa [159, Theorem 1.3], and of our main result, Theorem 4.3.

The keystone in our scheme is the stability of singularities for the Navier-Stokes equations. Stability of singularities and compactness arguments were also used: (i) to prove the existence of potential blow-up solutions with minimal $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ data [287], or L^3 data [196]; (ii) to abstractly quantify Escauriaza-Seregin-Šverák type results as in the papers [7, Theorem 4.1 and Remark 4.2] and [9, Theorem 3.1 and Lemma 3.3]; see also Chapter 6; (iii) to give an abstract proof of a mild criticality breaking result in [37]; see also Lemma 6.7 in Subsection 6.3.5 below. The following lemma is taken from Barker's Ph.D. thesis [32, Proposition 5.5]; see also [9, Proposition A.5] and [280, Proposition 5.21] for subsequent generalisations. The stability of singular points was first proved in the interior case by Rusin and Šverák [287]. It is a consequence of ε-regularity type results.

Lemma 4.6 (persistence of singularities; [36, Lemma 3])**.** *Suppose (U_k, P_k) are suitable solutions to the Navier-Stokes equations in $Q_+(1)$. Suppose that there exists a finite $M \in (0,\infty)$ such that*

$$\sup_k \Big(\|U_k\|_{L^2_t L^\infty_x(Q_+(1))} + \|\nabla U_k\|_{L^2(Q_+(1))} + \|P_k\|_{L_{\frac{3}{2}}(Q_+(1))} \Big) = M < \infty. \tag{4.38}$$

Furthermore assume that

$$\lim_{k\to\infty} \|U_k - U_\infty\|_{L_3(Q_+(1))} = 0, \tag{4.39}$$

$$P_k \rightharpoonup P_\infty \;\; in \;\; L_{\frac{3}{2}}(Q_+(1)), \tag{4.40}$$

$$(0,0) \quad is\ a\ singular\ point\ of \quad U_k \quad for\ all \quad k$$

Then the above assumptions imply that (U_∞, P_∞) is a suitable solution to the Navier-Stokes equations in $Q_+(1)$ with $(0,0)$ being a singular point of U_∞.

To show more precisely how our proof of Theorem 4.3 works, let us assume for contradiction that U is singular at the space-time point $(0,T)$. Let $R_k \downarrow 0$ be any sequence. Schematically our method is in three steps.

Step 1: rescaling and a priori bounds We rescale in the following way

$$U_k(y,s) := R_k U(R_k y, T + R_k^2 s) \quad \text{and } P_k(y,s) := R_k^2 P(R_k y, T + R_k^2 s), \tag{4.41}$$

for all $y \in \mathbb{R}^3_+$ and $s \in (-T/R_k^2, 0)$. Theorem 4.2 enables us to show that the ODE blow-up Type I condition (4.20) implies the boundedness of the following scale-invariant quantities

$$\sup_{0<r<1} \left\{ \frac{1}{r} \sup_{T-r^2<t<T} \int\limits_{B_+(r)} |U(x,t)|^2 dx + \frac{1}{r} \int\limits_{T-r^2}^{T} \int\limits_{B_+(r)} |\nabla U|^2 dxdt \right.$$
$$\left. + \frac{1}{r^2} \int\limits_{T-r^2}^{T} \int\limits_{B_+(r)} |P - (P(\cdot,t))_{B_+(r)}|^{\frac{3}{2}} dxdt \right\} < \infty. \tag{4.42}$$

Hence (4.38) holds for (U_k, P_k) and we can apply Lemma 4.6.

Step 2: passing to the limit and persistence of singularities Applying Lemma 4.6 about the persistence of singularities gives us the following. Namely the blow-up profile U_∞ is an ancient mild solution in $\mathbb{R}^3_+ \times (-\infty, 0)$, which has a singularity at $(0,0)$, satisfies the Type I assumption (4.20) with $T = 0$ and has bounded scaled energy.

Step 3: continuous alignment and contradiction via regularity of 2D flows The continuous alignment condition for the vorticity implies that the vorticity direction of U_∞ is constant in a large ball. We can then reach a contradiction by the following lemma for $\bar{U} = U_\infty$, proved in [36, Section 4].

Lemma 4.7 (regularity via reduction to 2D; [36, Proposition 4])**.** *Suppose that $(\bar{U}, \bar{P})$ is an ancient mild solution to the Navier-Stokes equations on $\mathbb{R}^3_+ \times (-\infty, 0)$ with $\bar{U}|_{\partial\mathbb{R}^3_+} = 0$. Let $\bar{\omega} := \nabla \times \bar{U}$ and suppose $(\bar{U}, \bar{P})$ is a suitable solution to the Navier-Stokes equations on $Q_+(r)$ for any $r > 0$. Furthermore suppose that $\bar{U}$ satisfies the ODE blow-up Type I assumption (4.20) with $T = 0$ and*

$$\sup_{0<r<\infty} \left\{ \frac{1}{r} \sup_{-r^2<t<0} \int\limits_{B_+(r)} |\bar{U}(x,t)|^2 \, dx + \frac{1}{r} \int\limits_{Q_+(r)} |\nabla \bar{U}|^2 \, dxdt \right.$$
$$\left. + \frac{1}{r^2} \int\limits_{Q_+(r)} |\bar{P} - (\bar{P}(\cdot,t))_{B_+(r)}|^{\frac{3}{2}} \, dxdt \right\} \leq M'. \tag{4.43}$$

Under the above assumptions the following holds true. For all $M,\ M' \in (0,\infty)$, there exists $\gamma(M,M') \in (0,\infty)$ such that if

$$\bar{\omega}(x, -t_0) \cdot \vec{e}_i = 0 \quad \textit{in} \quad B_+(\gamma(M,M')\sqrt{t_0}) \textit{ for } i = 2,3 \quad \textit{and some} \quad t_0 \in (0,\infty) \tag{4.44}$$

then $(x,t) = (0,0)$ is a regular point for $\bar{U}$.

The proof of Lemma 4.7 goes again by contradiction. Notice that suitability of $(\bar{U}, \bar{P})$ is needed to use Lemma 4.6 on the persistence of singularities. Notice that arguments from [39] demonstrate that $\bar{U}$ is C^∞ in space-time on $\overline{\mathbb{R}^3_+} \times (-\infty, 0)$, hence the pointwise condition on the vorticity (4.44) is well defined.

Remark 4.8 (main flexibility in our method). The main flexibility of our method lies in the fact that we can use *any* sequence $R_k \downarrow 0$ in the rescaling procedure. Therefore, we can tune the sequence to our needs. In the case of the whole-space, we can take advantage of this versatility on time slices of the cone $\mathcal{C}_{t_n,\delta,\bar{x}}$ defined in (4.22), see [36, Theorem 3 in Section 5], and not the whole-cone $\mathcal{C}_{\delta,\bar{x}}$ defined in (4.21) as in Theorem 4.3.

Remark 4.9 (on the ODE blow-up rate Type I). Under the ODE blow-up rate Type I assumption (4.20) with $T = 0$, we gain additional information about the blow-up profile. Namely, $U_\infty \in L^\infty_{loc}(-\infty, 0; L^\infty(\mathbb{R}^3_+))$. This plays an important role in our proof. Specifically, it ensures that for $t_0 > 0$, U_∞ is the unique strong solution to the Navier-Stokes equations on $\mathbb{R}^3_+ \times (-t_0, 0)$ with initial data $U_\infty(\cdot, -t_0)$. It is not known if such considerations apply to the case when U_∞ satisfies (4.43) only.

Remark 4.10 (a difficulty for localized results and a new idea). We discuss some new idea needed to prove a localized version of Theorem 4.3, see [36, Theorem 4]. For the whole proof, we assume that (U, P) is a suitable solution in $Q_+(1)$. We assume that $(0, 0)$ is a space-time blow-up point. Our aim is to reach a contradiction under a localized continuous alignment condition, see [36, equation (134)]. In the local setting close to boundaries, we can have boundedness of the velocity but unboundedness of the vorticity, as can be seen from the construction of bounded flows with unbounded derivative in the half-space; see Subsection 4.1.4 above. Hence, contrary to mild solutions in the half-space there is no a priori bound for the space-time regularity of the solution that ensures that the vorticity direction converges in the space of continuous functions. But this is a crucial ingredient to use the vorticity alignment condition and to thus show that the limiting flow resulting from the rescaling procedure is two-dimensional. We go around the use of strong convergence for the vorticity by using Egoroff's theorem.

Chapter 5

Norm concentration

This chapter relies mainly on the papers:

— [10], with Dallas Albritton and Tobias Barker, Localized smoothing and concentration for the Navier-Stokes equations in the half space, *submitted* (2021).

— [35], with Tobias Barker, Localized Smoothing for the Navier–Stokes Equations and Concentration of Critical Norms Near Singularities, *Arch. Ration. Mech. Anal.* (2020).

Contents

This Chapter is devoted to the study of concentration effects for certain critical norms near potential singularities of the three-dimensional Navier-Stokes equations in the whole-space and in the half-space. The central thread is to identify scales involved in singularity formation, describe possible asymptotic self-similarity or understand the shape of blow-up solutions, if they exist. This study complements the analysis of Chapter 4, where we studied geometric concentration phenomena.

Norm concentration is a by-product of local smoothing properties for 'local energy solutions' with locally critical or subcritical initial data. Local energy solutions, that have uniformly bounded local energy may arise from blow-up limits around Type I singularities. In order to investigate norm concentration in the half-space, we therefore need to extend the existence results for local energy solutions, that were previously known in the whole-space, to the half-space. This poses considerable difficulties related to the strong nonlocal effects of the pressure in the half-space. The problem was mentioned as an open question in several papers [40, 41]. In collaboration with Maekawa and Miura [247], we solve this problem by introducing new estimates for the pressure with data having uniformly bounded local energy.

5.1 Context: state of the art and obstacles

5.1.1 Why study infinite energy solutions?

Let us list a number of reasons that motivate the study of non-decaying solutions:

(1) The global energy is supercritical for the scaling of the Navier-Stokes equations in 3D. Hence the energy blows-up when zooming in on potential singularities. In certain settings, for example if one considers a Type I singularity, the blow-up limit is of uniformly locally finite energy which is for instance used in the work of Seregin [294] concerned with the blow-up of the critical L^3 norm and in my work with Barker [35] concerned with concentration near Type I singularities.

(2) Such solutions are adapted to the study of forward self-similar solutions and discretely self-similar solutions, which are good candidates for producing non-unique Leray-Hopf solutions [197, 198, 179].

(3) Such solutions are interesting from the point of view of dynamics because they can generate their own dynamics. The large-scale effects of the pressure are stronger and may even drive the flow, see the examples of parasitic solutions (5.1) in the whole-space and (4.2) in the half-space. They can include a strain representing the effect of other vortices on a vortex filament modeled for instance by the Burgers vortex, see [74, 145, 162].

(4) In the same spirit as the last item, one can easily find examples of blow-up solutions with infinite energy. There are for instance singular Burgers vortices [263, 275] that blow-up in a scale-invariant way and grow linearly at space infinity. In collaboration with Maekawa and Miura we constructed a family of blow-up solutions around these singular Burgers vortices [248]. It is however unclear if such solutions have any relevance for potential singularities of Leray-Hopf solutions since they have constant vorticity directions, see Chapter 4.

(5) The framework of spatially homogeneous statistical solutions is a good setting to study probabilistic solutions with a law invariant under space translations. Basson [43] studies the existence of homogeneous statistical solutions without source term and proves their suitability (see also Višik and Fursikov [345, 346]). In [44] Basson proves the existence of spatially homogeneous statistical solutions with additive white noise source term.

5.1.2 Local energy solutions in the whole-space

The study of 'local energy solutions' was pioneered by Lemarié-Rieusset [235] in the early 2000s. Roughly speaking, these are weak solutions with locally finite energy, that satisfy a local energy inequality. Hence they form a larger class than the Leray-Hopf solutions [238, 189] that have finite global energy. The notion of local energy solutions became very popular for a number of reasons that we detail above in Subsection 5.1.1. In a nutshell, they provide a framework for the study of solutions with special symmetries, such as forward (discretely) self-similar solutions, while keeping enough properties to have standard tools such as ε-regularity.

To work with the local energy inequality (see (5.20) for the half-space), it is important to be able to compute the pressure to estimate the term

$$\int_0^t \int_{\mathbb{R}^3_+} \big(|U|^2 + 2P\big) U \cdot \nabla\Phi \, dx ds.$$

Since the energy is assumed to be barely locally finite, the availability of pressure formulas becomes a non trivial question. Indeed, there are many examples of solutions of locally finite energy that have zero initial data

$$\mathcal{U}(x,t) = f(t) \quad \text{and} \quad \mathcal{P}(x,t) = -f'(t) \cdot x \tag{5.1}$$

for $f \in C^\infty_0((0,\infty);\mathbb{R}^3)$; see (4.2) for an example in the half-space. These are the so-called 'parasitic solutions' or Serrin's examples [311] driven by the pressure. Decomposing the pressure $P = P_{Helm} + P_{harm}$ into a Helmholtz pressure that is formally equal to

$$P_{Helm} = (-\Delta)^{-1}\nabla\cdot\nabla\cdot(U \otimes U)$$

(this formula requires some adaptation for the case of non-decaying data see (5.2)) and a harmonic pressure P_{harm} that is harmonic on $\mathbb{R}^3$, the parasitic flows correspond to $P_{Helm} = 0$ and to a non trivial harmonic pressure that grows linearly. In order to have a pressure that is equal to the Helmholtz pressure up to a constant harmonic pressure, one needs to rule out the parasitic solutions. There are two approaches for this, which lead to two different definitions of local energy solutions:

(i) Either some mild decay of the solution is assumed at space infinity, such as

$$\lim_{|\bar{x}|\to+\infty} \int_0^{R^2} \int_{B_{\bar{x}}(R)} |U(x,t)|^2 \, dx dt = 0 \qquad \text{for any} \quad R \in (0,\infty).$$

A Liouville theorem [194] enables then to rule out the parasitic solutions and ensures the validity of the pressure formula. This approach is adopted for instance by Jia and Šverák [196, 197] and Tsai [339, page 160].

(ii) Or one includes the pressure formula directly in the definition of the solutions. For all

$\bar{x} \in \mathbb{R}^3$, there exists a function $c_{\bar{x}}(t) \in L^{\frac{3}{2}}(0,T)$ such that for all $(x,t) \in \mathbb{R}^3 \times (0,T)$,

$$\begin{aligned} P(x,t) - c_{\bar{x}}(t) = & -\frac{1}{3}|U(x,t)|^2 + \frac{1}{4\pi}\int\limits_{B_{\bar{x}}(2)} K(x-y)\cdot U(y,t)\otimes U(y,t)dy \\ & + \frac{1}{4\pi}\int\limits_{\mathbb{R}^3\setminus B_{\bar{x}}(2)} (K(x-y)-K(\bar{x}-y))\cdot U(y,t)\otimes U(y,t)dy, \end{aligned} \tag{5.2}$$

where the kernel is defined by $K := \nabla^2(\frac{1}{|x|})$ and the splitting between a 'small-scale' or a 'local' part and a 'large-scale' or a 'nonlocal' part is needed to have enough decay on the kernel to handle the non-decaying solution U. This thread is followed by for instance by Kikuchi and Seregin [217] and Kwon and Tsai [226].

The equivalence between the class of solutions satisfying a pressure formula and the class of mild solutions is thoroughly investigated in several papers: by Furioli, Lemarié-Rieusset and Terraneo [139], Lemarié-Rieusset [235] and Bradshaw and Tsai [69] for data in L^2_{uloc}, by Fernández-Dalgo and Lemarié-Rieusset [135] for data in the weighted space $L^2(\mathbb{R}^3,(1+|x|)^{-4}\,dx)$ that allows for growth at space infinity.

As for the global-in-time existence of local energy solutions, it was proved for data decaying mildly in different senses. Lemarié-Rieusset [235] proved the global-in-time existence for data $U_0 \in L^2_{uloc}(\mathbb{R}^3)$ satisfying

$$\lim_{\substack{|\bar{x}|\to+\infty \\ \bar{x}\in\mathbb{R}^3}} \int\limits_{B_{\bar{x}}(1)} |U_0|^2\,dx = 0. \tag{5.3}$$

This is equivalent to taking the data in the closure of $C^\infty_0(\mathbb{R}^3)$ for the L^2_{uloc} norm. That class contains $L^{3,\infty}(\mathbb{R}^3)$ [67, Lemma 6.3]. Bradshaw and Tsai [67, Theorem 1.5] prove the global existence under the assumption that $U_0 \in L^2_{uloc}(\mathbb{R}^3)$ satisfies

$$\lim_{R\to\infty}\sup_{\bar{x}\in\mathbb{R}^3}\frac{1}{R^2}\int\limits_{B_{\bar{x}}(1)} |U_0|^2\,dx = 0. \tag{5.4}$$

That class includes critical Morrey data $\dot{M}^{2,1}$, hence data (even discretely self-similar) that does not satisfy the mild decay (5.3). In the work [226], Kwon and Tsai show the existence of global local energy solutions solutions for data that satisfies the following mild decay of oscillation

$$\lim_{|\bar{x}|\to\infty}\int\limits_{B_{\bar{x}}(1)} |U_0-(U_0)_{B_{\bar{x}}(1)}|^2\,dx = 0.$$

Beyond the class of uniformly locally finite energy, global-in-time local energy solutions were studied for locally finite data $U_0 \in L^2_{loc}(\mathbb{R}^3)$ that in addition is discretely self-similar. In that vein, let us cite the work of Chae and Wolf [85], Bradshaw and Tsai [66] and the book of Lemarié-Rieusset [237]. Fernández-Dalgo and Lemarié-Rieusset [134] prove global-in-time existence of weak solutions for data in the space $U_0 \in L^2(\mathbb{R}^3,(1+|x|)^{-\gamma}\,dx)$, $\gamma \in (0,2]$. Notice that constants are excluded from that space. Bradshaw and Kukavica [62] (local-in-time) and Bradshaw, Kukavica and Tsai [64] (global-in-time) introduced a class of initial data that they call 'intermittent data' that contains all L^2_{loc} discretely self-similar data. Such data is allowed to grow at space infinity in an intermittent way. This class includes the class considered in [67, Theorem 1.5], see (5.4), and the data considered in [134].

Let us point to two approaches used in these works to extend the solutions globally in time:

(1) An approach that is inspired from Calderón's decomposition [77]. For example in case of $L^2_{uloc}(\mathbb{R}^3)$ data such that (5.3), one proves the existence of a local-in-time solution that has improved integrability at time $t_1 > 0$ and then splits $U(\cdot, t_1)$ into a large part in $C_0^\infty(\mathbb{R}^3)$ for which one has a global-in-time Leray-Hopf solution [238] and a small part in the uniformly local Lebesgue space $L^3_{uloc}(\mathbb{R}^3)$ for which one has a global mild solution [250]. Fundamental to this strategy is to be able to transfer the mild decay (5.3) of the data to the solution itself via local energy estimates. This scheme, or variants, is used by Lemarié-Rieusset [235] and Kwon and Tsai [226]; see also our Theorem 5.10 below with Maekawa and Miura for local energy solutions in the half-space and its proof in Subsection 5.3.2.

(2) The other approach is based on scaling. One combines the local existence with local energy inequalities on expanding balls $B(n)$ with $n \to \infty$ which push the existence time like n^2. The assumption on mild decay on the data is used to get a representation formula for the pressure. This scheme, or variants, is used by Lemarié-Rieusset [236] for critical Morrey-Campanato spaces, Bradshaw and Tsai [67]. Scaling also plays a key role in the argument of Fernández-Dalgo and Lemarié-Rieusset [134].

5.1.3 Regularity results for local energy solutions

The regularity of Leray-Hopf solutions starting from smooth Schwartz or $C_0^\infty(\mathbb{R}^3)$ data being a famous open problem, it is obvious that the regularity problem is even more difficult for local energy solutions. However, local energy solutions form a large enough class in which it is interesting to have even partial results, for at least two reasons:

(i) this class contains forward self-similar and discretely self-similar data,

(ii) it is a good framework for local-in-space short-time smoothing.

Regularity of self-similar solutions Forward self-similar solutions were shown to be smooth by Grujić [170] using the partial regularity of Caffarelli, Kohn and Nirenberg [75]. This argument does not apply to discretely self-similar solutions though. Regularity of discretely self-similar solutions for data in $L^{3,\infty}(\mathbb{R}^3)$ and a similarity parameter $\bar{\lambda}$ close to 1 was proved by Kang, Miura and Tsai [212, Theorem 1.8].

For Leray-Hopf solutions, Leray proved eventual regularity in [238, paragraph 34]. In the context of local energy solutions, such results are not known and it is actually conjectured that there are counter-examples to such properties, for instance for a discretely self-similar solution having a local singularity with data in $L^{3,\infty}$ [65] (provided that such an object exists). Bradshaw and Tsai [67] investigated several initial regularity, eventual regularity, far field regularity properties for local energy solutions under smallness of critical Morrey-type quantities. In [68], Bradshaw and Tsai studied a class, intermediate between the Leray-Hopf class and the local energy class for which eventual regularity holds.

Local-in-space short-time smoothing This topic pioneered by Jia and Šverák [197] is a part of the general questioning about what data produce smooth solutions. Here we consider global supercritical data such as $U_0 \in L^2_{uloc}(\mathbb{R}^3)$. In addition, we assume that locally, the data belongs to a subcritical space $U_0 \in L^m(B_0(1))$ for $m > 3$. Then Jia and Šverák [197, Theorem 3.2] prove that local energy solutions with such data are Hölder continuous in space and time away from the initial time (up to initial time if instead of $U_0 \in L^m(B_0(1))$ one assumes that the data is Hölder continuous) locally in space, i.e. in a cylinder $B_0(\frac{1}{2}) \times (\beta, S)$ for some $S \in (0, \infty)$ and any $0 < \beta < S$. This result follows from a decomposition of the solution reminiscent of Calderón [77] into a mild solution a originating roughly from $U_0|_{B_0(1)} \in L^m(\mathbb{R}^3)$ and V a local energy solution to a perturbed Navier-Stokes system with subcritical drift originating from $U_0|_{\mathbb{R}^3 \setminus B_0(1)} \in L^2_{uloc}(\mathbb{R}^3)$. The mild solution a is smooth, while the perturbation V is smooth locally in space and time, up to initial time, thanks to an ε-regularity result for the Navier-Stokes system with subcritical drift. A key point is to show that the local energy of the perturbation V is small in $B_0(\frac{1}{2}) \times (0, S)$ by controlling the non-local effects of the pressure. In [197], the ε-regularity result is proved via a compactness scheme inspired from Lin's paper [243].

Extension of this result to the critical case, see below Theorem 5.1 and Subsection 5.3.1 for the whole-space and Theorem 5.11 and Subsection 5.3.3 for the half-space, requires new tools due to the lack of improvement of flatness at the limit.

In the past decade local-in-space smoothing results became an important tool for several applications:

Forward self-similar solutions In [197], the local-in-space regularity is used to prove certain a priori estimates for forward self-similar solutions [197, Theorem 4.1]. Using the self-similarity one can transfer local regularity away from the origin into decay at space infinity. This enabled the authors to prove the breakthrough result about the existence of large-data forward self-similar solutions.

Concentration In collaboration with Barker [35], see Theorem 5.5 below for the whole-space, and with Albritton and Barker [10], see Theorem 5.13 below for the half-space, we show concentration of certain scale-critical norms near potential Type I singularities. Such results are direct corollaries of local-in-space smoothing results.

Quantitative regularity Local-in-space smoothing is the pivotal tool in our scheme for quantitative regularity developed in [38] in collaboration with Barker, see Chapter 6.

5.2 Main results

5.2.1 Local smoothing in the whole-space

The following result shows that certain regularization properties that are known globally for mild solutions also hold locally in space and time. Our theorem below is stated in the locally finite-energy framework in view of the applications to concentration, see Theorem 5.5 below. For a definition of local energy solutions in the whole-space, we refer to [197, Definition 3.1]; see also [235, Chapter 32 and 33] and [217] and Definition 5.8 below for the half-space. Notice that contrary to [217], the definition that we use does not incorporate the pressure formula. This is the reason for the mild decay assumption (5.5) in the theorem below.

Theorem 5.1 (local-in-space short-time smoothing in the whole-space; [35, Theorem 1], in collaboration with Barker)**.** *For all $M \in (0, \infty)$, there exists $S(M) \in (0, \frac{1}{4}]$ and an independent universal constant $\gamma_{univ} \in (0, \infty)$ such that the following holds true.*

Consider any local energy solution U to the Navier-Stokes equations with initial data $U_0 \in L^2_{uloc,\sigma}(\mathbb{R}^3)$ satisfying

$$\begin{aligned} &\|U_0\|_{L^2_{uloc}(\mathbb{R}^3)} \leq M, \\ &\lim_{\substack{|\bar{x}|\to+\infty \\ \bar{x}\in\mathbb{R}^3}} \sup_{t\in(0,T')} \int_{B_{\bar{x}}(1)} |U(\cdot,t)|^2\,dx = 0, \\ &U_0 \in L^3(B_0(1)) \quad \textit{and} \quad \|U_0\|_{L^3(B_0(1))} \leq \gamma_{univ}. \end{aligned} \tag{5.5}$$

Then the above assumptions imply that

$$U \in L^\infty(B_0(\tfrac{1}{2}) \times (\beta, S(M))), \tag{5.6}$$

for all $\beta \in (0, S(M))$.

This result asserts that the regularity of local energy solutions is a somewhat local property, near initial time. Indeed the solution U is bounded hence smooth in space. In some sense this means that the nonlocal effects of the pressure are not strong enough to perturb the regularizing properties of the parabolic part of the equation in short time; see also the comment **(S)** in [197, p. 234].

Remark 5.2 (a stronger statement). The result proved in [35] is actually stronger than the statement above. Indeed, considering the mild solution a associated to an L^3 continuous divergence-free extension of the critical data $U_0|_{B_0(1)}$, we prove that

$$U - a \in C^{0,\nu}_{par}(\overline{B_0(\tfrac{1}{2})} \times [0, S(M)]), \tag{5.7}$$

for some $\nu \in (0, \frac{1}{2})$.

Remark 5.3 (quantitative version). Theorem 5.1 is qualitative. It is possible to state a quantitative version of this theorem as is done below for the analogous result in the half-space, see Theorem 5.11. Such quantitative versions of local-in-space smoothing, especially for locally subcritical data, are key to the strategy developed in Chapter 6 to prove quantitative regularity estimates.

Remark 5.4 (local smoothing in borderline endpoint Lorentz and Besov spaces). In addition we prove local-in-space short-time smoothing for data locally small in borderline endpoint spaces. Namely, we prove a version of Theorem 5.1: (i) for $U_0 \in L^{3,\infty}(B_1(0))$, see [35, Appendix B], and (ii) for $U_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(B_1(0))$ for $p \in (3, \infty)$, see [35, Appendix B]. We also state a localized version of Theorem 5.1 in the framework of suitable solutions covering the case of critical Lebesgue, Lorentz and Besov data, see [35, Theorem 3].

Novelty of our results

Our theorem can be seen as an extension to the scale-critical case of the pioneering result of Jia and Šverák [197, Theorem 3.1] for subcritical $U_0 \in L^m(B_0(1))$, $m > 3$. Such an extension poses certain challenges. Compactness arguments as in [197, Theorem 3.1] cannot be used directly to prove the Höder regularity (5.7) of the perturbation or even the boundedness (5.6) of U. Indeed, in the critical case there is a lack of improvement of flatness for the limiting perturbed linear equation with scale-critical drift terms. Difficulties with using compactness arguments are also found when proving ε-regularity statements for the Navier-Stokes equations in higher dimensions, see [127]-[126]. This is the main difficulty we have to overcome to prove Theorem 5.1. We handle this difficulty by: (i) proving a subcritical Morrey bound thanks to a Caffarelli-Kohn-Nirenberg type scheme, (ii) bootstrapping the regularity via linear estimates. These points are explained in more details in Subsection 5.3.1 below.

Further development

Shortly after our paper [35] appeared on arXiv, Kang, Miura and Tsai released a paper where they prove a localized version of Theorem 5.1 above, see [211, Theorem 1.1]. Their theorem is stated for suitable solutions to the Navier-Stokes equations and for data locally in L^3. They manage to obtain the boundedness property via a compactness scheme, but not the Hölder continuity of the perturbation as in (5.7). This is consistent with the remark above and the lack of improvement of flatness for the linear limit equation (5.27). Their method actually relies on improvement of integrability for the linear limit system (5.27) in the Lebesgue scale [211, proposition 4.1], rather than improvement of flatness. This then turns into a subcritical Morrey type bound for the solution of the perturbed Navier-Stokes system (5.26). The boundedness of the perturbation follows from standard ε-regularity for the Navier-Stokes equation without perturbation term. We use this method in our work with Albritton and Barker [10] to establish a version of Theorem 5.1 for the half-space, see Theorem 5.11 below.

5.2.2 Concentration in the whole-space

We investigate accumulation behavior of norms near blow-up times on balls whose radius shrinks to zero as time approaches the singularity. In the next theorem we state 'norm concentration' near scale-critical potential singularities of the Navier-Stokes equations. We assume that the solution satisfies the following generalized Type I bound: for fixed M, $T^*\in(0,\infty)$ and a fixed radius $r_0\in(0,\infty]$,

$$\sup_{\bar{x}\in\mathbb{R}^3}\ \sup_{r\in(0,r_0)}\ \sup_{T^*-r^2<t<T^*} r^{-\frac{1}{2}}\left(\int\limits_{B_{\bar{x}}(r)}|U(x,t)|^2dx\right)^{\frac{1}{2}}\leq M. \tag{5.8}$$

In order to make sense of the condition (5.8) in the case when $r_0>\sqrt{T^*}$, we may extend U by zero in negative times. In the Type I blow-up regime the diffusion is heuristically balanced with the nonlinearity. Despite this it remains a long-standing open problem whether or not Type I blow-ups can be ruled out when M is large.

Theorem 5.5 (concentration near scale-critical singularities in the whole-space; [35, Theorem 2], in collaboration with Barker)**.** *Let γ_{univ} and $S(M)$ be given by Theorem 5.1 above. There exists $t_*(T^*,M,r_0)\in[0,\infty)$ such that the following holds true. Let U be a Leray-Hopf solution to the Navier-Stokes equations in $\mathbb{R}^3\times(0,\infty)$ satisfying the generalized Type I bound (5.8). Furthermore, suppose U first blows-up at T^* and has a singular point at the space-time point $(0,T^*)$.*
Then the above assumptions imply that

$$\|U(\cdot,t)\|_{L^3\left(B_0\left(\sqrt{\frac{T^*-t}{S(M)/2}}\right)\right)}>\gamma_{univ}, \tag{5.9}$$

for all $t\in(t_*(T^*,M,r_0),T^*)$.

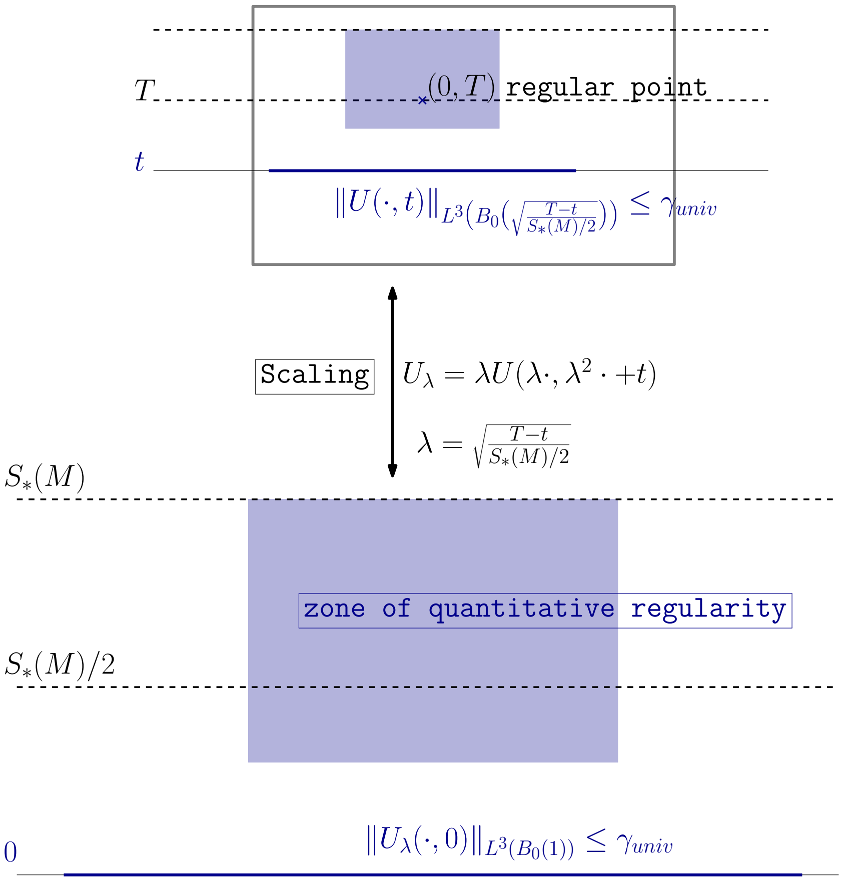

Figure 5.1 – Quantitative local-in-space short-time smoothing and concentration near potential Type I singularities

Figure 5.1 illustrates how the concentration result of Theorem 5.5 follows from the local-in-space short-time smoothing stated in Theorem 5.1.

Remark 5.6 (on the generalized Type I condition). Let $M' \in (0,\infty)$. It is clear that the generalized Type I condition (5.8) is satisfied by Leray-Hopf solutions blowing-up at time $T^* > 0$ and such that

$$|U(x,t)| \leq \frac{M'}{|x|}, \quad \text{for all } (x,t) \in \mathbb{R}^3 \times (0,T^*).$$

More generally, it is also satisfied for Leray-Hopf solutions U blowing-up at time $T^* > 0$ and satisfying a scale-critical Morrey-type bound, i.e.

$$\|U(\cdot,t)\|_{\dot{M}^{2,3}} := \sup_{\bar{x}\in\mathbb{R}^3} \sup_{r\in(0,\infty)} r^{-\frac{1}{2}} \Big(\int_{B_0(r)} |U(x,t)|^2 dx \Big)^{\frac{1}{2}} \leq M'$$

for all $t \in (0,T^*)$. This condition corresponds to (5.8) with $r_0 = \infty$ and $M = M'$. In this case, the concentration in Theorem 5.5 holds for any $t \in (0,T^*)$. It is less obvious to see that type I blow-ups satisfying the bound

$$\sqrt{T^* - t}|U(x,t)| \leq M' \tag{5.10}$$

or

$$\sqrt{T^*-t}^{\theta}|x|^{1-\theta}|U(x,t)| \leq M' \tag{5.11}$$

for some $\theta \in (0,1)$ also enter the framework of Theorem 5.5. Yet, (5.10) and (5.11) imply that there exists $r_0 \in (0,\infty)$ and $M(M',U_0,r_0) \in (0,\infty)$ such that (5.5) holds. This is proved in [310]; see also the review article [303, pages 844-849].

Remark 5.7 (concentration in borderline endpoint in Lorentz and Besov spaces). This remark is the pendant of Remark 5.4. Local-in-space short-time smoothing for data locally small in the borderline endpoint spaces $L^{3,\infty}$ and $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ for $p \in (3,\infty)$ implies concentration in the same spaces on a paraboloïd of concentration near potential singularities satisfying the generalized Type I bound (5.8). The proof goes along the same lines as for the L^3 case, following the rescaling used showed in Figure 5.1.

Novelty of our result

Norm concentration phenomena were investigated for dispersive equations in the wake of the pioneering work by Merle and Tsutsumi [255], see [165, 256, 330, 186, 188].

In [240] an interesting concentration result is proved for a weak Leray-Hopf solution U which first blows-up at $T^*>0$. In particular, this result imply that there exists $t_n \uparrow T^*$ and $x_n \in \mathbb{R}^3$ such that

$$\|U(\cdot,t_n)\|_{L^m(B_{x_n}(\sqrt{C(m)(T^*-t_n)}))} \geq \frac{C(m)}{(T^*-t_n)^{\frac{1}{2}(1-\frac{3}{m})}}, \quad 3 \leq m \leq \infty. \tag{5.12}$$

We are not aware of any prior such results of this type for the Navier-Stokes equations.

By using a rescaling argument and an estimate of the existence time of mild solutions in terms of the size of the initial data in $L^m_{uloc}(\mathbb{R}^3)$, $m>3$, in collaboration with Maekawa and Miura we improved (5.12). In particular, see [249, Corollary 1.1], we showed that for every $t \in (0,T^*)$ (not just a sequence $t_n \uparrow T$) there exists $x(t) \in \mathbb{R}^3$ such that

$$\|U(\cdot,t)\|_{L^m(B_{\sqrt{x(t)}(C(m)(T^*-t)}))} \geq \frac{C(m)}{(T^*-t)^{\frac{1}{2}(1-\frac{3}{m})}} \quad 3 \leq m \leq \infty. \tag{5.13}$$

This result holds for the whole-space relying on the mild solutions constructed by Maekawa and Terasawa [250] and in the half-space relying on the solutions we contructed with Maekawa and Miura in [249]. Notice that both (5.12) and (5.13) hold without any additional Type I assumption.

In (5.12) or (5.13) no information is provided on x_n and $x(t)$. It is natural to ask whether the concentration phenomenon occurs on balls $B_x(R)$ with $R=O(\sqrt{T^*-t})$ and with (x,T^*) being a singular point. Theorem 5.5 answers this in the affirmative for the L^3 for Leray-Hopf solutions which first blow-up at time T^* and which satisfy the Type I bound.

Further developments

In [212, Theorem 1.6], Kang, Miura and Tsai state two results that can be read as concentration/accumulation results for the supercritical L^2 norm near potential singularities of the Navier-Stokes equations. The first result is in the spirit of the result (5.13) from [249].

There exists $\gamma_{univ} \in (0,\infty)$, $S \in (0,\infty)$ and a function $x = x(t) \in \mathbb{R}^3$ such that for all $t \in (0,T^*)$,

$$\frac{1}{\sqrt{T^*-t}} \int\limits_{B_{x(t)}\left(\sqrt{\frac{T^*-t}{S}}\right)} |U(x,t)|^2\,dx > \gamma_{univ}. \tag{5.14}$$

This result is in the vein of the one of Grujić and Xu [173, Theorem 4.1] and Bradshaw and Tsai [69, Theorem 8.2]. The second result of [212, Theorem 1.6] is in the spirit of (5.5) from [35]. It holds under the generalized Type I condition (5.8). There exists $\gamma_{univ} \in (0,\infty)$ and $S(M) \in (0,\infty)$ such that for all $t \in (0,T^*)$,

$$\frac{1}{\sqrt{T^*-t}} \int\limits_{B_0\left(\sqrt{\frac{T^*-t}{S(M)}}\right)} |U(x,t)|^2\,dx > \gamma_{univ}. \tag{5.15}$$

Notice that both results involve scale-invariant Morrey type quantities. Considering their contrapositive, they can also be read as dynamically restricted regularity criteria since the scales on which smallness is needed are smaller and smaller as one approaches the final time.

In a different direction, let us also mention the result of Miller [260, Theorem 1.10] that states accumulation behavior of $\|U(x,t)\mathbf{1}_{\{|U(x,t)|>h(t)\}}\|_{L^3(\mathbb{R}^3)}$, where $h = h(t)$ is any function $L^2(0,T^*;\mathbb{R}_+)$.

5.2.3 Local energy solutions in the half-space

In collaboration with Maekawa and Miura we extended the notion of local energy solution to the half-space $\mathbb{R}^3_+$. We prove local-in-time as well as global-in-time existence results. For simplicity, let us only define global-in-time local energy solutions. The definition of local energy solutions on $\mathbb{R}^3_+ \times (0,T)$ can be adapted mutatis mutandis; see [247, Definition 1.1].

Definition 5.8 (local energy solutions in the half-space; [247, Definition 1.1] in collaboration with Maekawa and Miura)**.** A pair (U,P) is called a 'local energy solution' to the Navier-Stokes equations in $\mathbb{R}^3_+ \times (0,\infty)$ with no-slip boundary condition on $\partial\mathbb{R}^3_+ \times (0,\infty)$ and the initial data $U_0 \in L^2_{uloc,\sigma}(\mathbb{R}^3_+)$ satisfying the mild decay assumption

$$\lim_{\substack{|\bar{x}|\to+\infty \\ \bar{x}\in\overline{\mathbb{R}^3_+}}} \int\limits_{B_{\bar{x}}(1)\cap\mathbb{R}^3_+} |U_0|^2\,dx = 0, \tag{5.16}$$

if (U,P) satisfies the following conditions:

(1) We have $U \in L^\infty_{loc}([0,\infty); L^2_{uloc,\sigma}(\mathbb{R}^3_+))$, $P \in L^{\frac{3}{2}}_{loc}\big(\overline{\mathbb{R}^3_+} \times (0,\infty)\big)$ and

$$\sup_{\bar{x}\in\mathbb{R}^3_+} \int_0^{T'} \|\nabla U\|^2_{L^2(B_{\bar{x}}(1)\cap\mathbb{R}^3_+)} dt + \sup_{\bar{x}\in\mathbb{R}^3_+} \left(\int_\delta^{T'} \|\nabla p\|^{\frac{3}{2}}_{L^{\frac{9}{8}}(B_{\bar{x}}(1)\cap\mathbb{R}^3_+)} dt \right)^{\frac{2}{3}} < \infty, \tag{5.17}$$

$$\lim_{\substack{|\bar{x}|\to+\infty \\ \bar{x}\in\overline{\mathbb{R}^3_+}}} \sup_{t\in(0,T')} \int\limits_{B_{\bar{x}}(1)\cap\mathbb{R}^3_+} |U(\cdot,t)|^2\,dx = 0, \tag{5.18}$$

for all $T' \in (0,\infty)$ and $\delta \in (0,T')$.

(2) For any $\varphi \in C_0^\infty(\overline{\mathbb{R}^3_+})$, $\varphi U \in L^2(0,T;H^1_0(\mathbb{R}^3_+))$.

(3) The pair (U,P) is a solution in the sense of distributions.

(4) The function $t \mapsto \int_{\mathbb{R}^3_+} U(\cdot,t)\cdot w\,dx$ belongs to $C([0,T))$ for any compactly supported $w \in L^2(\mathbb{R}^3_+)^3$. Moreover, for any compact set $K \Subset \overline{\mathbb{R}^3_+}$,

$$\lim_{t\to 0}\|U(\cdot,t)-U_0\|_{L^2(K)} = 0. \tag{5.19}$$

(5) The pair (U,P) satisfies the local energy inequality: for all non-negative $\Phi \in C_0^\infty\big((0,T)\times\overline{\mathbb{R}^3_+}\big)$ and for a.e. $t \in (0,T)$,

$$\begin{aligned} &\int_{\mathbb{R}^3_+} |U(x,t)|^2\Phi(x,t)\,dx + 2\int_0^t\int_{\mathbb{R}^3_+} |\nabla U|^2\Phi(x,t)\,dxds \\ &\leq \int_0^t\int_{\mathbb{R}^3_+} |U|^2(\partial_t+\Delta)\Phi\,dxds + \int_0^t\int_{\mathbb{R}^3_+} \big(|U|^2+2P\big)U\cdot\nabla\Phi\,dxds. \end{aligned} \tag{5.20}$$

Remark 5.9 (on the mild decay assumption). Our definition is in the vein of the one for the whole-space used in [196, 197], where the authors defined local energy weak solutions in $\mathbb{R}^3\times(0,\infty)$ which decay at spatial infinity. Contrary to [217] they do not include the representation formula for the pressure. Similarly, in the class considered here, the solutions decay mildly at spatial infinity thanks to (5.18), and thus the parasitic solutions are automatically excluded (via Liouville theorems, see for instance [194, 195, 249]), which guarantees the validity of a representation formula for the pressure. For a definition, in the spirit of [217], of local energy solutions in the half-space that avoids any mild decay assumption and hence includes the representation formula for the pressure, we refer to [63, Definition 1.1].

Theorem 5.10 (global-in-time local energy solutions in the half-space; [247, Theorem 1] in collaboration with Maekawa and Miura)**.** *For any initial data $U_0 \in L^2_{uloc,\sigma}(\mathbb{R}^3_+)$ satisfying the mild decay assumption* (5.16)*, there exists a local energy weak solution (U,P) to the Navier-Stokes equations in $\mathbb{R}^3_+\times(0,\infty)$ in the sense of Definition 5.8.*

Novelty of our result

This result states the global-in-time existence of local energy weak solutions in the sense of Definition 5.8. It is the analog for the half-space of the theorem of Lemarié-Rieusset [235, Theorem 33.1] and of Kikuchi and Seregin [217, Theorem 1.5] for the whole-space $\mathbb{R}^3$; see also Subsection 5.1.2. Our Theorem 5.10 answers an open problem mentioned by Barker and Seregin in [40, Section 1]:

> Unfortunately, and analogue of Lemarié-Rieusset type solutions for the half-space is not known yet. In fact it is an interesting open problem.

The main innovation making this progress possible is a new pressure estimate based on resolvent pressure formulas for non-decaying data, see Subsection 5.3.2. This pressure estimate gives some Lebesgue integrability in time of the harmonic pressure near initial time, hence the term involving the pressure can be estimated in the local energy inequality.

Interestingly, such pressure estimates do not seem to be obtained via previously known pressure representation formulas which give a non-integrable singularity near initial time. This was the main block to the open problem mentioned above. We refer to Subsection 5.3.2 below for further insights on this point.

In addition, let us remark that our tools enable us to reprove the result of Barker and Seregin [40] on the extension of Seregin's result [294] to solutions in the half-space bounded in $L^3(\mathbb{R}^3_+)$ along a sequence of times.

Further developments

In the work [63], Bradshaw, Kukavika and Ożański prove the global existence of local energy solutions in the half-space for non-uniformly locally square integrable data that may grow at large-scales in an intermittent sense. In this setting, one has a Morrey-type local energy. Such data was also considered for instance in [62, 64, 135] in the whole-space. This framework is adapted to the study of forward discretely self-similar solutions, see also Subsection 5.1.2 above. In [63, Theorem 1.6] existence of forward discretely self-similar solutions is stated for the Navier-Stokes equations in the half-space with a no-slip boundary condition. The analysis of the paper [63] builds upon our work with Maekawa and Miura, namely the pressure formulas discovered in [247] and the kernel estimates from [249]. The decomposition of the pressure, see [249, Section 2] and Subsection 5.3.2 below, is actually included in the definition of local energy solutions. These formulas are very useful to Bradshaw, Kukavika and Ożański, as they were to us, in order to avoid the singularity in short-time of the harmonic pressure estimates [163, 324]. The pressure estimates though differ from ours, because contrary to [247] (see Definition 5.8 and Theorem 5.10 above) the local kinetic plus dissipation energy functional in [63] is not uniformly locally finite.

Finally, let us emphasize that the pressure formulas from our work [247] with Maekawa and Miura are central to the proof of local-in-space short-time smoothing in the half-space, see Theorem 5.11 in the next subsection.

5.2.4 Local smoothing and concentration in the half-space

In the paper [10] in collaboration with Albritton and Barker, we prove the analog of Theorem 5.1 (local-in-space short-time smoothing) and Theorem 5.5 (concentration near a scale-critical singularity) for solutions of the Navier-Stokes equations in the half-space with no-slip boundary conditions.

The following result states local smoothing for locally critical and subcritical data at the level of local energy solutions defined in Definition 5.8 and shown to exist in Theorem 5.10 above.

Theorem 5.11 (local-in-space short-time smoothing in the half-space; [10, Theorem 1.1], in collaboration with Albritton and Barker)**.** *There exists $\gamma_{univ} \in (0,\infty)$ such that the following holds. Let $T \in (0,\infty)$ and $m \in [3,\infty)$ be fixed. Let U be a local energy solution on $\mathbb{R}^3_+ \times (0,T)$ with initial data $U_0 \in L^2_{uloc}(\mathbb{R}^3_+)$ satisfying*

$$\|U_0\|_{L^2_{uloc}(\mathbb{R}^3_+)} \leq M,$$

and the mild decay condition (5.16)*. Assume that*

$$\text{either} \quad \|U_0\|_{L^m(B_+(1))} \leq N, \qquad \text{if} \quad m \in (3, \infty), \tag{5.21}$$

$$\text{or} \quad \|U_0\|_{L^3(B_+(1))} \leq N \leq \gamma_{univ}, \qquad \text{if} \quad m = 3. \tag{5.22}$$

Then there exists $S = S(M, N, m) \in (0, T]$ *satisfying the following property:*

$$\sup_{t \in (0,S)} t^{\frac{3}{2m}} \|U(\cdot, t)\|_{L^\infty(B_+(\frac{1}{2})))} \leq C(m)\big(N + N^{\kappa(m)}\big), \tag{5.23}$$

for constants $C(m) \in (0, \infty)$ *and* $\kappa(m) \geq 1$.

The result above is a global result. We also obtain localized results for suitable solutions, see [10, Theorem 1.4]. Such a result is in the spirit of the result of Kang, Miura and Tsai [211, Theorem 1.1]. Moreover, contrary to the whole-space result Theorem 5.1 that we stated above in a qualitative form, the half-space result (5.11) is stated in a quantitative way.

We now turn to the application of the above result to concentration of critical norms near potential scale-critical singularities in the half-space. As in the case of the whole-space, we state our result for solutions satisfying a generalized Type I condition

$$\sup_{\bar{x} \in \mathbb{R}^3_+} \sup_{r \in (0, r_0)} \sup_{T^* - r^2 < t < T^*} r^{-\frac{1}{2}} \left(\int_{B_{\bar{x},+}(r)} |U(x,t)|^2 dx \right)^{\frac{1}{2}} \leq M. \tag{5.24}$$

In order to make sense of the condition (5.24) in the case when $r_0 > \sqrt{T^*}$, we may extend U by zero in negative times.

Remark 5.12 (on the generalized Type I condition in the half-space)*.* As consequence of the unification of Type I blow-ups in the half-space achieved in [36, Theorem 2] in collaboration with Barker, see Theorem 4.2, we see that the 'ODE blow-up Type I condition'

$$\sqrt{T^* - t}|U(x,t)| \leq M'$$

implies that (5.24) is satisfied for some $r_0 \in (0, \infty)$ and $M(M', U_0, r_0) \in (0, \infty)$.

Theorem 5.13 (concentration near scale-critical singularities in the half-space; [10, Theorem 1.3], in collaboration with Albritton and Barker)**.** *Let* $T^* > 0$*. Let* γ_{univ} *and* $S(M, \gamma_{univ}, 3)$ *be given by Theorem 5.11 above.*
There exists $t_*(T^*, M, r_0) \in [0, \infty)$ *such that the following holds true. Let* U *be a Leray-Hopf solution to the Navier-Stokes equations in* $\mathbb{R}^3_+ \times (0, \infty)$ *with the no-slip boundary condition on* $\partial\mathbb{R}^3_+ \times (0, T^*)$ *and satisfying the generalized Type I bound* (5.24)*. Furthermore, suppose* U *first blows-up at* T^* *and has a singular point at the space-time point* $(0, T^*)$*.*
Then the above assumptions imply that

$$\|U(\cdot, t)\|_{L^3\left(B_+\left(\sqrt{\frac{T^*-t}{S(M)/2}}\right)\right)} > \gamma_{univ}, \tag{5.25}$$

for all $t \in (t_*(T^*, M, r_0), T^*)$.

In addition to this global concentration result, let us note that we obtained also a localized version of this concentration result for suitable solutions, see [10, Theorem 1.5].

Novelty of our results

As we mentioned previously, one heuristic interpretation of local-in-space short-time smoothing is the following. The nonlocal effects of the pressure do not substantially hinder the parabolic smoothing properties of the Navier-Stokes equations, at least locally in space and time. It is not clear at first sight that such a property holds even in $\mathbb{R}^3$. It is all the more difficult to prove such local smoothing properties in the half-space, for a number of fundamental reasons related to the fact that the nonlocal effects of the pressure are much stronger in $\mathbb{R}^3_+$ than in $\mathbb{R}^3$. We refer to the difficulties outlined in Section 4.1. Our results Theorem 5.11 and Theorem 5.13 are the first results of this type for the half-space. These results build upon the step forward done in my paper [247] with Maekawa and Miura about the existence of local energy solutions in the half-space, see Theorem 5.10 above.

5.3 New ideas and strategy for the proofs

5.3.1 Local-in-space short-time smoothing in the whole-space via a Caffarelli-Kohn-Nirenberg scheme

In the subcritical case, the main part of the proof of Jia and Šverák [197] for proving local-in-space short-time smoothing relies upon establishing an ε-regularity criteria for suitable solutions of the perturbed Navier-Stokes equations with subcritical a, i.e. $a \in L^\infty_t L^m_x(B_0(1)\times(-1,0))$, $m>3$ and $\nabla\cdot a=0$,

$$\begin{aligned}&\partial_t V-\Delta V+V\cdot\nabla V+a\cdot\nabla V+V\cdot\nabla a+\nabla Q=0,\\&\nabla\cdot V=0\quad\text{in}\ B_0(1)\times(-1,0).\end{aligned}\tag{5.26}$$

Notice that a is the mild solution associated to $U_0|_{B_0(1)}$ properly extended to the whole-space in a divergence-free manner. In particular, they show that if certain scale-critical quantities involving V and Q on the unit cube $B_0(1)\times(-1,0)$ are small then one has decay of the oscillation:

$$\frac{1}{r^5}\int\limits_{t_0-r^2}^{t_0}\int\limits_{B_{\bar{x}}(r)}\Big|V-\int\limits_{t-r^2}^{t}\int\limits_{B_{\bar{x}}(r)}V\,dyds\Big|^3dxds'\leq Cr^\alpha$$

for all $(\bar{x},t)\in B_0(\frac12)\times(-\frac14,0)$ and for some $\alpha>0$. This implies parabolic Hölder continuity of V by Campanato's characterisation. The proof of the decay of the oscillation in Jia and Šverák's paper [197] is achieved by contradiction and by compactness arguments. Related arguments were previously used in the context of the Navier-Stokes equations by Lin [243] and by Ladyženskaja and Seregin in [230]. Such arguments applied to the system (5.26) crucially use that for a in subcritical spaces, we have parabolic Hölder continuity in $B_0(\frac12)\times(-\frac14,0)$ for the following limit linear system:

$$\begin{aligned}&\partial_t W-\Delta W+a\cdot\nabla W+W\cdot\nabla a+\nabla Q=\nabla\cdot G,\\&\nabla\cdot W=0\quad\text{in}\quad B_0(1)\times(-1,0),\end{aligned}\tag{5.27}$$

for a subcritical forcing term $G\in L^{\frac{5m}{3}}(B_0(1)\times(-1,0))$.

Unfortunately, when U_0 is critical and hence a and G belong to scale-invariant spaces with respect to the Navier-Stokes scaling such as $L^5(B_0(1)\times(-1,0))$, we do not expect solutions of (5.27) to be Hölder continuous. This lack of improvement for the perturbed linear system (5.27) to prevents us from relying on compactness arguments to directly prove the boundedness of the perturbation V.

As is the case in [197] paper, the key point is to take advantage of the smallness of the local energy of the perturbation V locally in space near initial time, i.e. in $B_0(\frac{7}{8}) \times (0, S(M))$ for some $S(M) \in (0, \infty)$. There are then two main bricks in the proof:

(Step-1) We prove a subcritical Morrey bound on the perturbation. Smallness of the local energy together with the smallness of $\|a\|_{L^5_{t,x}}$ enables us to prove a subcritical Morrey bound on V: for $\delta \in (0, 3)$ fixed, for $(\bar{x}, t) \in B_0(\frac{3}{4}) \times (0, S(M))$,

$$\sup_{r \in (0, \frac{3}{4})} \frac{1}{r^{5-\delta}} \int_{t-r^2}^{t} \int_{B_{\bar{x}}(r)} |V|^3 dx ds < \infty, \tag{5.28}$$

with V extended by 0 in negative times. Estimate (5.28) is based on a Caffarelli-Kohn-Nirenberg type iteration. The proof requires some technical innovations, in particular concerning the treatment of the pressure and of the perturbation terms $a \cdot \nabla V$ and $V \cdot \nabla a$. A major difficulty is that the decay of $\|a\|_{L^5(B_{\bar{x}}(r) \times (t-r^2, t))}$ does not improve when $r \to 0$. The $L^5_{t,x}$ norm for a is the critical threshold for the iteration to work. Exploiting that a is a solution to the Navier-Stokes equations and the bound (5.28), one could directly apply standard ε-regularity away from the initial time to get smoothness of the solution U. This is what we do for the half-space, see Subsection 5.3.3 below. Instead, in the whole-space, we aim at obtaining the boundedness of V up to the initial time, see (5.7).

(Step-2) We obtain the boundedness of the perturbation $V = U - a$ and eventually Hölder continuity in the parabolic metric up to initial time (5.7). It goes through the use of the Morrey bound (5.28) to control the nonlinear term in (5.26) and a bootstrap to get the boundedness. Related arguments were used by Seregin in [307]. We now comment on the final bootstrap to get the Hölder regularity. Let us point out that for the subcritical case $U_0 \in L^m(B_0(1))$ (which corresponds to a belonging to subcritical spaces), Jia and Šverák prove in [197] that the perturbation $V = U - a$ is Hölder continuous in the parabolic metric up to the initial time. Moreover, in [197] the Hölder exponent arising from the proof degenerates as m approaches the critical case $m = 3$. Perhaps at first sight it appears somewhat unexpected that one still obtains Hölder continuity of V up to the initial time, for the critical initial data case. Our proof for showing this relies upon the structure of estimates for the mild solution a, in particular $\sup_{s \in (0, S(M))} s^{\frac{1}{5}} \|a(\cdot, s)\|_{L^5(\mathbb{R}^3)} \ll 1$ and the fact that V has zero initial data. Such points allow us to obtain a decay of the L^∞ norm of V near the initial time, which is key for going from V being bounded to Hölder continuous.

The extension of our results to the Besov case $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ for $p \in (3, \infty)$, see Remark 5.4, relies on some ideas which are new as far as we know. In particular, in the Caffarelli-Kohn-Nirenberg-type iteration, we need to exploit the local decay of the kinetic energy near the initial time, because the critical drift is more singular in the Besov case than in the L^3 case. It satisfies namely for $S \in (0, \infty)$ and sufficiently small data

$$\sup_{s \in (0, S)} \left(s^{\frac{1}{2}(1-\frac{3}{p})} \|a(\cdot, s)\|_{L^p} + s^{\frac{1}{2}} \|a(\cdot, t)\|_{L^\infty} \right) \le K(p) \sup_{s \in (0, S)} s^{\frac{1}{2}(1-\frac{3}{p})} \|e^{t\Delta} U_0\|_{L^p}.$$

In particular, we don't have a global $L^5_{t,x}$ bound on a. Rather than carrying out the Caffarelli-Kohn-Nirenberg iteration at the level of the subcritical Morrey-type bound (5.28), we propagate subcritical Morrey bounds (and local energy bound) that have decay near initial time: for some $\eta \in (0,1)$, we show that

$$\sup_{r\in(0,\frac{3}{4})} \frac{1}{r^{5-\delta}} \int_{t-r^2}^{s} \int_{B_{\bar{x}}(r)} |V|^3 dx d\hat{s} \leq C_{univ} s_+^{\frac{3}{2}\eta}, \qquad t-r^2 < s < t.$$

Such an insight was used before for global estimates by Barker [33] to prove weak-strong uniqueness, in Barker, Seregin and Šverák's paper [41] for global $L^{3,\infty}$ solutions and by Albritton and Barker [7] for global Besov solutions. However, to the best of our knowledge, it is the first time that the decay of the kinetic energy near initial time is used in local estimates, such as a Caffarelli-Kohn-Nirenberg type iteration. We believe this point is of independent interest.

5.3.2 Existence of global-in-time local energy solutions in the half-space

The key point of our work [247] and of the proof of the existence of local energy solutions in the half-space, see Theorem 5.10, is a new estimate for the pressure of Navier-Stokes in the half-space with non-decaying data. We rely on a decomposition of the pressure derived from the explicit representation obtained in collaboration with Maekawa and Miura [249]. This representation formula for the pressure of the Stokes resolvent problem builds upon an idea of Desch, Hieber and Prüss [124] that enables to circumvent the use of the Helmholtz-Leray projector known to be ill-behaved on spaces of non-decaying functions. The resolvent kernel is decomposed into a Dirichlet-Laplace part and a part corresponding to the nonlocal pressure, see [249, Subsection 2.3].

Indeed, we generalize the whole-space representation formula (5.2) to the case of the half-space. Due to the boundary $\partial\mathbb{R}^3_+$, in addition to the Helmholtz pressure P_{Helm}, a harmonic pressure P_{harm} has to be taken into account. Indeed, the nonlinear term in the Navier-Stokes equations can be decomposed into

$$\nabla\cdot(U(\cdot,t)\otimes U(\cdot,t)) = \nabla P_{Helm} + \mathbb{P}\nabla\cdot(U(\cdot,t)\otimes U(\cdot,t)).$$

Hence the harmonic pressure solves the Neumann boundary value problem

$$\begin{cases} -\Delta P_{harm}(\cdot,t) = 0 & \text{in } \mathbb{R}^3_+, \\ \partial_3 P_{harm}(\cdot,t) = \gamma|_{x_3=0}\Delta U_3(\cdot,t) & \text{on } \partial\mathbb{R}^3_+. \end{cases} \tag{5.29}$$

where $\mathbb{P}$ is the Helmholtz-Leray projection on divergence-free vector fields. We are able to provide an explicit representation for the Helmholtz part of the pressure, as well as for the harmonic part. Since we handle non-decaying data, each pressure part has to be splitted, as in the case of the whole-space, into a local part and a nonlocal part. Notice that the formulas for the nonlocal pressure terms, see $P^{U_0}_{nonloc}$, $P^{U\otimes U}_{Helm,nonloc}$ and $P^{U\otimes U}_{harm,nonloc}$ below, are obtained by substracting a constant, as for the whole-space. This is done in order to get a kernel with enough decay to handle non-decaying data.

Let $T > 0$ and $\bar{x} \in \overline{\mathbb{R}^3_+}$. We decompose the solution (U,P) to the Navier-Stokes equations into (U^{U_0}, P^{U_0}) the solution of the homogeneous Stokes system

$$\begin{cases} \partial_t U^{U_0} - \Delta U^{U_0} + \nabla P^{U_0} = 0 & \text{in } \mathbb{R}^3_+ \times (0,T), \\ \nabla \cdot U^{U_0} = 0, & \\ U^{U_0} = 0 & \text{on } \mathbb{R}^3_+ \times (0,T), \\ U^{U_0}(\cdot, 0) = U_0, & \end{cases} \tag{5.30}$$

and $(U^{U\otimes U}, P^{U\otimes U})$ the solution of the Stokes system with source term $-\nabla \cdot (U \otimes U)$,

$$\begin{cases} \partial_t U^{U\otimes U} - \Delta U^{U\otimes U} + \nabla P^{U\otimes U} = -\nabla \cdot (U \otimes U) & \text{in } \mathbb{R}^3_+ \times (0,T), \\ \nabla \cdot U^{U\otimes U} = 0, & \\ U^{U\otimes U} = 0 & \text{on } \partial\mathbb{R}^3_+ \times (0,T), \\ U^{U\otimes U}(\cdot, 0) = 0. & \end{cases} \tag{5.31}$$

The pressure P^{U_0} is a harmonic pressure since there is no source term in (5.30), hence no Helmholtz part. We split it into a local part around $\bar{x}$ and a nonlocal part away from $\bar{x}$: for all $(x,t) \in \mathbb{R}^3_+ \times (0,T)$,

$$\begin{aligned} &P^{U_0}(x,t) \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma} e^{\lambda t} \int\limits_{\mathbb{R}^3_+} q_\lambda(x'-z', x_3, z_3) \cdot U_0'|_{B_{\bar{x},+}(2)}(z', z_3) dz' dz_3 d\lambda \\ &\quad + \frac{1}{2\pi i} \int\limits_{\Gamma} e^{\lambda t} \int\limits_{\mathbb{R}^3_+} \big(q_\lambda(x'-z', x_3, z_3) - q_\lambda(\bar{x}'-z', \bar{x}_3, z_3)\big) \cdot U_0'|_{\mathbb{R}^3_+ \setminus B_{\bar{x},+}(2)}(z', z_3) dz' dz_3 d\lambda \\ &=: P^{U_0}_{loc}(x,t) + P^{U_0}_{nonloc}(x,t). \end{aligned} \tag{5.32}$$

This formula is obtained by Dunford's formula which gives an inverse of the Laplace transform. Here for all $x' \in \mathbb{R}^2$ and $x_3,\ z_3 > 0$,

$$q_\lambda(x', x_3, z_3) := i \int\limits_{\mathbb{R}^2} e^{ix'\cdot\xi} e^{-|\xi| x_3} e^{-\sqrt{\lambda + |\xi|^2} z_3} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\sqrt{\lambda + |\xi|^2}} \right) d\xi \tag{5.33}$$

is the harmonic pressure kernel for the resolvent problem defined in [249, (2.8e) and Proposition 3.7], and $\Gamma = \Gamma_\kappa$ with $\kappa \in (0,1)$ is the curve of $\mathbb{C}$ defined previously in (4.26).

Remark 5.14 (on the formulas for the harmonic pressure terms). Notice that the formula (5.32) involves the tangential component of U_0 rather than a vertical component as in the formula (4.25). Hence the resolvent pressure kernels q_λ and $\tilde{q}_\lambda$ that appear in the formulas are different. The two formulas are derived from [249]. The same comment applies to $P^{U\otimes U}_{harm}$ defined below in (5.35).

As for the pressure term $P^{U\otimes U}$, we decompose it into a Helmholtz pressure $P^{U\otimes U}_{Helm}$ and a harmonic pressure $P^{U\otimes U}_{harm}$ and split again into local and nonlocal terms. We have

$$P^{U\otimes U} = P^{U\otimes U}_{Helm} + P^{U\otimes U}_{harm},$$

with for the Helmholtz pressure

$$
\begin{aligned}
&P^{U\otimes U}_{Helm}(x,t)\\
&= \int\limits_{\mathbb{R}^3_+} \nabla_z^2 N(x'-z',x_3,z_3) U\otimes U|_{\mathbb{R}^3_+\setminus B_{\bar{x},+}(2)}(z',z_3,t)dz'dz_3\\
&\quad + \int\limits_{\mathbb{R}^3_+} \nabla_z^2 \big(N(x'-z',x_3,z_3) - N(\bar{x}'-z',\bar{x}_3,z_3)\big) U\otimes U|_{\mathbb{R}^3_+\setminus B_{\bar{x},+}(2)}(z',z_3,t)dz'dz_3\\
&= p^{U\otimes U}_{Helm,loc}(x,t) + p^{U\otimes U}_{Helm,nonloc}(x,t)
\end{aligned}
\tag{5.34}
$$

where N is the Neumann kernel in the half-space and for the harmonic pressure

$$
\begin{aligned}
&P^{U\otimes U}_{harm}(x,t)\\
&= \frac{1}{2\pi i}\int\limits_0^t \int\limits_\Gamma e^{\lambda(t-s)} \int\limits_{\mathbb{R}^3_+} q_\lambda(x'-z',x_3,z_3)\cdot \left(\mathbb{P}\nabla\cdot(U\otimes U|_{B_{\bar{x},+}(2)})\right)'(z',z_3,s)dz'dz_3d\lambda ds\\
&\quad + \frac{1}{2\pi i}\int\limits_0^t \int\limits_\Gamma e^{\lambda(t-s)} \int\limits_{\mathbb{R}^3_+} \big(q_\lambda(x'-z',x_3,z_3) - q_\lambda(\bar{x}'-z',\bar{x}_3,z_3)\big)\\
&\qquad\qquad \cdot \Big(\mathbb{P}\nabla\cdot(U\otimes U|_{\mathbb{R}^3_+\setminus B_{\bar{x},+}(2)})\Big)'(z',z_3,s)dz'dz_3d\lambda ds\\
&=: p^{U\otimes U}_{harm,loc}(x,t) + p^{U\otimes U}_{harm,nonloc}(x,t),
\end{aligned}
\tag{5.35}
$$

where q_λ and Γ are defined as above in (5.33) and (4.26). It is becomes clear from the formula (5.35) that the pressure in the half-space is more nonlocal than in the whole-space. Indeed, in addition to being nonlocal in space, there is nonlocality in time, because the harmonic pressure depends on the history of the flow.

It is essential to keep in mind that every term in decomposition above depends on $\bar{x}$. However, for two points $\bar{x}$ and $\hat{x}$, the difference of the associated pressures is a constant that depends only on time. Finally, let us emphasize that this level of precision in the description of the pressure can be achieved due to the special structure of $\mathbb{R}^3_+$, which allows to use the Fourier transform in the horizontal direction and hence to obtain explicit formulas. Notice that the (5.32) and (5.35) for the harmonic pressure terms P^{U_0} and $P^{U\otimes U}_{harm}$ and (5.34) for the Helmholtz pressure have a convolution structure in the in the direction tangential to the boundary, but not in the vertical direction.

For estimating the pressure terms, we rely on the results for the linear theory in the half-space obtained in or companion paper [249]. The local pressure $P^{U_0}_{loc}$ is certainly the most subtle term to analyze. Interestingly, if one uses the formulas obtains by Solonnikov [324, formula (1.9)], see also the papers [321, 323], one gets a non-integrable singularity near initial time for points $\bar{x}\in\partial\mathbb{R}^3_+$,

$$\|P^{U_0}_{loc}\|_{L^2(B_{\bar{x},+}(1))} \leq C\frac{(1+t)^{\frac{1}{2}}}{t}\|U_0\|_{L^2(B_{\bar{x},+}(2))}.$$

We are able to improve this bound near initial time by computing the integral in space variables with the resolvent pressure kernel q_λ first, and then computing the Dunford integral over the curve Γ. We obtain the following proposition.

Proposition 5.15 (estimates for the linear pressure terms; [247, Proposition 2.1]). *Let $T > 0$. There exists a constant $C(T) \in (0,\infty)$ such that for $t \in (0,T)$,*

$$\frac{t}{\log(e+t)}\|\nabla P^{U_0}(\cdot,t)\|_{L^2_{uloc}(\mathbb{R}^3_+)} \leq C\|U_0\|_{L^2_{uloc}(\mathbb{R}^3_+)}, \tag{5.36}$$

$$t^{\frac{3}{4}}\|P^{U_0}_{loc}(\cdot,t)\|_{L^2(B_{\bar{x},+}(1))} \leq C(T)\|U_0\|_{L^2(B_{\bar{x},+}(2))}, \tag{5.37}$$

$$t^{\frac{3}{4}}\|P^{U_0}_{nonloc}(\cdot,t)\|_{L^\infty(B_{\bar{x},+}(1)))} + t^{\frac{3}{4}}\|\nabla P^{U_0}_{nonloc}(\cdot,t)\|_{L^\infty(B_{\bar{x},+}(1)))} \leq C(T)\|U_0\|_{L^2_{uloc}(\mathbb{R}^3_+)}. \tag{5.38}$$

Moreover, (5.36) *holds with C independent of T.*

Remark 5.16 (erratum). In the published version of this result [247, Proposition 2.1], there is a mistake in the estimate (5.38). We claimed a singularity like $t^{\frac{1}{2}}$ instead of $t^{\frac{3}{4}}$. The slightly worse singularity in (5.38) does not change however substantially the other estimates in the paper since what matters is that one has some integrability of $\|P^{U_0}_{nonloc}(\cdot,t)\|_{L^\infty}$ in short-time. The latest version on arXiv `https://arxiv.org/abs/1711.04486` is correct.

Estimate 5.36 above is directly obtained from the estimates for the Stokes semigroup in L^2_{uloc} studied in our companion paper [249, Proposition 5.3]. The estimate (5.37) of $P^{U_0}_{loc}$ leads to the bound

$$\|P^{U_0}_{loc}\|_{L^{\frac{4}{3}-}(0,T;L^2(B_{\bar{x},+}(1)))} \leq C(T)\|U_0\|_{L^2(B_{\bar{x},+}(2))}.$$

This integrability in time, although with a small Lebesgue exponent, enables to use the local energy inequality (5.20) to control the solution U of the Navier-Stokes equations.

Let us notice that the singularity $O(t^{-\frac{3}{4}})$ in (5.37) is consistent with the one obtained in [273, 336] for the Stokes resolvent problem in a bounded domain Ω with no-slip boundary condition and source term f:

$$\|p\|_{L^2(\Omega)} \lesssim_\alpha |\lambda|^{-\alpha}\|f\|_{L^2(\Omega)} \qquad \text{for} \quad \alpha \in [0,1/4). \tag{5.39}$$

In [335] the optimality of the threshold $1/4$ is established. The bound (5.39) turns, via Dunford's formula, into a short-time estimate for the pressure associated to the unsteady Stokes problem with a singularity $O(t^{-3/4-\delta})$ for $\delta > 0$. We emphasize that in the case of bounded domains on the one hand and of localized estimates on the other hand, the estimates that are obtained break the natural scaling of the equations in the whole-space. This is not a contradiction because in both situations there is no scale-invariance. For more discussion on this topic, we refer to Subsection 4.1.2 above.

We do not rewrite here the estimates for $P^{U\otimes U}_{Helm,loc}$, $P^{U\otimes U}_{Helm,nonloc}$, $P^{U\otimes U}_{harm,loc}$ and $P^{U\otimes U}_{harm,nonloc}$ that can be found in the paper, see [247, Proposition 2.2 and Proposition 2.3].

With these estimates for the pressure, the proof of Theorem 5.10 goes roughly as follows. The evolution starts with a rough data barely locally integrable, $U_0 \in L^2_{uloc,\sigma}(\mathbb{R}^3_+)$, with the mild decay condition (5.16). The local-in-time local energy solution instantly becomes slightly more regular: $U(\cdot,t_0) \in L^4_{uloc,\sigma}(\mathbb{R}^3_+)$ for almost all t_0 in the existence time interval with the mild decay

$$\lim_{\substack{|\bar{x}|\to+\infty\\ \bar{x}\in\overline{\mathbb{R}^3_+}}} \int\limits_{B_{\bar{x},+}(1)\cap\mathbb{R}^3_+} |U(\cdot,t_0)|^4\,dx = 0. \tag{5.40}$$

This allows to decompose the data $U(\cdot,t_0)$ into a large $C^\infty_c(\mathbb{R}^3_+)$ part for which we have global-in-time Leray-Hopf solutions, and a small part in $L^4_{uloc,\sigma}(\mathbb{R}^3_+)$ for which we have local-in-time existence of mild solutions thanks to our work in collaboration with Maekawa and Miura [249, Proposition 7.1]. The difficult part of this reasoning is to transfer the decay of the initial data U_0 satisfying (5.16) to the solution U, i.e. to prove (5.40). This issue is addressed in [235, Proposition 32.2] and [217, Theorem 1.4] in the case of the whole-space. We handle this question for the half-space, see (5.41) below, which is more involved due to the strong nonlocal nature of the pressure. Our main results in this direction are summarized in the following proposition.

Proposition 5.17 (main proposition for the existence of global-in-time local energy solutions; [247, Corollary 5.9, Lemma 4.1 and Theorem 2])**.** *This proposition is divided into two parts:*

(1) For all $M>0$, there exist $T(M)\in(0,\infty)$ and $A_M\in[1,\infty)$ such that for all $U_0\in L^2_{uloc,\sigma}(\mathbb{R}^3_+)$ satisfying the mild decay (5.16), for all local energy solution U to the Navier-Stokes equations on $\mathbb{R}^3_+\times(0,T(M))$, if $\|U_0\|_{L^2_{uloc}(\mathbb{R}^3_+)}\leq M$, then

$$\sup_{\bar{x}\in\overline{\mathbb{R}^3_+}}\ \sup_{t\in(0,T(M))} \int\limits_{B_{\bar{x},+}(1)} |U(\cdot,t)|^2 + \int\limits_0^{T(M)} \int\limits_{B_{\bar{x},+}(1)} |\nabla U|^2 + \Bigg(\int\limits_0^{T(M)} \int\limits_{B_{\bar{x},+}(1)} |U|^3\Bigg)^{\frac{2}{3}} \leq A_M.$$

(2) For M, $T(M)$, U_0 and U as above we have: for all $R\geq 1$,

$$\begin{aligned}\sup_{\overline{\mathbb{R}^3_+}}\ \sup_{t\in(0,T(M))} &\int\limits_{B_{\bar{x},+}(1)} |(1-\theta(\tfrac{\cdot}{R})U(\cdot,t)|^2 + \int\limits_0^T \int\limits_{B_{\bar{x},+}(1)} |(1-\theta(\tfrac{\cdot}{R}))\nabla U|^2\\ &+\Bigg(\int\limits_0^{T(M)} \int\limits_{B_{\bar{x},+}(1)} |(1-\theta(\tfrac{\cdot}{R}))U|^3\Bigg)^{\frac{2}{3}} + \Bigg(\int\limits_\delta^{T(M)} \int\limits_{B_{\bar{x},+}(1)} |P|^{\frac{3}{2}}\Bigg)^{\frac{2}{3}} \overset{R\to\infty}{\longrightarrow} 0,\end{aligned} \tag{5.41}$$

for all $\delta\in(0,T(M))$. Here $\theta\in C^\infty_c(\mathbb{R}^3)$ is a non-negative cut-off such that $\theta\equiv 1$ on $B_0(1)$ and $\mathrm{supp}(\theta)\subseteq B_0(2)$.

The proof of the decay estimate at large-scales (5.41) goes through the proof of the following Gronwall-type inequality: for all $R\geq 1$, for all $t\in[0,T(M)]$,

$$\alpha_R(t)+\beta_R(t)\leq C(M)\Bigg(\Bigg(\int\limits_0^t \alpha_R^{21}(s)ds\Bigg)^{\frac{1}{21}} + R^{-1}(\log R) + \|(1-\theta(\tfrac{\cdot}{R}))U_0\|_{L^2_{uloc}(\mathbb{R}^3_+)}\Bigg).$$

Here

$$\alpha_R(t) := \sup_{\bar{x}\in\overline{\mathbb{R}^3_+}} \int\limits_{B_{\bar{x},+}(1)} |(1-\theta(\tfrac{\cdot}{R}))U(\cdot,t)|^2,\quad \beta_R(t) := \sup_{\bar{x}\in\overline{\mathbb{R}^3_+}} \int\limits_0^t \int\limits_{B_{\bar{x},+}(1)} |(1-\theta(\tfrac{\cdot}{R}))\nabla U|^2.$$

5.3.3 Local-in-space short-time smoothing in the half-space via a compactness method

For the proof of local-in-space smoothing in the half-space, Theorem 5.11, the general idea is the same as for the whole-space, see Subsection 5.3.1. We decompose the solution U to the Navier-Stokes equation into $U = a + V$, where a is the mild solution roughly corresponding to the initial data $U_0|_{B_+(1)}$ and V is a solution to a perturbed equation. The key is again to prove smallness of the perturbation V locally near $B_+(1) \times \{0\}$ via local energy estimates and then smoothness of V by an ε-regularity result. Instead of relying on a Caffarelli-Kohn-Nirenberg type scheme as in Subsection 5.3.1 for the whole-space, we use here a compactness proof, inspired by Lin [243], Seregin and Ladyženskaja [230] for the ε-regularity of solutions to the Navier-Stokes equations on the one hand, and Jia and Šverák [197] and Kang, Miura and Tsai [211] for the ε-regularity of solutions to the Navier-Stokes equations with subcritical or scale-critical drift terms on the other hand.

A novelty of our approach compared to [211] is that we build smallness of a into our compactness arguments, which completely bypasses the estimates in [211, Section 4] for the linear perturbed equation (5.27). Therefore our limit system in the compactness proof is simply a linear Stokes system.

We prove a one-scale ε-regularity criteria for suitable solutions of the perturbed Navier-Stokes equation. In the half-space, it is standard that ε-regularity also depends on the pressure P, and a typical choice of quantity is $\|P\|_{L^{3/2}}$. This choice is convenient for treating the term involving UP in the local energy inequality. This comment brings us to our first difficulty, namely, that for solutions of the linear Stokes equations in the half-space with initial data in L^2, the pressure estimates are only known in $L^{4/3-}_{t,loc}L^2_x$, see Proposition 5.15 above. We already commented on the low time integrability in connection with the proof of existence local energy solutions in the half-space in Subsection 5.3.2. Therefore, we must prove ε-regularity for the perturbed Navier-Stokes system under a new assumption on the pressure. Essentially, we require smallness of the pressure in $L_t^{1+\delta_t}L_x^{2-\delta_x}$ with $0 < \delta_x \leq \delta_t \ll 1$. The integrability of the velocity with the Hölder conjugate exponents compensates for the low time integrability of the pressure but remains controlled, with room to spare, by the energy space $L_t^\infty L_x^2 \cap L_t^2 H_x^1$.

There is a second difficulty, which concerns only the critical case $m = 3$ and was also encountered in [35, 211]. In this case, due to the lack of improvement of flatness, see Subsection 5.3.1 about the proof of local-in-space short-time smoothing in the whole-space, the compactness argument yields a subcritical Morrey bound just below L^∞, but does not give boundedness let alone Hölder continuity. In [35], we overcome this difficulty by using parabolic regularity theory to bootstrap the regularity of the perturbation V, from subcritical Morrey to Hölder. In principle this is also possible here, with some more technicalities because we have to bootstrap the regularity in the half-space. However, it is not necessary for our application. Rather, following [211], we combine (i) the subcritical Morrey estimates for the perturbation V, (ii) the critical estimates for a, and (iii) the standard ε-regularity criterion (without lower order terms) for the solution U of the Navier-Stokes equations so as to conclude the L^∞-smoothing (5.23).

Chapter 6

Quantitative regularity

This chapter relies mainly on the papers:

— [37], with Tobias Barker, Mild criticality breaking for the Navier-Stokes equations, *J. Math. Fluid Mech.* (2021).

— [38], with Tobias Barker, Quantitative regularity for the Navier-Stokes equations via spatial concentration, *Comm. Math. Phys.* (2021).

Contents

Our interest in quantitative estimates under a priori control of critical norms goes back to the summer of 2019 and the publication by Tao [334] of his breakthrough paper on the quantification of the Escauriaza, Seregin and Šverák [129] result, see Subsection 6.1.4 below. Our focus is on regularity estimates of the generic form

$$\|U\|_{L^\infty(\mathbb{R}^3\times(-\frac{1}{2},0))} \leq \mathscr{G}(\|U\|_A, \|U(\cdot,0)\|_B) \tag{6.1}$$

for solutions U of the Navier-Stokes equations, with $A \subseteq L^1(-1,0;C)$ and B, C are certain Banach spaces contained in $\mathcal{S}'(\mathbb{R}^3)$. The point is to derive an explicit formula for $\mathscr{G}$. Together with Barker we develop a generic approach based on the concentration of local scale-critical quantities in the physical space. We study quantitative regularity:

(i) **Under a Type I (scale-critical) control**, see Theorem 6.1, in the vein of Tao's result; in our result $A = L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))$ and $B \subseteq L^3(\mathbb{R}^3)$, while in Tao's result $A = L^\infty(-1,0;L^3(\mathbb{R}^3))$ and there is no dependence in the second variable of $\mathscr{G}$;

(ii) **Outside a scale-critical regime** under control of the L^3 norm at discrete times, see Theorem 6.3; in this case the form of the quantitative estimate (6.1) has to be adapted a bit, see (6.22).

Explicit quantitative estimates in the critical case also enable us to slighlty break the criticality barrier using ideas from nonlinear dispersive equations, see Theorem 6.5. This whole chapter is concerned with the three-dimensional Navier-Stokes equations in the whole-space.

6.1 Context: state of the art and obstacles

6.1.1 Blow-up of borderline critical norms near potential singularities

This paragraph is concerned with qualitative results. We focus on borderline critical norms that cannot be made small by shrinking the time interval. We make a difference between two families:

(i) **Non endpoint spaces** such as $L^\infty_t L^3_x$ or $L^\infty_t L^{3,r}_x$, $r \in (3,\infty)$, for which regularity is known for arbitrarily large norm; this is the case for norms where a certain smallness is hidden that enables to prove that the limit 'blow-up profile', obtained by zooming in on a potential singularity, is zero; for instance in the case of the L^3 norm, we have the property that $\|g\|_{L^3(B(r))} \to 0$ when $r \downarrow 0$;

(ii) **Endpoint spaces**, sometimes called 'ultra-critical spaces' (see [47]) such as $L^\infty_t L^{3,\infty}_x$, $L^\infty_t \dot{B}^{-1+\frac{3}{p}}_{p,\infty,x}$, $p \in (3,\infty)$, or $L^\infty_t BMO^{-1}_x$ that do require some extra assumption to ensure the vanishing of the limiting blow-up profile.

Notice that these two categories of spaces also appear in the mild solution theory. Indeed the type of results that we can prove are of a different nature: (i) local-in-time existence for any data, however large, and global-in-time existence for small data in the first category, (ii) only global-in-time existence results for small data are known in the second category. Note though that it is possible to prove the existence of global weak solutions for arbitrary data in all cases [304, 41, 6, 7].

Let us first consider the case of non endpoint norms. In the breakthrough paper [129], Escauriaza, Seregin and Šverák showed that if $(\bar{x}, T^*)$ is a singular point then

$$\limsup_{t\uparrow T^*} \|U(\cdot,t)\|_{L^3(B_{\bar{x}}(r))} = \infty \quad \text{for any fixed } r > 0. \tag{6.2}$$

This is equivalent to showing regularity under $L^\infty_t L^3_x$ control. The result was reproved by Gallagher, Koch and Planchon [141] via a profile decomposition technique inspired from Kenig and Koch [215]. The result was also extended to non endpoint Lorentz spaces $L^\infty_t L^{3,r}_x$, $r \in (3,\infty)$, by Phuc [281] and to non endpoint Besov spaces $L^\infty_t \dot{B}^{-1+\frac{3}{p}}_{p,q,x}$, p, $q \in$

$(3,\infty)$, by Gallagher, Koch and Planchon [142] using a profile decomposition method and by Albritton [6] using the stability of global weak Besov solutions. About teen years after (6.2) was proved, Seregin [294] improved (6.2):

$$\lim_{t\uparrow T^*}\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}=\infty. \tag{6.3}$$

This result goes beyond the critical case because it tells that boundedness of the $L^3(\mathbb{R}^3)$ norm along a sequence of times, rather than for all times, is enough to get the regularity.

Second we consider the case of endpoint spaces. Let us point that it is unknown in general whether the Type I blow-up condition $U\in L^\infty_t L^{3,\infty}_x$ implies regularity. Exceptions are when this quasinorm is small [221], when the flow is axisymmetric [94, 93, 219], or when the blow-up profile vanishes thanks to some smallness condition, see [41, 297]. Finally, Albritton and Barker [7] show the analog of Seregin's result [294] for endpoint Besov norms. Namely they prove that if the endpoint Besov norm $\dot B^{-1+\frac{3}{p}}_{p,\infty}$, $p\in(3,\infty)$, is bounded along a sequence of times $t_n\uparrow T$ and if at final time T the blow-up profile satisfies

$$\sqrt{T-t_n}U\big(\sqrt{T-t_n}(\cdot-\bar x),T\big)\overset{*}{\longrightarrow}0\quad\text{in}\quad\mathcal{D}'(\mathbb{R}^3)$$

then $(\bar x,T)$ is a regular point.

6.1.2 Effective quantitative regularity in the subcritical and non borderline critical case

It is known since the major work of Leray [238] that subcritical L^p norms, $p\in(3,\infty]$ blow-up with the following rate

$$(T^*-t)^{\frac{3}{2}(\frac{1}{3}-\frac{1}{p})}\|U(\cdot,t)\|_{L^p}\leq c_p\quad\text{for all}\quad t\in(0,T^*), \tag{6.4}$$

for $c_p\in(0,\infty)$ and T^* a blow-up time. These blow-up estimates are the other side of the coin of regularity estimates. We present here two methods to obtain quantitative regularity estimates under subcritical and critical controls. A further strategy is presented in Subsection 6.3.1.

Subcritical case If U is a finite-energy solution on $\mathbb{R}^3\times(0,1)$ with a finite subcritical norm, then it is known that U must belong to $C^\infty(\mathbb{R}^3\times(0,1])$. See, for example, [227]. Moreover, one typically has a quantitative estimate of the form (6.1) with

$$\mathscr{G}(x,y)=cx^\beta\ \text{ with }\ \beta>0.$$

To demonstrate this, consider U belonging to $L^{5+\delta}_{x,t}(\mathbb{R}^3\times(0,1))$ for $\delta>0$. An application of Caffarelli, Kohn and Nirenberg's result [75] gives that (6.1) holds true with $\mathscr{G}(x,y)\simeq x^{\frac{\delta+5}{\delta}}$. Such a quantitative estimate is invariant with respect to the Navier-Stokes scaling.

Critical case In the subcritical norm case, we saw that seeking estimates of the form (6.1) that are invariant with respect to the scaling gives a suitable candidate for $\mathscr{G}$. The case when the norm is critical is more subtle, since a scaling argument does not provide a suitable candidate for $\mathscr{G}$. We first mention that the case of sufficiently small critical norms, for example

$$\|U\|_{L^5(\mathbb{R}^3\times(0,1))}<\varepsilon_*, \tag{6.5}$$

is essentially of a similar category to the subcritical case (though a scaling argument is not applicable). Indeed, a similar argument as before based on [75] gives that in this case we have (6.1) with $\mathscr{G}(x,y) \simeq x$. This is consistent with the fact that solutions with small scale-invariant norms exhibit similar behavior to the linear system and hence are typically expected to satisfy linear estimates. For obtaining quantitative estimates of the form (6.1) when the scale-invariant norm is large, it is less clear what the candidate for $\mathscr{G}$ might be. For the case of a smooth finite-energy solution U having finite but large scale-invariant $L^5(\mathbb{R}^3 \times (0,1))$ norm, one way to obtain quantitative estimates is to consider the vorticity equation (4.12) with initial vorticity $\omega_0 \in L^2(\mathbb{R}^3)$; see also [237, Chapter 11]. Performing an energy estimate yields for $t \in [0,1]$

$$\|\omega(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t\int_{\mathbb{R}^3} |\nabla\omega|^2 dxdt' \leq \|\omega_0\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t\int_{\mathbb{R}^3} (\omega\cdot\nabla U)\cdot\omega dxdt', \quad (6.6)$$

where the second term in the right-hand side is due to the vortex stretching term $\omega\cdot\nabla U$ in (4.12). For the case that $U \in L^5(\mathbb{R}^3 \times (0,1))$, application of Hölder's inequality, Sobolev embedding theorems and Young's inequality lead to

$$\|\omega(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t\int_{\mathbb{R}^3} |\nabla\omega|^2 dxdt' \leq \|\omega_0\|^2_{L^2(\mathbb{R}^3)} + C\int_0^t \|U(\cdot,t')\|^5_{L^5(\mathbb{R}^3)}\|\omega(\cdot,t')\|^2_{L^2(\mathbb{R}^3)}dt'. \quad (6.7)$$

Gronwall's lemma, followed by arguments similar to the subcritical case, yields

$$\|U\|_{L^\infty(\mathbb{R}^3\times(\frac{1}{2},1))} \lesssim \|\omega\|^2_{L^\infty(0,1;L^2(\mathbb{R}^3))} \leq \|\omega_0\|^2_{L^2(\mathbb{R}^3)} \exp\left(\|U\|^5_{L^5(\mathbb{R}^3\times(0,1))}\right). \quad (6.8)$$

Though this is not exactly of the form (6.1), a slightly different argument gives that for any finite-energy solution U in $L^5(\mathbb{R}^3 \times (0,1))$ we get that (6.1) holds with $\mathscr{G}(x,y) \simeq \exp(Cx^5)$. In particular, this can be achieved using the strategy of Subsection 6.3.1.

The above argument (6.6)-(6.8) shows that being able to substantially improve upon $\mathscr{G}(x,y) \simeq \exp(Cx^5)$ would most likely require the utilization of a nonlinear mechanism that reduces the influence of the vortex stretching term $\omega\cdot\nabla U$ in (4.12) (such a mechanism is used for instance in Constantin and Fefferman's result [107], see Subsection 4.1.5). It seems plausible that the discovery of such a mechanism would have implications for the regularity theory of the Navier-Stokes equations.

6.1.3 Abstract quantitative regularity in the borderline critical case

The argument of Escauriaza, Seregin and Šverák [129] is by contradiction and hence qualitative. It can, though, be quantified abstractly using the 'persistence of singularities' lemma in [287, Lemma 2.2] (note that the persistence of singularities was also a key tool in our strategy for regularity under vorticity alignement, see Subsection 4.3.3). Namely, there exists a function $\mathscr{G}$ such that if U is a finite-energy solution to the Navier-Stokes equations then

$$\|U\|_{L^\infty(0,1;L^3(\mathbb{R}^3))} < \infty \Rightarrow \|U\|_{L^\infty(\mathbb{R}^3\times(\frac{1}{2},1))} \leq \mathscr{G}(\|U\|_{L^\infty(0,1;L^3(\mathbb{R}^3))}). \quad (6.9)$$

Such an argument is by contradiction and hence gives no explicit information about $\mathscr{G}$.

On a tangential note, it is demonstrated [32] by an elementary scaling argument inspired from a talk of Seregin at the University of Sussex on 03 March 2014, that if the set of finite-energy solutions to the Navier-Stokes equations (with Schwartz class initial data) that blows-up is non empty, there cannot exist a positive universal function $\mathscr{F}$ such that the following analogue of (6.4) holds true:

$$\lim_{s\to 0^+} \mathscr{F}(s) = \infty, \tag{6.10}$$

and for all $T^* > 0$, if U is a finite-energy solution to the Navier-Stokes equations (with Schwartz class initial data) that first blows-up at $T^* > 0$ then U necessarily satisfies

$$\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)} \geq \mathscr{F}(T^*-t) \tag{6.11}$$

for all $t \in [0,T^*)$.

6.1.4 Tao's quantitative result

In a remarkable recent development [334], Tao used a new approach to provide the first explicit quantification of the seminal result of Escauriaza, Seregin and Šverák [129]. In particular, Tao shows that for classical solutions to the Navier-Stokes equations on $\mathbb{R}^3 \times (-1,0)$ belonging to the critical space $L^\infty(-1,0;L^3(\mathbb{R}^3))$,

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq \exp\big(\exp\big(\exp\big(\|U\|^C_{L^\infty_t L^3_x(\mathbb{R}^3\times(0,t))}\big)\big)\big)(-t)^{-\frac{1}{2}} \quad \text{for all} \quad -1 \leq t < 0, \tag{6.12}$$

where $C \in (0,\infty)$ is a universal constant. Here $\mathscr{G}(x,y) \simeq \exp(\exp(\exp(x^C)))$ in the notation of the general form estimate (6.1).

Combining these quantitative estimates and the Leray blow-up rate (6.4) for the subcritical L^∞ norm, Tao showed that if a finite-energy solution U first blows-up at $T^* > 0$ then for some universal constant $c \in (0,\infty)$,

$$\limsup_{t\uparrow T^*} \frac{\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{\big(\log\log\log \frac{1}{T^*-t}\big)^c} = \infty. \tag{6.13}$$

Since there cannot exist $\mathscr{F}$ such that (6.10)-(6.11) holds true, at first sight (6.13) may seem somewhat surprising, though it is not conflicting with such a fact. Notice that

$$\frac{\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{\big(\log\log\log \frac{1}{T^*-t}\big)^c}$$

is not invariant with respect to the Navier-Stokes scaling but is slightly supercritical due to the presence of the logarithmic denominator. The $\limsup_{t\uparrow T^*}$ in (6.13) is due to the fact that the whole $L^\infty_t L^3_x(\mathbb{R}^3 \times (-1,0))$ norm appears in the quantitative regularity (6.12).

We refer to Subsection 6.3.3 for a description of Tao's scheme and a comparison to ours.

6.2 Main results

6.2.1 Localized blow-up rate for a scale-critical singularity

Theorem 6.1 (rate of blow-up, Type I; [38, Theorem A], in collaboration with Barker)**.** *For all $M \in [1,\infty)$ sufficiently large, the following holds true.*
Assume that U is a mild solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [0,T^*)$ with $U \in$

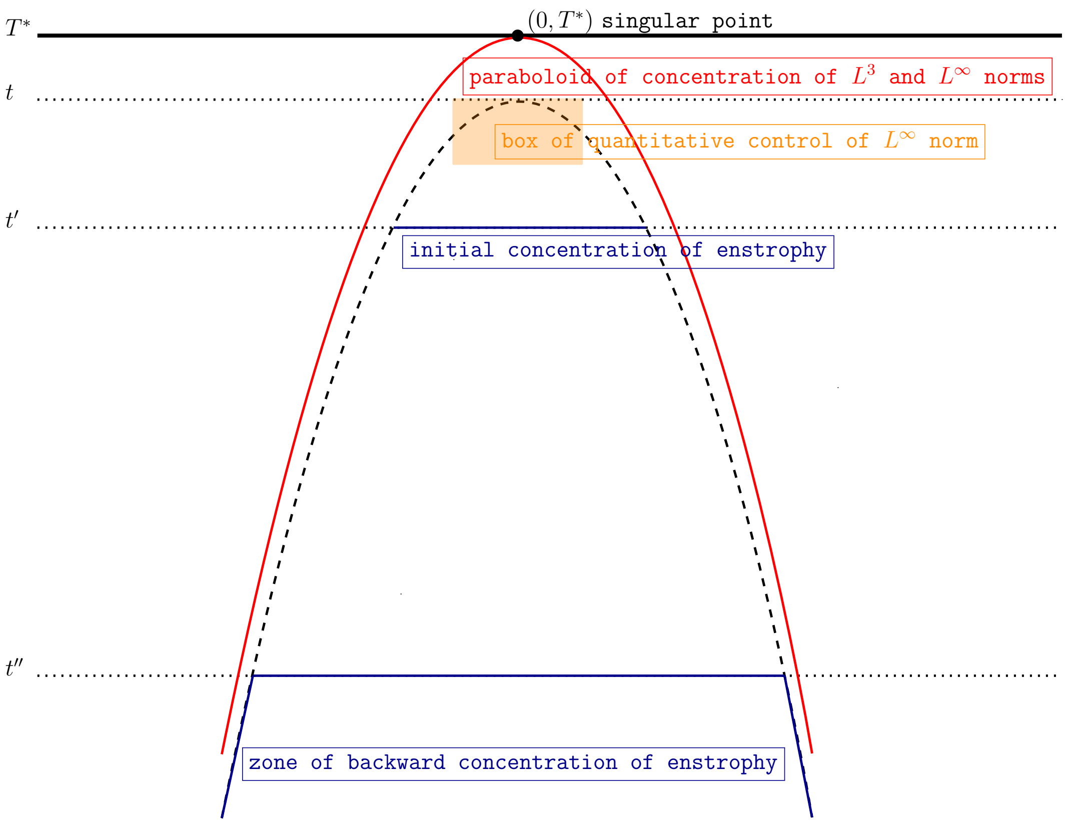

Figure 6.1 – Blow-up of the L^3 norm on concentrating sets

$L^\infty_{loc}([0,T^*);L^\infty(\mathbb{R}^3))$.
Assume in addition that $(0,T^*)$ *is a Type I blow-up, i.e.* U *has a singular point at* $(x,t) = (0,T^*)$*, in particular* $U \notin L^\infty_{x,t}(Q_{(0,T^*)}(r))$ *for all sufficiently small* $r > 0$*, and*

$$\|U\|_{L^\infty_t L^{3,\infty}_x(\mathbb{R}^3\times(0,T^*))} \leq M. \tag{6.14}$$

Then the above assumptions imply that there exists $S(M) \simeq M^{-30} \in (0,\frac{1}{4}]$ *such that for any* $t \in (\frac{T^*}{2},T^*)$ *and*

$$R \in \left(\sqrt{\frac{T^*-t}{S(M)}}, e^{M^{1022}}\sqrt{T^*}\right) \tag{6.15}$$

we have

$$\int\limits_{|x|<R} |U(x,t)|^3 dx \geq \frac{\log\left(\frac{R^2}{M^{802}|T^*-t|}\right)}{\exp(\exp(M^{1025}))}. \tag{6.16}$$

The proof of Theorem 6.1 relies on the combination, as is showed in Figure 6.1, of the quantitative regularity in the Type I case on the one hand, see Subsection 6.3.2 below and [38, Proposition 2.1], with concentration estimates near a potential singularity for the local L^3 norm and for the L^∞ norm on the other hand, see the results of Chapter 5.

Remark 6.2 (further quantitative results). In addition to Theorem 6.1, our techniques enable us to obtain two further results:

(1) a regularity criteria based on the relative smallness of the $L^{3,\infty}$ quasinorm at final time, see [38, Proposition 4.1]. Namely, we prove that for a global-in-time suitable finite-energy solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, \infty)$ satisfying the Type I bound (6.14), if for some $T^* \in (-1, 0]$,

$$\lim_{r \to 0} \|U(\cdot, T^*)\|_{L^{3,\infty}(B_0(r))} \leq \exp(-\exp(M^{1023})), \tag{6.17}$$

then, $(0, T^*)$ is a regular point.

(2) a quantification of the number of singularities in a Type I blow-up scenario, see [38, Corollary 4.3]. We prove that a global-in-time suitable finite-energy solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [0, \infty)$ satisfying the Type I bound (6.14) has at most $\exp(\exp(M^{1024}))$ blow-up points at time T^*.

Novelty of our results

The innovations in the results above lie in the following aspects, that all rely on the fact that we assume the scale-invariant assumption (6.14).

First, in Theorem 6.1 not only is the rate new but also the fact that the L^3 norm blows up on a ball concentrating on the potential singularity. Indeed, given the range (6.15), it is possible to take $R = O\big((T^* - t)^{\frac{1-\delta}{2}}\big)$ for $\delta \in (0, 1)$ and times t sufficiently close to the blow-up time T^*. Previously, see [240, Theorem 1.3], it was shown that if a solution blows up without a Type I bound then the L^3 norm blows up on certain non-explicit concentrating sets. This localization is possible because we work with quantities defined locally in physical space, rather than global quantities involving the Fourier transform as in [334].

Second, the lower bound (6.16) holds for all times $t \in (\frac{T^*}{2}, T^*)$ and not only for a sequence tending to T^* as is the case in (6.13) obtained in [334]. Indeed, under the Type I condition (6.14), we are able to get a quantitative regularity result that only involves the L^3 norm at final time.

Third, we manage to remove two logarithms from the lower bound (6.13). Using a result by Chae and Wolf [84], we notice that our rate is optimal for backward discretely self-similar solutions with sufficient decay, see [38, Corollary 1.1]. On this topic notice that our estimates do not succeed in ruling our this blow-up scenario, because there is a huge gap between the lower bound in 6.16 in which the constant behaves like $\exp(-\exp(M^{1025}))$ and the upper bound one gets using the Type I a priori bound in which the constant behaves like M^3, see [38, equation (12)]. Moreover, notice that estimate (6.16) is written in a scale-invariant form. Hence this estimate does not contradict Seregin's remark, see Subsection 6.1.3, about the non-existence of a universal blow-up rate in the borderline case.

Fourth, we are able to explicitly quantify the number of blow-up points in the Type I scenario, see remark 6.2, which improves upon the non effective bounds in [100, 297].

Further developments

In the recent paper [276], Palasek was also able to improve upon the triple log rate (6.13) obtained by Tao in [334]. In the case of axisymmetric solutions for instance, the triple log is replaced by a double log. Without any symmetry assumption on the solution, a similar improvement can be obtained by replacing the L^3 norm by the norm $\|r^{1-\frac{3}{q}} U\|_{L^\infty_t L^q_x}$ for $q \in (3, \infty)$ and $r := |x_h|$.

The quantitative regularity for solutions $U \in L^\infty_t L^d_x$ to the Navier-Stokes equations in higher dimensions $d \geq 4$ was handled by Palasek in [277]. This work gives an effective quantification of the qualitative result by Dong and Du [125]. For the blow-up rate, one pays the price of an additional logarithm compared to the result in dimension three (6.13).

Let us also mention the result of Feng, He and Wang [133] which quantifies the blow-up of borderline non-endpoint Lorentz norms $L^{3,q}(\mathbb{R}^3)$, for $q \in (3,\infty)$, hence quantifying the result by Phuc [281].

Concerning the quantification of the number of blow-up points in a Type I blow-up scenario, Barker [34, Theorem 2] was able to drastically improve the double exponential bound that we obtained in [38], see Remark 6.2. Indeed, the bound for the number of blow-up points is reduced to $O(M^{20})$ under the weaker assumption that the $L^{3,\infty}$ quasi-norm of the solution is uniformly bounded by M on a sequence tending to the blow-up time.

6.2.2 Quantification of Seregin's 2012 result

Theorem 6.3 (quantification of Seregin's result; [38, Theorem B] in collaboration with Barker)**.** *For all $M \in [1,\infty)$ sufficiently large, the following holds true.*
We define

$$M^\flat := \exp\Big(\frac{LM^5}{2}\Big), \tag{6.18}$$

for an appropriate universal constant $L \in (0,\infty)$. Let U be a finite-energy $C^\infty(\mathbb{R}^3 \times (-1,0))$ solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1,0]$.
Assume that there exists $t_k \in [-1,0)$ such that

$$t_k \uparrow 0 \quad \textit{with} \quad \sup_k \|U(\cdot,t_k)\|_{L^3(\mathbb{R}^3)} \leq M. \tag{6.19}$$

Select any 'well-separated' subsequence (still denoted t_k) such that

$$\sup_k \frac{-t_{k+1}}{-t_k} < \exp(-2(M^\flat)^{1223}). \tag{6.20}$$

Then for

$$j := \lceil \exp(\exp((M^\flat)^{1224}))\rceil + 1, \tag{6.21}$$

we have the bound

$$\|U\|_{L^\infty\left(\mathbb{R}^3 \times \left(\frac{t_{j+1}}{4},0\right)\right)} \leq \frac{CM^{-23}}{(-t_{j+1})^{\frac12}}, \tag{6.22}$$

for a universal constant $C \in (0,\infty)$.

Figure 6.2 is an illustration of this theorem. Of course the zone of quantitative regularity depends on the sequence t_k on which the assumption (6.19) holds via the time t_{j+1}, j being the index defined by (6.21). The quadruple exponential bound

$$0 > t_{j+1} \gtrsim -\exp(-\exp\exp\exp(M^6))$$

is a consequence of the definition (6.21) of j, the well seperation property (6.20) and the definition (6.18) for $M^\flat$.

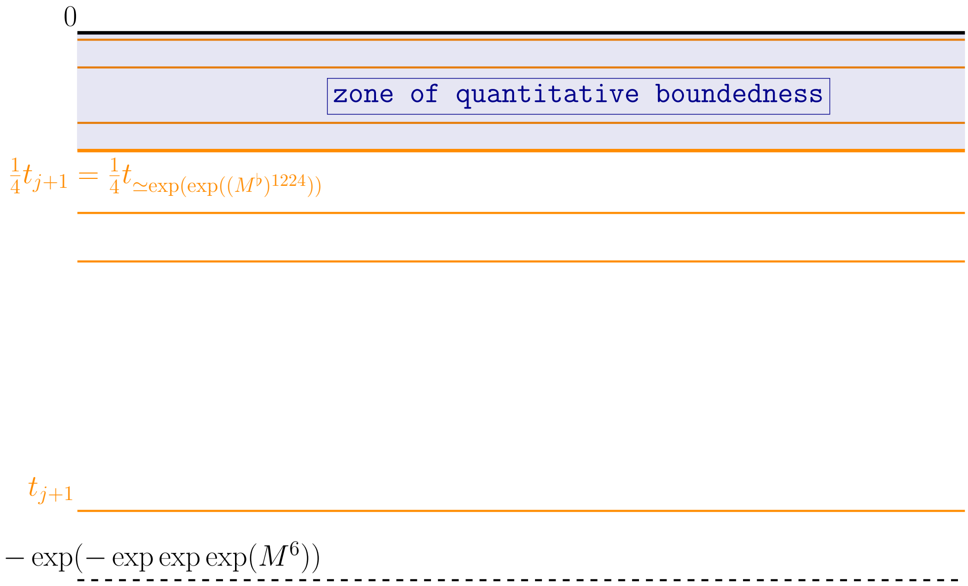

Figure 6.2 – Quantification of Seregin's 2012 result

Remark 6.4 (a further quantitative result). We also obtain a quantitative regularity criteria in terms of the relative smallness of the solution at final time and initial time. Namely, if

$$\|U(\cdot,-1)\|_{L^3(\mathbb{R}^3)} \le M,$$

then

$$\|U(\cdot,0)\|_{L^3(B_0(\exp((M^\flat)^{1221}))\setminus B_0(1))} \le \exp(-\exp((M^\flat)^{1223})),$$

implies that $(0,0)$ is a regular point. A 'non-effective' version of this result is in [7, Theorem 4.1 (i)].

Novelty of our result

Theorem 6.3 is a quantitative version of the result of Seregin [294, Theorem 1.1]. Let us emphasize that our result, as well as Seregin's previously, holds without a scale-critical condition because the control (6.19) of the L^3 norm is only on time slices. This is in stark contrast with Theorem 6.16, Tao's [334] quantitative regularity see (6.12) above and the quantitative regularity results of [276, 277, 133].

6.2.3 Slight breaking of criticality in the borderline case: a conjecture of Tao

We know discuss the possibility of transferring the slight supercriticality in time of Tao's result (6.13) to slight supercriticality in space.

Theorem 6.5 (blow-up of slightly supercritical Orlicz norms; [37, Theorem 2])**.** *There exists a universal constant $\theta \in (0,1)$ such that the following holds.*
Let U be a weak Leray-Hopf solution to the Navier-Stokes equations on $\mathbb{R}^3 \times (0,\infty)$ with initial data $U_0 \in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$. Assume that U first blows-up at $T^* > 0$, namely

$$U \in L^\infty_{loc}(0,T^*;L^\infty(\mathbb{R}^3)) \ \textit{and} \ U \notin L^\infty(\tfrac{1}{2}T^*,T^*;L^\infty(\mathbb{R}^3)).$$

Then the above assumptions imply that

$$\limsup_{t\uparrow T^*}\int\limits_{\mathbb{R}^3}\frac{|U(x,t)|^3}{\Big(\log\log\log\big((\log(e^{e^{3e^e}}+|U(x,t)|))^{\frac{1}{3}}\big)\Big)^{\theta}}dx=\infty. \tag{6.23}$$

Novelty of our result

Recently, in [334, Remark 1.6], Tao conjectured that if a solution first loses smoothness at time $T^*>0$, then the Orlicz norm $\|U(\cdot,t)\|_{L^3(\log\log\log L)^{-c}(\mathbb{R}^3)}$ must blow-up as t tends to T^*. Theorem 6.5 provides a positive answer to Tao's conjecture, albeit with an extra logarithm in the denominator.

As far as we know, Theorem 6.5 is the first result of this type concerned with slight criticality breaking in borderline spaces. Previously, it was shown in [87] that if U is a weak Leray-Hopf solution satisfying

$$\int\limits_0^\infty\int\limits_{\mathbb{R}^3}\frac{|U|^5}{\log(1+|U|)}dxdt<\infty$$

then U is smooth on $\mathbb{R}^3\times(0,\infty)$. Subsequent improvements were obtained in [234] and [52]; see also [265]. Let us mention that the techniques used in these papers cannot be used to treat the borderline case considered in Theorem 6.5. We also mention the paper of Chan and Yoneda [88] that mixes the boundedness of a strongly supercritical norm $L^\infty_t L^{\alpha,\infty}_x$ for $\alpha\in(2.343,3)$ with a geometrical information involving $\nabla\cdot(U/|U|)$.

For other partial differential equations, it is often the case that a refined understanding of critical regimes can be used to prove 'slightly supercritical' results. Such slightly supercritical results occur for the nonlinear wave equation [331, 104], the hyperdissipative Navier-Stokes equations [332, 31, 102, 103], the supercritical SQG equation [113, 109] and the fractional Burgers equation [112] to name a few. In these works the slight supercriticality is obtained by varying the power of the nonlinearity or the strength of the fractional dissipation.

Our method is indeed inspired by the recent result of Bulut [73] for the nonlinear supercritical Schrödinger equation. In particular our proof of Theorem 6.5 relies on the proof of another statement, see Lemma 6.7 below, that we name 'mild criticality breaking', which is the counterpart for Navier-Stokes of Bulut's result for Schrödinger.

6.3 New ideas and strategy for the proofs

The goal of our paper [38] is to develop a new robust strategy for obtaining new quantitative estimates of the Navier-Stokes equations. The main novelty is that our strategy allows us to obtain local quantitative estimates, Theorem 6.1, and applies to certain situations where we are outside the regime of scale-invariant controls, Theorem 6.3. Before showing how our strategy can be applied to prove our two main results, let us outline the main idea on a toy model.

6.3.1 A new strategy for quantitative estimates: a toy model

In this section, we outline the strategy that is used for quantitative regularity in the borderline case, see Subsection 6.3.2, in the simpler non borderline critical space $L^5_{t,x}$.

Schematically, there are two main parts in the reasoning:

(1) Assume certain critical a priori bounds, such as

$$\|U\|_{L^5(\mathbb{R}^3\times(-1,0))} < \infty \quad \text{or} \quad \|U\|_{L^\infty(-1,0;L^3(\mathbb{R}^3))} < \infty$$

as in Tao's result (see Subsection 6.1.4), or the Type I assumption

$$\|U\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))} \le M < \infty \quad \text{and} \quad \|U(\cdot,0)\|_{B_0(R(M))} < \infty$$

where $R(M) \gg 1$ as in Subsection 6.3.2 below. Then, certain scale-invariant quantities (Kato-type norms, Fourier-based quantities, scale-invariant enstrophies) cannot concentrate for times $0 > t \ge t_*$ or for frequencies $N \ge N_*$, where t_* (resp. N_*) is a given quantitative time (resp. frequency). That time t_* or frequency N_* can be interpreted as Kolmogorov dissipative scales. Tao's point and ours is to quantify them explicitly in terms of M. Such results are achieved using various tools (bilinear estimates, local smoothing results, Carleman inequalities). The main insight is to propagate the concentration from a given time slice t' to backward time slices t'' and to have enough such slices that one can then sum to gain coercivity.

(2) A regularity result: non concentration implies smallness implies quantitative regularity via ε-regularity type results.

Let us now sketch the proof in the toy model case, see Figure 6.3. Assume that

$$\|U\|_{L^5(\mathbb{R}^3\times(-1,0))} < \infty.$$

For the first part of the reasoning, we argue in the following two-step way:

(Step-1) **Backward propagation of Kato-norm concentration**
Using bilinear estimates for the heat semigroup [78] we get

$$\|U(\cdot,t')\|_{L^5(\mathbb{R}^3)} > \frac{\varepsilon}{(-t')^{\frac15}} \Rightarrow \|U(\cdot,t'')\|_{L^5(\mathbb{R}^3)} > \frac{\varepsilon/2}{(-t'')^{\frac15}}, \quad \text{for all} \quad t'' \in (-1,2t').$$

(Step-2) **Summation of scales**
Summing the concentration for $t'' \in (-1,2t')$,

$$\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))} \ge \int_{-1}^{2t'} \|U(\cdot,t'')\|^5_{L^5(\mathbb{R}^3)}\, dt'' \ge -\frac{\varepsilon^5}{32}\log(-2t').$$

Hence,

$$\frac12 > -t' > \frac12 \exp\left(-\frac{32\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))}}{\varepsilon^5}\right) =: -t_*.$$

For the second part of the reasoning we use ε-regularity on $Q_{(\bar{x},0)}(\sqrt{-t_*/2}) = B_{\bar{x}}(\sqrt{-t_*/2})\times(-\frac{t_*}{2},0)$ for all $\bar{x}\in\mathbb{R}^3$ to obtain

$$\|U\|_{L^\infty(\mathbb{R}^3\times(-\frac{t_*}{2},0))} \lesssim \frac{\varepsilon^{\frac13}}{\sqrt{-t_*}} \lesssim \varepsilon^{\frac13}\exp\left(\frac{16\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))}}{\varepsilon^5}\right). \tag{6.24}$$

Notice that in the quantitative estimate (6.24) the subcritical $L^\infty_{t,x}$ norm is estimated in terms of a single exponential of the $L^5_{t,x}$ norm; compare to (6.12) which involves a triple exponential.

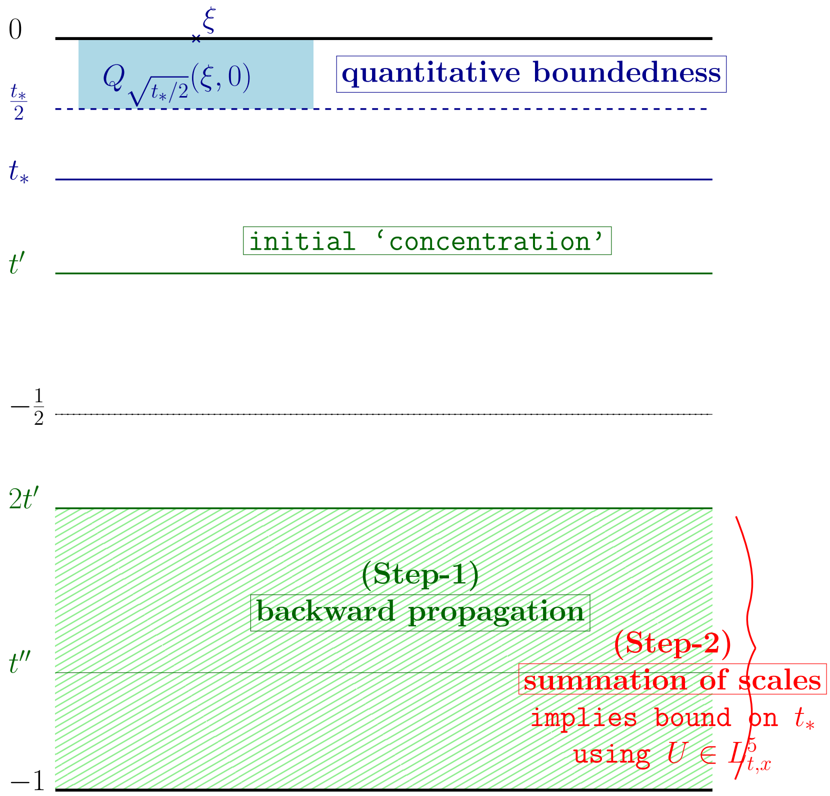

Figure 6.3 – Quantitative regularity via concentration of scale-critical quantities: a toy model

6.3.2 A new strategy for quantitative estimates under a Type I bound

We outline here the strategy for the case when one has a scale-invariant control in the form

$$\|U\|_{L^\infty_t L^{3,\infty}_x(\mathbb{R}^3\times(-1,0))} \leq M. \tag{6.25}$$

Local-in-space local-in-time smoothing in our approach Fundamental to our strategy is the use of local-in-space smoothing near the initial time for the Navier-Stokes equations pioneered by Jia and Šverák in [197] (see Subsection 5.1.3). In particular, the result of [197], together with rescaling arguments from [35], implies the following. Assume U is a smooth solution with sufficient decay of the Navier-Stokes equations on $\mathbb{R}^3 \times [-1,0]$ and satisfies the Type I bound (6.25). If the scale-invariant enstrophy is small for a certain time $t' \in (-1,0)$ in the following way

$$(-t')^{\frac{1}{2}} \int\limits_{B_0(4\sqrt{S(M)}^{-1}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 dx \leq M^2\sqrt{S(M)}, \tag{6.26}$$

then

$$\|U\|_{L^\infty_{x,t}\left(B_0\left(\frac{1}{2}\sqrt{S(M)}^{-1}(-t')^{\frac{1}{2}}\right)\times\left(\frac{3}{4}(-t')+t',0\right)\right)} \tag{6.27}$$

can be estimated explicitly in terms of M and $-t'$ via an ε-regularity result (local-in-space smoothing for subcritical data). Here, $S(M) = CM^{-100}$. We refer to Figure 6.4 where $t' = \frac{t_*}{2}$.

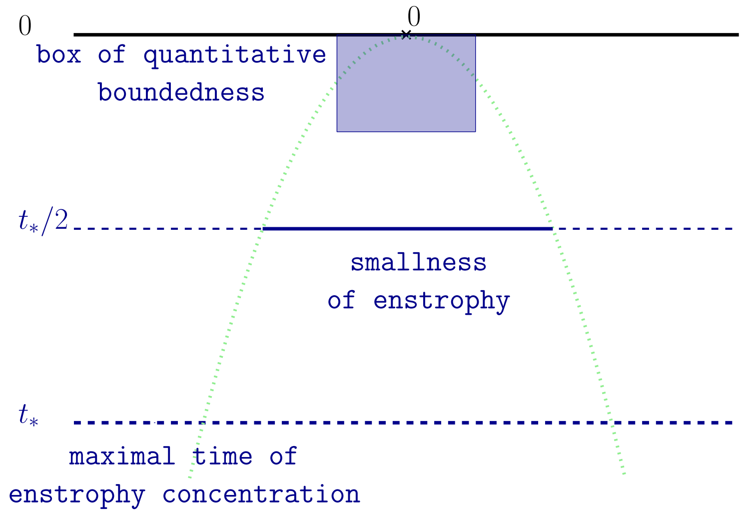

Figure 6.4 – Quantitative regularity via concentration of a scale-invariant enstrophy: maximal time of enstrophy concentration and quantitative boundedness

Our main goal In this perspective, the aim of our strategy is the following

> **Our goal:** If (6.26) fails for t', what is a lower bound $-t_*$ for $-t'$ in terms of M and $\|U(\cdot,0)\|_{L^3(B_0(R(M)))}$ for a certain $R(M) \gg 1$?

This upper bound $-t_*$ for the enstrophy concentration can be interpreted as the time at which dissipation effects take over the nonlinearity. Notice that for that to happen, we need that the solution belongs to L^3 at final time, because a mere Type I assumption is at this point not enough to beat the scaling. Our strategy for obtaining a lower bound of $-t_*$, stated in [38, Proposition 2.1], can be summarized in three steps.

Three steps strategy to show our main goal

(Step-1) **Backward propagation of vorticity concentration**
For this step and the next one, we refer to Figure 6.6. Suppose $t' \in (-1,0)$ is not too close to -1 and is such that

$$\int_{B_0(4\sqrt{S(M)}^{-1}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 dx > \frac{M^2\sqrt{S(M)}}{(-t')^{\frac{1}{2}}}. \tag{6.28}$$

We show that for all $t'' \in (-1,t')$ such that $-t''$ is well-separated from $-t'$, we have

$$\int_{B_0(4\sqrt{S(M)}^{-1}(-t'')^{\frac{1}{2}})} |\omega(x,t'')|^2 dx > \frac{M^2\sqrt{S(M)}}{(-t'')^{\frac{1}{2}}}. \tag{6.29}$$

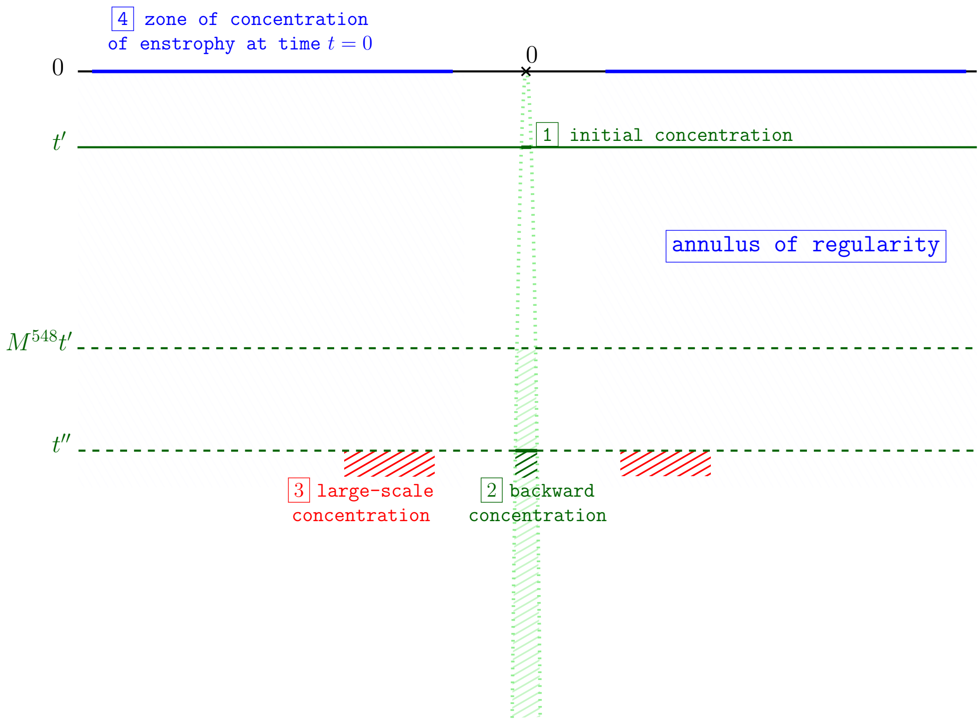

Figure 6.5 – Quantitative regularity via concentration of a scale-invariant enstrophy: backward, large-scale and forward propagation of enstrophy concentration

(Step-2) **Lower bound on localized L^3 norm at the final moment in time**
Using the previous step, together with the same arguments as in Tao's paper [334] involving quantitative Carleman inequalities, we show that for certain permissible annuli that

$$\int_{R\leq|x|\leq R'} |U(x,0)|^3 dx \geq \exp(-\exp(M^C)). \tag{6.30}$$

The role of the Type I bound is to show that the solution U obeys good quantitative estimates in certain space-time regions, 'epochs of regularity' and 'annuli of regularity', which is needed to apply the Carleman inequalities to the vorticity equation.

(Step-3) **Conclusion: summing scales to bound $-t'$ from below**
We refer to Figure 6.6 for this step. Summing (6.30) over all permissible disjoint annuli finally gives us the desired lower bound for $-t'$. We note that the localized L^3 norm of U at time 0 plays a distinct role to that of the Type I condition described in the previous step. Its sole purpose is to bound the number of permissible disjoint annuli that can be summed. This turns into a lower bound of $-t'$, which reads:

$$-t' \geq CM^{-749} \exp\left(-4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(M^{1023})} |U(x,0)|^3 \, dx \right). \tag{6.31}$$

The single exponential of $\int_{B_0(M^{1023})} |U(x,0)|^3 \, dx$ in the lower bound (6.31) is why the lower bound in Theorem 6.1 on the localized L^3 norm near a Type I singularity is a single

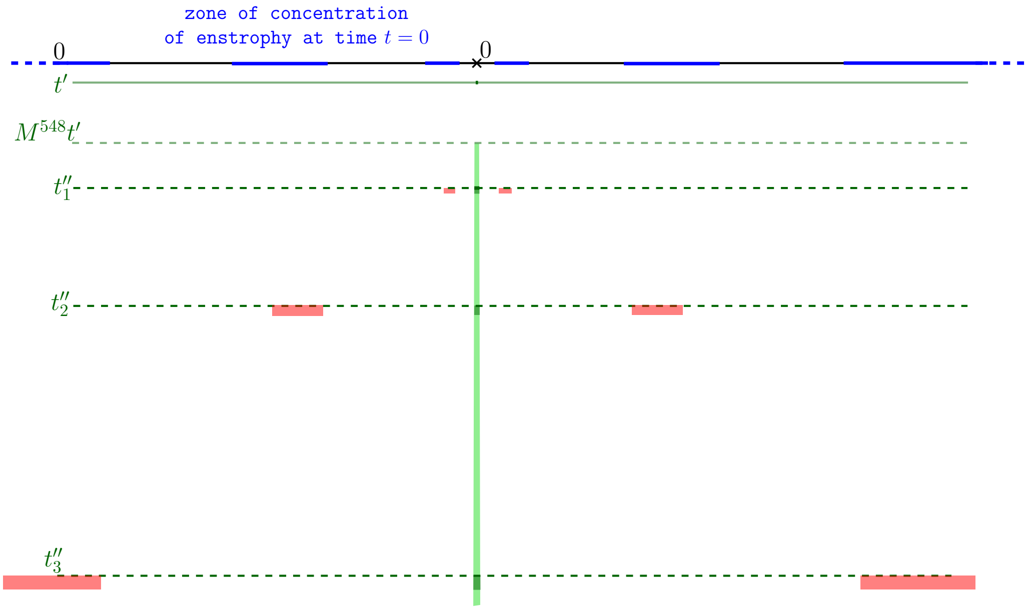

Figure 6.6 – Quantitative regularity via concentration of a scale-invariant enstrophy: summation of scales

logarithm. The fact that (6.31) just depends on U at the final time explains why (6.16) holds at pointwise times.

Remark 6.6 (on the exponentials). We emphasize that under the Type I assumption (6.25), the single exponential dependence in $\int_{B_0(M^{1023})} |U(x,0)|^3\,dx$ in the estimate (6.31) is due to the summation of scales in the third step 'Conclusion: summing scales to bound $-t'$ from below' above. This is hard to improve because the fact that the concentration zones at final time need to be non-overlapping, see Figure 6.6, causes that we can only sum a geometric sequence of times t''. The double exponential dependence in the Type I bound M in (6.31) comes from the fact that we use linear tools, namely quantitative Carleman inequalities for backward uniqueness and unique continuation, and pigeonhole to locate good scales, such as for instance to find a good annulus of quantitative regularity. This double exponential in M may be improved in principle but this does not seem to be possible with the techniques of our paper.

6.3.3 A comparison to Tao's strategy

The high-level strategies of Tao and ours are almost parallel. The differences between the two approaches lie in the quantities (concentration in Fourier space vs. local concentration in physical space) that are considered and in the tools that we use.

Global-in-space local-in-time smoothing in Tao's approach Fundamental to Tao's approach for showing (6.12) is the following fact, see [334, Section 6]. If U is a classical solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, 0]$ with

$$\|U\|_{L^\infty_t L^3_x(\mathbb{R}^3\times(-1,0))} \le A \tag{6.32}$$

$$\text{and} \quad N^{-1}\|P_N U\|_{L^\infty_{x,t}(\mathbb{R}^3\times(-\frac{1}{2},0))} < \varepsilon(A) \ \text{ for all } \ N \ge N_*, \tag{6.33}$$

for a certain $0 < \varepsilon(A) \ll 1$, then $\|U\|_{L^\infty_{x,t}(\mathbb{R}^3\times(-\frac{1}{8},0))}$ can be estimated explicitly in terms of A and N_*. Here P_N is the Littlewood-Paley projection on the frequency $N > 0$. Related observations were made previously by Cheskidov and Shvydkoy [98, 99] and Cheskidov and Dai [97], but without the bounds explicitly stated. There the frequency N_* is called the Kolmogorov scale and denoted Λ. Heuristically, see [333], if (6.33) holds, that is if $N^{-1}\|P_N U\|_{L^\infty_{x,t}} \ll 1$ is small, then $N\|P_N U\|^2_{L^\infty_{t,x}} \ll N^2\|P_N U\|_{L^\infty_{t,x}}$ so that the diffusion dominates the nonlinearity.

Notice that assumption (6.32) replaces the Type I assumption (6.25) and the smallness of the scale-invariant quantity (6.33) at high frequencies replaces the smallness of the scale-invariant enstrophy (6.26).

Tao's main goal In this perspective, Tao's aim is the following:

> **Tao's goal**: Under the scale-invariant assumption (6.32), if (6.33) fails for $\varepsilon(A) = A^{-C}$ and $N = N_0$, what is an upper bound N_* for N_0?

In Tao's paper [334, Theorem 5.1], it is shown that $N_0 \lesssim \exp\exp\exp(A^C)$, which implies (6.12) by means of the quantitative regularity mechanism (6.33). We emphasize that since the regularity mechanism (6.33) is global, all quantitative estimates obtained in this way are in terms of globally defined quantities.

One might think of the main goal of our strategy as a physical space analogy to Tao's goal with

$$N_0 \simeq (-t')^{-\frac{1}{2}}.$$

In contrast to (6.33), the regularity mechanism (6.26)-(6.27) produces quantitative estimates that are in terms of locally defined quantities, which is crucial for obtaining the localized results as in Theorem 6.1.

Four steps strategy to show Tao's main goal The strategy in [334] for showing Tao's goal with $N_0 \lesssim \exp(\exp(\exp(A^C)))$ can be summarized in four steps. We refer the reader to the Introduction in [334] for more details.

(Step-1) **Frequency bubbles of concentration [334, Proposition 3.2]**
Suppose $\|U\|_{L^\infty_t L^3_x(\mathbb{R}^3\times(-1,0))} \le A$ is such that

$$N_0^{-1}|P_{N_0}u(x_0,0)| > A^{-C}. \tag{6.34}$$

Then for all $n \in \mathbb{N}$ there exists $N_n > 0$, $(x_n, t_n) \in \mathbb{R}^3 \times (-1, t_{n-1})$ such that

$$N_n^{-1}|P_{N_n}u(x_n,t_n)| > A^{-C} \tag{6.35}$$

with

$$x_n = x_0 + O((-t_n)^{\frac{1}{2}}), \quad N_n \simeq |-t_n|^{-\frac{1}{2}}. \tag{6.36}$$

This is the analogue of (Step-1) above in our proof in collaboration with Barker.

(Step-2) **Localized lower bounds on vorticity [334, p.37]**
For certain scales $S > 0$ and an 'epoch of regularity' $I_S \subseteq [-S, -A^{-\alpha}S]$, where the solution enjoys 'good' quantitative estimates on $\mathbb{R}^3 \times I_S$ (in terms of A and S), Tao shows the following. The previous step and $\|U\|_{L^\infty_t L^3_x(\mathbb{R}^3\times[-1,0])} \leq A$ imply

$$\int\limits_{B_{x_0}(A^\beta S^{\frac{1}{2}})} |\omega(x,t)|^2\,dx \geq A^{-\gamma}S^{-\frac{1}{2}} \quad \text{for all} \quad t \in I_S. \tag{6.37}$$

Here, α, β and γ are positive universal constants.
This is an additional step where frequency information is transferred to the scale-invariant enstrophy. This is not needed in our proof because we work directly with the scale-invariant enstrophy.

(Step-3) **Lower bound on the L^3 norm at the final moment in time t_0 [334, p.37-40]**
Using quantitative versions of the Carleman inequalities in [129], see [334, Proposition 4.2 and Proposition 4.3], Tao shows that the lower bounds in (Step-2) can be transferred to a lower bound on the L^3 norm of U at the final moment of time 0. The applicability of the Carleman inequalities to the vorticity equation requires the 'epochs of regularity' in the previous step and the existence of 'good spatial annuli' where the solution enjoys good quantitative estimates. Specifically, Tao shows that (Step-2) on I_S implies

$$\int\limits_{R_S\leq|x-x_0|\leq R'_S} |U(x,0)|^3 dx \geq \exp(-\exp(A^C)). \tag{6.38}$$

This is the analogue of (Step-2) above in our proof.

(Step-4) **Conclusion: summing scales to bound TN_0^2**
Letting S vary for certain permissible S, the annuli in (6.38) become disjoint. The sum of (6.38) over such disjoint annuli is bounded from above by $\|U(\cdot,0)\|_{L^3(\mathbb{R}^3)}$ and the lower bound due to the summing of scales is $\exp(-\exp(A^C))\log(TN_0^2)$. This gives the desired bound on N_0, namely

$$TN_0^2 \lesssim \exp(\exp(\exp(A^C))).$$

This is the analogue of (Step-3) above in our proof.

Let us emphasize once more that the approach in [334] produces quantitative estimates involving globally defined quantities, since the quantitative regularity mechanism (6.33) is inherently global. We also emphasize that the fact that $\|U\|_{L^\infty_t L^3_x} \leq A$ is crucial for showing (Step-1) and (Step-2) in the above strategy.

6.3.4 Robustness of our new strategy: quantification without a Type I bound

Our strategy above, see Subsection 6.3.2 is robust enough (with certain adjustments) to apply to certain settings without a Type I control as for the quantification of Seregin's result [294] in Theorem 6.3.

In the strategy of Tao [334] the lower bound on vorticity (6.37), which is needed for getting a lower bound on the localized L^3 norm at final time 0 via quantitative Carleman inequalities, is obtained from the frequency bubbles of concentration. In order for this transfer of scale-invariant information to take place, it appears essential that the solution has a scale-invariant control such as $\|U\|_{L^\infty_t L^3_x} \leq A$, see [334, Proposition 3.1]. In our strategy, we instead work directly with quantities similar to (6.37) involving vorticity, which are tailored for the immediate use of quantitative Carleman inequalities. In this way, we crucially avoid any need to transfer scale-invariant information, giving our strategy a certain degree of robustness.

Recall that Theorem 6.3 is concerned with quantitative estimates, where we assume

$$t_k \uparrow 0 \ \text{ with } \ \sup_k \|U(\cdot, t_k)\|_{L^3(\mathbb{R}^3)} \leq M. \tag{6.39}$$

First we remark that the local quantitative regularity statement (6.26)-(6.27) remains true (with t' replaced by t_k) if U is a $C^\infty(\mathbb{R}^3 \times [-1,0])$ finite-energy solution and the Type I condition is replaced by the weaker assumption that $\|U(\cdot, t_k)\|_{L^3(\mathbb{R}^3)} \leq M$. Our goal then becomes the following

Our second goal: If (6.26) fails for $t' = t_j$, what is an upper bound for j?

A key point is that we are able to backward propagate concentration as in (Step-1) in Subsection 6.3.2 despite the fact that the L^3 norm is bounded only at a countable number of times. As above, this requires a separation between the time t' where initial concentration happens and the times s of backward propagation of concentration, see (Step-1 time slices). This is why in Theorem 6.3 we need to take a sufficiently well-separated subsequence of t_k, see (6.20).

(Step-1 time slices) **Backward propagation of vorticity concentration in the time slices case**
Fix any $\alpha \geq M^\flat$ and let $t', t'' \in [-1, 0)$ be such that

$$\frac{t''}{\alpha^{1051}} < t' < 0.$$

Assume that

$$\|U(\cdot, t')\|_{L^3} \leq M \quad \text{and} \quad \|U(\cdot, t'')\|_{L^3} \leq M.$$

If the vorticity concentrates at time t' in the following way

$$\int\limits_{B_0(4\sqrt{S(M)}^{-1}(-t')^{\frac{1}{2}})} |\omega(x, t')|^2 \, dx > M^2 (-t')^{-\frac{1}{2}} \sqrt{S(M)},$$

then for any $s \in [t'', \frac{t''}{8\alpha^{201}}]$ the vorticity concentrates in the following sense

$$\int\limits_{B_0(4(-s)^{\frac{1}{2}}\alpha^{106})} |\omega(x, s)|^2 \, dx > \frac{(M+1)^2}{(-s)^{\frac{1}{2}}\alpha^{106}}.$$

To show this we use energy estimates in [304] for solutions to the Navier-Stokes equations with $L^3(\mathbb{R}^3)$ initial data. We decompose

$$U(\cdot, s) = e^{(s-t_0)\Delta} U(\cdot, t'') + V(\cdot, s) \qquad \text{for any} \quad 0 \geq s \geq t''$$

and get

$$\|V(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_{t''}^{s} \int\limits_{\mathbb{R}^3} |\nabla V(x, t)|^2 dx dt \leq C(M^\flat)^4 (s - t'')^{\frac{1}{2}}.$$

The price one pays in this setting when compared to the estimates in Tao's paper [334], is a gain of an additional exponential in the estimates. The reason is the control on the energy of $U(\cdot,s)-e^{(s-t_0)\Delta}U(\cdot,t'')$ with $U(\cdot,t'')\in L^3(\mathbb{R}^3)$ requires the use of Gronwall's lemma. Such estimates are also central to gain good quantitative control of the solution in certain space-time regions, which are required for applying the quantitative Carleman inequalities.

6.3.5 Mild criticality breaking

Our method for proving Theorem 6.5 relies on the following lemma and on a careful tuning of the parameters (estimating the $L^{3-\mu}$ norm for a well-chosen parameter μ). Lemma 6.7 is directly inspired by the recent result of Bulut [73] for a nonlinear supercritical defocusing Schrödinger equation.

Lemma 6.7. *For all $M\in[1,\infty)$ and $E\in[1,\infty)$ sufficiently large, there exists $\delta(M,E)\in(0,\frac{1}{2}]$ such that the following holds. Let U be a suitable weak Leray-Hopf solution to the Navier-Stokes equations on $\mathbb{R}^3\times(0,\infty)$ with initial data $U_0\in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$. Assume that*

$$\|U_0\|_{L^2},\ \|U_0\|_{L^4}\leq M,$$

and that

$$\|U\|_{L^\infty(0,\infty;L^{3-\delta(M,E)}(\mathbb{R}^3))}\leq E. \tag{6.40}$$

Then, the above assumptions imply that U is smooth on $\mathbb{R}^3\times(0,\infty)$. Moreover, there is an explicit formula for $\delta(M,E)$, see [37, equation (26)], and $\delta(M,E)\to 0$ when $E\to\infty$ or $M\to\infty$.

We call the result of Lemma 6.7 a 'mild breaking of the criticality', or a 'mild supercritical regularity criteria' as opposed to strong criticality breaking results obtained for instance in the axisymmetric case [279, 298, 299, 95]. Indeed, the supercritical space $L^\infty_t L^{3-\delta(M,E)}$ in which we break the scaling depends on the size E of the solution in this supercritical space via $\delta(M,E)$. In other words this can be considered as a non effective regularity criteria, hence the term 'mild'. Moreover, given a solution U, assume that you knew all the $L^\infty_t L^{3-\delta}_x$ norms for $\delta\to 0$. Then the question whether Lemma 6.7 applies to U or not becomes a question about how fast

$$\|U\|_{L^\infty(0,\infty;L^{3-\delta}(\mathbb{R}^3))}$$

grows when $\delta\to 0$. We also emphasize that the larger the M, the closer the exponent $3-\delta(M,E)$ is from the critical exponent 3. Of course we would have regularity if the solution was a priori bounded in the critical space $L^\infty_t L^3_x$. Our result shows that with L^4 initial data we can relax the exponent 3 to a slightly supercritical $3-\delta(M,E)$. Let us also remark that the condition $U_0\in L^4(\mathbb{R}^3)$ can be replaced by any subcritical condition $U_0\in L^{3+}(\mathbb{R}^3)$.

The main idea of the proof of Lemma 6.7 is to transfer subcritical information from the initial time forward in time. In that perspective our main goal is

> **Our main goal:** Prove that there exists $\delta(M,E)\in(0,\frac{1}{2}]$ and $K(M,E)\in[1,\infty)$ such that for all $U_0\in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$ and any suitable weak Leray-Hopf solution associated to the initial data U_0, if
>
> $$\|U_0\|_{L^2},\ \|U_0\|_{L^4}\leq M,$$

and

$$\|U\|_{L^\infty(0,\infty;L^{3-\delta(M,E)}(\mathbb{R}^3))} \leq E,$$

then

$$\|U\|_{L^\infty(0,T;L^4(\mathbb{R}^3))} \leq K(M,E). \tag{6.41}$$

This then obviously implies the result stated in Lemma 6.7. The crucial point is that $K(M,E)$ is uniform in time.

The only a priori globally controlled quantity is a supercritical $L^\infty_t L^{3-}$ norm. We are not aware of any regularity mechanism enabling to brake the critically barrier based on the sole knowledge of such a supercritical bound. Therefore, the idea, following Bulut [73] is to transfer the subcritical information coming from the initial data $U_0 \in L^4(\mathbb{R}^3)$ to arbitrarily large times by using three ingredients:

(1) the control of the critical $L^\infty_t L^3_x$ norm via interpolation between the supercritical norm $L^\infty_t L^{3-\delta(M,E)}_x$ and the subcritical $L^\infty_t L^4_x$ norm

$$\begin{aligned}\|U\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} &\leq \|U\|^{\frac{3-\delta}{3+3\delta}}_{L^\infty(0,T;L^{3-\delta}(\mathbb{R}^3))} \|U\|^{\frac{4\delta}{3+3\delta}}_{L^\infty(0,T;L^4(\mathbb{R}^3))}\\ &\leq E^{\frac{3-\delta}{3+3\delta}} K^{\frac{4\delta}{3+3\delta}} ;\end{aligned}$$

(2) the quantitative control of the critical non borderline $L^5_{t,x}$ norm (see [37, Proposition 3]) in terms of the critical norm $\|U\|_{L^\infty(0,\infty;L^3(\mathbb{R}^3))}$, and the supercritical L^2 and subcritical L^4 norms of the initial data U_0

$$\|U\|_{L^5(0,T;L^5(\mathbb{R}^3))} \leq C(M) \exp\exp\exp\left(C_{univ}\left(E^{\frac{3-\delta}{3+3\delta}} K^{\frac{4\delta}{3+3\delta}}\right)^c\right);$$

this hinges on the quantitative bounds on solutions belonging to the critical space $L^\infty_t L^3_x$, which were established by Tao in [334], see Subsection 6.1.4 above; this step enables the slicing of the interval $(0,T)$ into a T-independent number m of disjoint epochs $I_j = (t_j, t_{j+1})$,

$$\begin{aligned}\varepsilon^5 m = \sum_{j=1}^m \|U\|^5_{L^5(I_j;L^5(\mathbb{R}^3))} &\leq \|U\|^5_{L^5(0,T;L^5(\mathbb{R}^3))}\\ &\leq C(M) \exp\exp\exp\left(\left(E^{\frac{3-\delta}{3+3\delta}} K^{\frac{4\delta}{3+3\delta}}\right)^c\right);\end{aligned}$$

(3) an L^4 energy estimate [37, Proposition 4] under the $L^5_{t,x}$ control of U, which enables the transfer the subcritical information from time t_j to t_{j+1}

$$\begin{aligned}\mathcal{E}_{4,t_{j+1}} &\leq \|U(\cdot,t_j)\|^4_{L^4(\mathbb{R}^3)} + C\|U\|_{L^5(\mathbb{R}^3\times I_j)}\mathcal{E}_{4,t_{j+1}}\\ &\leq \|U(\cdot,t_{j+1})\|^4_{L^4(\mathbb{R}^3)} + C\varepsilon\mathcal{E}_{4,t_{j+1}},\end{aligned}$$

where $\mathcal{E}_{4,t_{j+1}}$ is the L^4 energy, see [37, equation (13)] and eventually to T

$$\|U\|^4_{L^\infty(0,T;L^4(\mathbb{R}^3))} = \max_{1\leq j\leq m+1}\{\|U\|^4_{L^\infty(I_j;L^4(\mathbb{R}^3))}\} \leq 64M^4 2^m.$$

One then designs the number $K(M,E)$ to bound the right hand side above.

Chapter 7

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