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présentée par

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Analyse asymptotique et couches limites : quelques problèmes en homogénéisation et en mécanique des fluides

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Rien, comme la vue de l'eau, ne donne la vision des nombres.
Victor Hugo, *Les travailleurs de la mer*

Résumé

Cette thèse est consacrée à l'analyse asymptotique de quelques équations aux dérivées partielles. Les modèles étudiés sont issus de la science des matériaux composites, de la mécanique des fluides géophysiques et de la mécanique des fluides viscoélastiques. Leur point commun est de faire intervenir un petit paramètre qui peut traduire des oscillations à l'échelle microscopique (hétérogénéités des matériaux composites, rugosités en mécanique des fluides), ou provenir d'un adimensionnement (nombre de Rossby, de Weissenberg). La problématique générale est de dériver des modèles limites, de montrer des résultats de convergence et (si possible) des estimations d'erreur entre une solution et son approximation. La présence d'un bord peut être la cause de singularités, ce qui conduit à étudier finement le comportement des solutions près du bord, dans une petite couche, la couche limite. Les deux axes de la thèse sont donc l'étude de problèmes de couches limites et la justification d'asymptotiques.

Pour l'étude des problèmes de couches limites (caractère bien posé et asymptotique loin du bord) le leitmotiv de nos travaux est de s'affranchir au maximum d'hypothèses de structure (périodicité, quasipériodicité) sur les données du problème. Nous étudions la convergence loin du bord d'un correcteur de couche limite provenant de l'homogénéisation d'un système elliptique à coefficients et donnée de Dirichlet oscillants (Chapitre 4). Nous démontrons le caractère bien posé du système de Stokes-Coriolis dans un demi-espace rugueux, pour un profil de rugosité arbitraire (Chapitre 5). Ce système est une version linéarisée du système des couches limites d'Ekman pour les fluides en rotation rapide.

Le second axe comprend la démonstration d'estimations d'erreur en homogénéisation périodique et l'étude de la limite newtonienne de fluides viscoélastiques. Nous étudions l'homogénéisation de systèmes elliptiques de type couche limite en domaines polygonaux convexes, et utilisons nos résultats pour établir un développement à l'ordre 1 des valeurs propres d'un système elliptique à coefficients oscillants (Chapitre 3). On s'intéresse enfin à des modèles de fluides viscoélastiques dans la limite de faible nombre de Weissenberg. Nous obtenons des résultats de convergence faible et forte vers le système de Navier-Stokes (Chapitre 6).

Liste des travaux rassemblés dans la thèse

- Chapitre 3 : article [Pra11] à paraître dans *Asymptotic Analysis*.
- Chapitre 4 : article [Pra13] publié dans *SIAM Journal on Mathematical Analysis*.
- Chapitre 5 : article [DP13] en collaboration avec Anne-Laure Dalibard, soumis.
- Chapitre 6 : en collaboration avec Didier Bresch.

Mots-clefs

Analyse multi-échelles. Homogénéisation. Couches limites. Queue de couche limite. Systèmes elliptiques. Valeurs propres. Estimations d'erreur. Fluides tournants. Couche limite d'Ekman. Rugosités. Estimations de Saint-Venant. Noyaux de Green et de Poisson. Théorie du potentiel. Operateur de Dirichlet to Neumann. Fluides viscoélastiques. Fluides non-newtoniens. Nombre de Weissenberg. Limite newtonienne. Entropie Relative.

Asymptotic Analysis and Boundary Layers:
A Few Problems in Homogenization and Fluid Mechanics

Abstract

This thesis is devoted to the asymptotical analysis of several partial differential equations. We study models from the science of composite materials, and from geophysical and viscoelastic fluid mechanics. They all involve a small parameter, which may either represent oscillations at the microscopic scale (heterogeneities in composite materials, rugosities in fluid mechanics), or a dimensionless parameter (Rossby or Weissenberg numbers). Generally speaking, our work is concerned with the derivation of limit models, with the proof of convergence results, and (if possible) of error estimates between a solution and its approximation. The presence of a boundary may cause singularities, which lead to a refined study of the behaviour of the solutions near the boundary, in a small layer, the boundary layer. Hence, the works gathered here focus on the one hand on boundary layer problems, and on the other hand on the mathematical justification of some asymptotics.

The leitmotiv of our studies of boundary layer problems (well-posedness and asymptotics far away from the boundary) is to free ourselves from structure assumptions (periodicity, quasiperiodicity) on the data of the problem. We study the convergence far away from the boundary of a boundary layer corrector coming from the homogenization of an elliptic system with oscillating coefficients and boundary data (Chapter 4). We show the well-posedness of the Stokes-Coriolis system in a rough half-space, for an arbitrary rugosity profile (Chapter 5). This system is a linearized version of the Ekman boundary layer system for highly rotating fluids.

The scope of our work on convergences is to prove error estimates in periodic homogenization and to investigate the newtonian limit of viscoelastic fluids. We study the homogenization of boundary layer type elliptic systems in convex polygonal domains, and use our results to show a first-order asymptotic expansion of the eigenvalues of an elliptic system with oscillating coefficients (Chapter 3). We finally analyse models of viscoelastic fluid flows in the low Weissenberg limit. We get weak and strong convergence results toward the Navier-Stokes system (Chapter 6).

List of Preprints and Publications in the Thesis

- Chapter 3 : paper [Pra11] to appear in *Asymptotic Analysis*.
- Chapter 4 : paper [Pra13] published in *SIAM Journal on Mathematical Analysis*.
- Chapter 5 : paper [DP13], joint work with Anne-Laure Dalibard, submitted.
- Chapter 6 : joint work with Didier Bresch.

Keywords

Multiscale analysis. Homogenization. Boundary layers. Boundary layer tail. Elliptic systems. Eigenvalues. Error estimates. Rotating fluids. Ekman boundary layer. Rugosity. Saint-Venant estimates. Green and Poisson kernels. Potential theory. Dirichlet to Neuman operator. Viscoelastic fluids. Non-newtonian fluid mechanics. Weissenberg number. Newtonian limit. Relative entropy.

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Chapter 1

Outline of results

Notice that this preliminary chapter is the translation (in English) of some parts of the Introduction, Chapter 2 (in French).

The works gathered in this thesis have in common to be devoted to the asymptotic analysis of partial differential equations. We study models from material science and fluid mechanics. To put it in a nutshell, the systems we analyse involve a parameter $\varepsilon \ll 1$, an equation

$$\mathcal{L}^\varepsilon u^\varepsilon = 0,$$

posed in a domain $\Omega^\varepsilon \subset \mathbb{R}^d$ (or $(0,T) \times \Omega^\varepsilon$), and come with boundary (and initial) conditions for u^ε. The parameter ε may have a physical meaning (heterogeneities in composite materials, rugosities in fluid mechanics, dimensionless parameters like the Rossby or the Weissenberg numbers), or a purely mathematical one (use of approximated solutions to prove existence results).

The asymptotical analysis when $\varepsilon \to 0$ is motivated by physical, numerical and mathematical aspects. From a physical point of view, one has to derive limit models, valid in several régimes. From the point of view of the numerical analysis, one has to build approximated solutions, which capture the effect of the small scales without solving them explicitely. From the mathematical point of view, the goal is to address the following issues:

1. What is the limit system when $\varepsilon \to 0$?
2. Is this limit system well-posed?
3. Can we show a convergence result for u^ε?
4. Can we refine the asymptotics of u^ε?

It may happen that the limit problem is incompatible with the boundary conditions for the original system. This leads to strong singularities in the vicinity of the boundary, and hence to the study of *boundary layers*.

This preliminary chapter is divided into two parts, which correspond to our main concerns:

Analysis of boundary layer systems

– In the Chapter 4 we analyse the behaviour, far away from the boundary, of a solution to a boundary layer system coming from the homogenization of elliptic systems with oscillating coefficients and Dirichlet data. We treat a general case, with methods from potential theory.

– In the Chapter 5 we prove the well-posedness of a linearized Ekman boundary layer system for an arbitrary rugosity profile.

Asymptotics

– In the Chapter 3 we demonstrate error estimates in homogenization. In particular, we show a convergence rate for the homogenization of elliptic systems with oscillating coefficients and Dirichlet data in some polygonal domains. Moreover, we establish a first-order asymptotic expansion for the eigenvalues of an elliptic system with oscillating coefficients.

– The Chapter 6 gathers newly obtained results on the newtonian limit for some weakly viscoelastic fluid flow models.

1.1 Analysis of boundary layer systems

These works are devoted to the mathematical analysis of boundary layer type systems. Such systems arise from the study of the solutions u^ε near the boundary $\partial\Omega^\varepsilon$. Generally speaking, one faces three questions:

1. Is the boundary layer system well-posed?
2. Can we describe the behaviour far away from the boundary?
3. Is the boundary layer stable?

We focus here only on the first two questions. Our leitmotiv is to free ourselves from structure assumptions on the data, or on the boundary (such as periodicity, or quasiperiodicity). This raises a lot of mathematical issues. For example, boundary layer problems are posed in infinite domains (typically half-space domains), while the data (and hence the solutions) do not decrease at space infinity. It is hard to find an appropriate functional framework, in order to construct these solutions of infinite energy. This is especially the case, when one does not assume any structural property on the data. For the asymptotic behaviour far from the boundary, another problem linked with ergodicity appears.

We address these issues relying on energy estimates, or on *Saint-Venant* estimates, when the functional framework makes the use of a Poincaré inequality possible, or on methods from potential theory (Green's and Poisson's kernels).

1.1.1 Asymptotic analysis of boundary layer correctors in homogenization (Chapter 4)

This work corresponds to the paper [Pra13], published in the SIAM Journal on Mathematical Analysis.

We consider the elliptic sytem

$$\left\{\begin{array}{rll} -\nabla\cdot A(y)\nabla v_{bl} & = 0, & y\cdot n - a > 0 \\ v_{bl} & = v_0(y), & y\cdot n - a = 0 \end{array}\right., \tag{1.1}$$

posed in the half-space domain $\{y\cdot n - a > 0\}\subset\mathbb{R}^d$, where $n\in\mathbb{S}^{d-1}$. Notice that $v_{bl} = v_{bl}(y)\in\mathbb{R}^N$, that $v_0 = v_0(y)\in\mathbb{R}^N$ is a periodic function of y, and that $A = A^{\alpha\beta}(y)\in M_N\left(\mathbb{R}^d\right)$ is a family of smooth uniformly elliptic and periodic matrices indexed by $1\leq\alpha,\beta\leq d$. The system (1.1) comes from the analysis of boundary layers in periodic homogenization, i.e. systems with oscillating coefficients and Dirichlet data

$$\left\{\begin{array}{rll} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon_{bl} & = 0, & x\in\Omega \\ u^\varepsilon_{bl} & = \varphi\left(x,\frac{x}{\varepsilon}\right), & x\in\partial\Omega \end{array}\right.. \tag{1.2}$$

Again, there are two important issues related to (1.1). Is the system well-posed? Can we describe the behaviour of v_{bl} when $y \cdot n \to \infty$? The latter is related to the homogenized limit of the oscillating Dirichlet data in (1.2).

We distinguish between three cases for the analysis of (1.1):

RAT $n \in \mathbb{R}\mathbb{Q}^d$. This case has been analysed by numerous authors [MV97, AA99, Naz92].

DIV $n \notin \mathbb{R}\mathbb{Q}^d$ satisfying a small divisors assumption: there exists $C,\ \tau > 0$, such that for all $\xi \in \mathbb{Z}^d \setminus \{0\}$, for all $i = 1, \dots\ d-1$,

$$|n_i \cdot \xi| \geq C|\xi|^{-d-\tau}, \tag{1.3}$$

where $(n_1, \dots\ n_{d-1}, n)$ is a orthogonal basis of $\mathbb{R}^d$. This case has been treated in [GVM11].

NON DIV the general case $n \notin \mathbb{R}\mathbb{Q}^d$ without the small divisors assumption. This case is the purpose of our work in Chapter 4 and in the paper [Pra13].

Let us say some words about the **RAT** and **DIV** cases. Keep in mind that in both cases one can show the existence and uniqueness of a solution which decays rapidly when $y \cdot n \to 0$ (exponentially fast in the rational case) toward a *boundary layer tail*. When $n \in \mathbb{R}\mathbb{Q}^d$ (resp. $n \notin \mathbb{R}\mathbb{Q}^d$) one has a periodic (resp. quasiperiodic) setting. The existence follows from energy estimates. Note that this step does not rely on the small divisors assumption. The convergence far away from the boundary is obtained through a Saint-Venant estimate. When $n \notin \mathbb{R}\mathbb{Q}^d$, the small divisors assumption (1.3) is decisive and yields a convergence speed.

What can be expected in the general case **NON DIV**? The quasiperiodicity of the Dirichlet data on the boundary $y \cdot n - a = 0$ yields some ergodicity, which makes it reasonable to conjecture the convergence. However, without small divisors, the distance between the lattice $\mathbb{Z}^d \setminus \{0\}$ and the hyperplane $y \cdot n = 0$ may be arbitrarily small. This indicates that fast convergence is not likely to take place. *Assume now that $a = 0$ for the sake of simplicity.*

Our main point is to prove the convergence far away from the boundary; the existence of a variational solution v_{bl} results from the work of Gérard-Varet and Masmoudi [GVM11]. We have seen that we cannot use an energy method, through Saint-Venant estimates, in the general case. We resort instead to a potential theoretical approach. We represent v_{bl} in terms of the Poisson kernel $P = P(y, \tilde{y}) \in M_N(\mathbb{R})$ associated to the operator $-\nabla \cdot A(y)\nabla \cdot$ and to the domain $\{y \cdot n > 0\}$:

$$v_{bl}(y) = \int_{\tilde{y}\cdot n = 0} P(y, \tilde{y}) v_0(\tilde{y}) d\tilde{y}. \tag{1.4}$$

Our first result addresses the asymptotics of the kernel $P(y, \tilde{y})$ for $|y - \tilde{y}| \gg 1$.

Theorem A (Theorem 4.18, Chapter 4)**.** *There exists a function $P_{exp} = P_{exp}(y, \tilde{y}) \in M_N(\mathbb{R})$, which is explicit, such that for all $0 < \kappa < \frac{1}{2d}$, for all y (resp. $\tilde{y}$) satisfying $y \cdot n > 0$ (resp. $\tilde{y} \cdot n = 0$),*

$$P(y, \tilde{y}) = P_{exp}(y, \tilde{y}) + O\left(\frac{1}{|y - \tilde{y}|^{1+\kappa}}\right).$$

Following Avellaneda and Lin [AL91], by the change of variables $x = \varepsilon y$, $\tilde{x} = \varepsilon \tilde{y}$, with $\varepsilon := 1/|x - \tilde{x}|$, the large scale asymptotic analysis of $P(y, \tilde{y})$ boils down to an homogenization problem for the oscillating kernel

$$P^\varepsilon(x, \tilde{x}) = \frac{1}{\varepsilon^{d-1}} P\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right)$$

for $|x-\tilde{x}|$ close to 1. We face this question using classical two-scale expansions and boundary layer correctors. Notice that we only use the latter in the vicinity of the boundary. To get an error estimate, we rely on the uniform (in ε) local elliptic estimates of Avellaneda and Lin [AL87a].

The expression of P_{exp} is explicit (see (4.64)) and exhibits quasiperiodicity in the variable $\tilde{y}$. The exponent $\kappa > 0$ is essential in order to address the convergence $y \cdot n \to \infty$ in (1.4).

Theorem B (Theorems 4.2 and 4.3, Chapter 4). *Assume that $n \notin \mathbb{R}\mathbb{Q}^d$. Let v_{bl} be the variational solution to* (1.1) *in the sense of Gérard-Varet and Masmoudi [GVM11]. Then,*

1. *There exists $V^\infty \in \mathbb{R}^N$ such that*
$$v_{bl}(y) \overset{y\cdot n\to\infty}{\longrightarrow} V^\infty$$
locally uniformly in the tangential variable.
2. *The boundary layer tail V^∞ is independent of a.*
3. *Assume that n does not meet the small divisors assumption* (1.3). *Take $d=2$, $N=1$ and $A=\mathrm{I}_2$. For all $l>0$, there exists a smooth Dirichlet data v_0, such that the associated solution v_{bl} tends to V^∞ slower than $O\left((y\cdot n)^{-l}\right)$.*

The last point of the theorem remains true for $n \in \mathbb{R}\mathbb{Q}^d$. The second is an important property of the case $n \notin \mathbb{R}\mathbb{Q}^d$. The last point confirms our feeling about the meaning of the small divisors assumption (1.3). It shows, in the particular case $d=2$, $N=1$ and $A=\mathrm{I}_2$ that the latter is necessary to get the fast decay. In the general case, the same phenomenon leading to the counter-example should take place. Let us finally state, that for an abitrary $n \notin \mathbb{R}\mathbb{Q}^d$ such that **NON DIV**, the fact that the small divisors assumption is not satisfied does not allow, in our opinion, to construct an example of v_0 for which convergence is slower than $O\left(\frac{1}{\ln n}\right)$, for instance.

The proof of the theorem yields an explicit expression for V^∞: see (4.68) in Chapter 4. This formula generalizes the one of Moskow and Vogelius [MV97, Proposition 6.6]. This explicit formula for V^∞ can be used for numerical computations.

1.1.2 Well-posedness of the Stokes-Coriolis system in the half-space over a rough surface (Chapter 5)

This work is joint with Anne-Laure Dalibard and corresponds to the paper [DP13].

The goal of the Chapter 5 is to study the Stokes-Coriolis system in the 3d setting with Dirichlet data u_0 in a space of infinite energy of Sobolev regularity *uloc*

$$\left\{\begin{aligned} -\Delta u + e_3\times u + \nabla p &= 0, \quad y_3 > \omega(y_h)\\ \nabla\cdot u &= 0, \quad y_3>\omega(y_h)\\ u(y_h,\omega(y_h)) &= u_0, \quad y_3=\omega(y_h)\end{aligned}\right. . \tag{1.5}$$

The velocity of the fluid in the boundary layer is denoted by $u = u(y_h, y_3) \in \mathbb{R}^3$, with $y_h \in \mathbb{R}^2$ and $y_3 > \omega(y_h)$. Notice that the vector e_3 is a unit vector directing the y_3 axis. This system comes from the study of highly rotating fluids in the linear régime in the vicinity of an horizontal rough bottom (typically the bottom of an ocean, or the Earth's core-mantle boundary). It is a linearized version of the so-called Ekman boundary layer system. The system (1.5) only differs from the Stokes system by the term $e_3 \times u$, which represents the Coriolis acceleration. We will see that this term is responsible for strong singularities at low tangential frequencies.

We aim at freeing ourselves from any structure assumption (periodicity, quasiperiodicity) on the rugosity profile ω. The analysis of the linear system (1.5) is a first step in our project to study nonlinear Ekman boundary layers near arbitrary rough boundaries. Such studies are important from both a mathematical and a physical viewpoint.

One encounters several difficulties in the mathematical analysis of (1.5):

1. The first one is linked to the unboundedness of the domain $\{y_3 > \omega(y_h)\}$ in every direction (no Poincaré inequality), to the infinite energy of the data and of the solution, to the rough bottom (one cannot use the Fourier transform in the tangential direction at every height) and to the lack of structure. These problems have already been encountered by Gérard-Varet and Masmoudi in their study of the 2d Stokes system in a rough half-plane [GVM10].
2. The second type of difficulties is due to the operator. Unlike the Stokes operator, we neither have an explicit formula nor an estimate of the kernel associated to the Stokes-Coriolis operator. Furthermore, the computation in Fourier space (for the tangential variable) exhibits singularities, which are not present for the Stokes operator.
3. We address the 3d case, not the 2d case as in [GVM10], which complicates our estimates.

Let us describe the general idea for the proof. We divide our half-space in two domains: a rough channel $0 > y_3 > \omega(y_h)$ bounded in the vertical direction, in which we can carry out energy estimates relying on a Poincaré inequality, and a flat half-space $y_3 > 0$, in which the system can be studied thanks to the Fourier transform. The connection between the two domains is done via a *transparent boundary condition* on the interface $y_3 = 0$. This condition involves the Dirichlet to Neumann operator DN defined by

$$-\partial_3 u + p e_3|_{y_3=0} = \mathrm{DN}(u|_{y_3=0}).$$

Following this scheme, we replace the study of (1.5), by

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p &= f, & \omega(y_h) < y_3 < 0 \\ \nabla \cdot u &= 0, & \omega(y_h) < y_3 < 0 \\ u(y_h, \omega(y_h)) &= 0, & y_3 = \omega(y_h) \\ -\partial_3 u + p e_3|_{y_3=0} &= \mathrm{DN}(u|_{y_3=0}) + F, & \end{cases}$$

where f and F are source terms coming from the lifting of the boundary data u_0. We then estimate the trunctated energy

$$E_k := \int_{|y_h|<k} \int_{\omega(y_h)}^0 |\nabla u|^2 \, dy_3 dy_h,$$

which leads to a Saint-Venant estimate on E_k: for all $m > 0$, for all $k \geq m$,

$$E_k \leq C \left[k^2 + E_{k+m+1} - E_k + \frac{k^4}{m^5} \sup_{j \geq m+k} \frac{E_{j+m} - E_j}{j} \right].$$

Note that this estimate of the energy of u in $\{|y_h| < k,\ 0 > y_3 > \omega(y_h)\}$ involves the large scales of u. This is due to the non-locality of the operator DN.

We now briefly describe how we face the second type of difficulties. The singularities in low frequencies prevent us from working with Dirichlet data in $H^{1/2}_{uloc}$ neither for the existence in the half-space, nor for the definition of the Dirichlet to Neumann operator. Let us recall that $H^{1/2}_{uloc}(\mathbb{R}^2)$ is the space of functions uniformly locally in $H^{1/2}$, i.e.

$$\|\cdot\|_{H^{1/2}_{uloc}(\mathbb{R}^2)} := \sup_{l\in\mathbb{Z}^2} \|\cdot\|_{H^{1/2}(l+[0,1]^2)}\,.$$

The Dirichlet to Neumann operator is defined for fields $v_0 \in H^{1/2}_{uloc}$ on the flat bottom $y_3 = 0$ such that there exists a field V_h

$$v_{0,3} = \nabla_h \cdot V_h. \tag{1.6}$$

We call $\mathbb{K}$ the space of such v_0. It plays a key role for the existence in the flat half-space and the definition of DN. If v_0 is the trace on $y_3 = 0$ of the incompressible field u,

$$\begin{aligned} v_{0,3}(y_h) = u_3|_{x_3=0}(y_h) &= \int_{\omega(y_h)}^0 \partial_3 u_3(y_h,t)dt + u_{0,3}(y_h) \\ &= -\nabla_h \cdot \left(\int_{\omega(y_h)}^0 u_h(y_h,t)dt\right) - \nabla_h\omega\cdot u_{0,h} + u_{0,3}. \end{aligned}$$

In order to meet the condition (1.6), we have to impose the existence of a field $U_h = U_h(y_h) \in \mathbb{R}^2$ such that

$$-\nabla_h\omega\cdot u_{0,h} + u_{0,3} = \nabla_h \cdot U_h. \tag{1.7}$$

We emphasize that the reason for such a compatibility condition of the Dirichlet data and the roughness profile is the control of the low frequencies.

Our main result is the well-posedness of (1.5):

Theorem C (Theorem 5.1, Chapter 5). *Assume that $\omega \in W^{1,\infty}(\mathbb{R}^2)$, that $u_0 \in H^2_{uloc}(\mathbb{R}^2)$ and that the compatibility condition* (1.7) *is satisfied.*
Then there exists a unique weak solution u to (1.5) *such that for all $a > 0$, and for all $m \geq 4$,*

$$\sup_{l\in\mathbb{Z}^2} \|u\|_{H^1(\{y_h\in l+[0,1]^2,\omega(y_h)<y_3<a\})} < \infty, \tag{1.8a}$$

$$\sup_{l\in\mathbb{Z}^2} \int_1^\infty \int_{l+[0,1]^2} |\nabla^m u|^2 < \infty. \tag{1.8b}$$

Our theorem has some similarities with the result [GVM11, Proposition 6 and 10] on the 2d Stokes system for an arbitrary roughness. As a byproduct of our proof, we generalize this result to the 3d case: *if $u_0 \in H^2_{uloc}(\mathbb{R}^2)$, then there exists a unique weak solution to the Stokes system in $y_3 > \omega(y_h)$, such that for all $a > 0$ and $m \geq 1$, the bounds* (1.8a) *and* (1.8b) *are satisfied.* There is no need for an assumption like (1.7) in this case. Notice also the fact that the a priori bound (1.8b) is only true for m sufficiently large, which is peculiar to the Stokes-Coriolis system.

The main interest of our theorem lies in the fact that nothing is prescribed on ω, appart from the regularity assumption and the boundedness. The counter-example in [GVM10, Proposition 11] constructed for Laplace's equation shows that the question of the convergence far away from the boundary is not relevant for such profiles without ergodicity properties. Treating arbitrary rugosity profiles requires to understand the way the differential operator acts on all tangential frequencies, which makes the difficulty of such results. Notice that under a periodicity assumption on ω, the bad behaviour of the Stokes-Coriolis operator in low frequencies is filtered out because of the gap in the spectrum of periodic functions. Our theorem is a first step toward the generalization of the work of Gérard-Varet [GV03] for the nonlinear Ekman system (i.e. (1.5) with an additional inertial acceleration term $u\cdot\nabla u$) for a periodic profile ω.

1.2 Asymptotics

The starting point of every convergence result is a uniform a priori bound. These bounds make it possible: to get weak convergence via the Banach-Alaoglu theorem, to use two-scale convergence methods in periodic homogenization, to have strong convergence in a weaker norm through a Rellich injection.

In the following, we encounter frequently two major problems. The first one is linked to the lack of uniform a priori bounds (at least obvious ones) as for the boundary layer problem (1.2)

$$\|u^\varepsilon_{bl}\|_{H^1(\Omega)} \leq C\varepsilon^{-\frac{1}{2}}.$$

A convergence result for u^ε_{bl} therefore relies on a refined description of u^ε_{bl} thanks to the boundary layer correctors, such as v_{bl}.

The second problem is linked to the bad behaviour of weak convergence with respect to nonlinearities. Indeed, there might be resonances in a product of weakly convergent sequences, which either prevent the product of being weakly convergent, or are responsible for the fact that the limit of the product is not the product of the limits. This happens when passing to the limit in the product $A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon_{bl}$, or (in fluid mechanics) in the product $\tau_n W(u_n)$ (see below). In general it is not possible to pass to the weak limit in the product. In some situations, however, we can determine the limit using: the structure of the nonlinear term (compensated compactness methods, such as the div-curl lemma), the structure of the solution (two-scale asymptotic expansions, oscillating test function methods and two-scale convergence), the structure of the equation (defect measures and propagation of the compactness).

Let us mention that for many applications, a weak convergent result is not satisfactory. A more substantial contribution is the proof of error estimates between a solution and its approximation. This is what we do in periodic homogenization, for example.

1.2.1 Homogenization of the eigenvalue problem (Chapter 3)

This work corresponds to the paper [Pra11], to appear in Asymptotic Analysis.

The main focus of the Chapter 6 is the homogenization of the eigenvalue problem

$$\left\{\begin{array}{rll} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon &= \lambda^\varepsilon v^\varepsilon, & x\in\Omega \\ v^\varepsilon &= 0, & x\in\partial\Omega \end{array}\right. \tag{1.9}$$

where $\Omega\subset\mathbb{R}^2$ is bounded and A is symmetric, i.e. for all $1\leq \alpha,\ \beta\leq 2,\ 1\leq i,\ j\leq N$, $A^{\alpha\beta}_{ij}=A^{\beta\alpha}_{ji}$. We aim at deriving a first-order asymptotic expansion for the eigenvalue λ^ε_l, when $\varepsilon\to 0$ and the mode l is fixed. Such a study can be used to investigate the low frequency propagation of waves in periodic composite media (see [SV93]).

One shows that λ^ε_l converges toward its homogenized limit λ^0_l, eigenvalue associated to the homogenized system

$$\left\{\begin{array}{rll} -\nabla\cdot\overline{A}\nabla\overline{v}^0_l &= \lambda^0_l\overline{v}^0_l, & x\in\Omega \\ \overline{v}^0_l &= 0, & x\in\partial\Omega \end{array}\right. . \tag{1.10}$$

This problem has been the topic of many works, which address, in particular, the question of error estimates between λ^ε_l and λ^0_l: see Kesavan [Kes79a, Kes79b], Jikov, Kozlov and Oleĭnik [JKO94], Allaire and Conca [AC98a], Kenig, Lin and Shen [KLS12c, KLS12a]. As underlined by Allaire and Amar [AA99], an improved approximation of λ^ε_l uses information on the boundary layers.

We consider convex polygonal domains Ω with sides H^k satisfying either the **RAT**, or the **DIV** assumption. Before coming to an expansion for the eigenvalues, we establish a key result on the homogenization of boundary layer systems of the type of (1.2), with a boundary data φ which splits into:

$$\varphi(x,y) = -v_0(y) \cdot \nabla \bar{v}_l^0(x). \tag{1.11}$$

This condition is not restrictive, as boundary layer systems in homogenization often arise in such a form. *We now drop the subscript l.* We denote by $V^{k,\infty}$ the boundary layer tail associated to the side k of the polygon.

Theorem D (Theorem 3.16, Chapter 3)**.** *Assume that $\bar{v}^0 \in H^{2+\omega}(\Omega)$.*
Then, u_{bl}^ε, solution of (1.2) *with boundary data* (1.11)*, converges in $L^2(\Omega)$ toward $\bar{u}$ solving the elliptic system*

$$\begin{cases} -\nabla \cdot \overline{A} \nabla \bar{u} &= 0, & x \in \Omega \\ \bar{u} &= -V^{k,\infty} \cdot \nabla \bar{v}^0, & x \in \partial\Omega \cap H^k \end{cases}.$$

Moreover, there exists $\gamma > 0$ such that

$$\|u_{bl}^\varepsilon - \bar{u}\|_{L^2(\Omega)} = O\left(\varepsilon^\gamma\right),$$

and if $\bar{v}^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$, one can take $\gamma = \frac{1}{2}$ in the preceding estimate.

Note that the convergence is only true up to a subsequence ε_n in the rational case. This theorem is a generalization of the theorem in [MV97] (restricted to rational slopes and to scalar equations), and of the theorem in [GVM11], which addresses systems and polygons under the small divisors assumption, but with the regularity $H^3(\Omega) \cap C^2(\overline{\Omega})$ on $\bar{v}^0$. Our main contribution is to weaken this smoothness assumption, which is crucial for the study of the eigenvalue problem.

In fact, a polygonal domain has corners, which generate singularities in the solution $\bar{v}^0$ of (1.10). In the scalar case, the regularity $H^{2+\omega}(\Omega)$ follows from the theory of Grisvard [Gri85, Theorem 3.2.1.2]. This result becomes false without convexity of Ω. For systems, it is more tricky to recover this regularity. We use the works of Dauge [Dau88]on the one hand, and of Kozlov, Maz′ya and Rossmann [KMR01] on the other hand. The Theorem 3.4 of Chapter 3, contains all the regularity we need.

Improving some error estimates in homogenization by the use of boundary layers, and relying on a formula due to Osborn [Osb75, Theorem 3.1], we prove the following expansion of λ^ε:

Theorem E (Theorems 3.6 and 3.7, Chapter 3)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, which is either smooth uniformly convex, or convex polygonal with sides satisfying* **RAT** *or* **DIV***. Let λ_l^0 be an eigenvalue of* (1.11) *with multiplicity m and* $\mathrm{Vect}\left(\bar{v}_{l+j}^0\right)_{j=0,\dots m}$ *the corresponding eigenspace.*
Then, there exists $\gamma > 0$, there exists $\bar{u}_j \in L^2(\Omega)$ for $j = 1 \dots m$ such that

$$\left[\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda_{l+j}^\varepsilon}\right]^{-1} = \lambda_l^0 + \varepsilon\frac{\lambda_l^0}{m}\sum_{j=0}^{m-1}\int_\Omega \bar{u}_j(x)\cdot\bar{v}_{l+j}^0(x)dx + O\left(\varepsilon^{1+\gamma}\right)$$

(for a subsequence ε_n when the polygonal domain has rational slopes).

The first-order correctors to the eigenvalues involve directly the solutions $\bar{u}_j$ of homogenized boundary layer systems. Moreover, one can replace the exponent γ by $\frac{1}{2}$ under the assumption

$$\mathrm{Vect}\left(\bar{v}^0_{l+j}\right)_{j=0,\dots\, m} \subset H^3\left(\Omega\right) \cap C^2(\overline{\Omega}).$$

We note that this theorem improves the results of Moskow and Vogelius in three directions: more general domains, elliptic systems and better convergence rate (the formula of [MV97] is stated for a remainder $o(\varepsilon)$ instead of $O\left(\varepsilon^{1+\gamma}\right)$).

Notice that the case of smooth domains is a consequence of the recent work of Gérard-Varet and Masmoudi [GVM12] on the homogenization of (1.2). Besides, one can wonder if our newly obtained Theorem B could make it possible to extend such a result to arbitrary polygonal domains. Our feeling is that the very slow convergence without the small divisors assumption, does not make it possible to achieve a theorem such as Theorem D in this case.

We finally stress that our work on error estimates in homogenization, and on the expansion for the eigenvalues opens the way to numerical analyses in the spirit of [SV06, SV08].

1.2.2 Newtonian limit for weakly viscoelastic fluid flows (Chapter 6)

Joint work with Didier Bresch

This work is concerned with viscoelastic fluid flows, which have an elastic behaviour in short times, and a viscous one in large times. Such non newtonian fluids are ubiquitous: glaciers, Earth's mantle, dough, paint, solutions of polymers. They have a complex dynamic: like a ball made of the modelling clay Silly Putty, which bounces several times before spreading on the floor. Because of elasticity, viscoelastic fluids remember their history, which means that the dynamic of the flow at a given time depends on the past. This is in strong constrast with newtonian fluids (i.e. purely viscous fluids). The viscoelastic relaxation time is roughly the time on which the flow remembers the past. The dimensionless number, which compares the viscoelastic relaxation time to a time scale relevant to the fluid flow, is the Weissenberg (or Deborah number) We. The bigger We, the more important is the elasticity with respect to the viscosity.

One of our purposes is to study the newtonian limit of some macroscopic models of viscoelastic fluid flows, that is to say the limit $\mathrm{We} \to 0$. The works presented here are a first step toward a better understanding of the effect of a small amount of elasticity on the newtonian dynamic of a fluid. The latter is one of our mid-term projects.

All macro-macro models we consider here are the coupling of a momentum equation on the incompressible velocity $u = u(t,x) \in \mathbb{R}^d$ and an equation for the symmetric stress tensor $\tau = \tau(t,x) \in M_d\left(\mathbb{R}\right)$ (or a structure tensor $A = A(t,x) \in M_d\left(\mathbb{R}\right)$, which has a microscopic meaning). In the sequel, we concentrate on two models: namely the corotational Johnson-Segalman model

$$\left\{\begin{array}{rl} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau, \\ \nabla\cdot u &= 0, \\ \mathrm{We}\left(\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau\right) + \tau &= 2\omega D(u). \end{array}\right. \tag{1.12}$$

and the FENE-P model

$$\begin{cases} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau, \\ \nabla\cdot u &= 0, \\ \tau &= \frac{(b+d)\omega}{b}\frac{1}{\mathrm{We}}\left(\frac{A}{1-\frac{\operatorname{Tr}A}{b}} - \mathrm{I}\right), \\ \partial_t A + u\cdot\nabla A - \nabla u A - A\left(\nabla u\right)^T + \frac{1}{\mathrm{We}}\frac{A}{1-\frac{\operatorname{Tr}A}{b}} &= \frac{1}{\mathrm{We}}\mathrm{I}\,. \end{cases} \tag{1.13}$$

Notice that these systems are posed in $\Omega\subset\mathbb{R}^d$ a bounded domain, $\Omega=\mathbb{R}^d$, or $\Omega=\mathbb{T}^d$. We recall that

$$D(u) := \frac{\nabla u + (\nabla u)^T}{2}, \qquad W(u) := \frac{\nabla u - (\nabla u)^T}{2}.$$

Moreover, we assume that u satisfies a noslip boundary condition on $\partial\Omega$. There is no condition for τ (nor A) on the boundary. We start from the initial conditions:

$$u(0,\cdot) := u_0, \qquad \tau(0,\cdot) := \tau_0, \qquad A(0,\cdot) := A_0.$$

We consider only the case of Jeffrey fluids, for which $0<\omega<1$, in the framework of global in time weak solutions. The case $\omega=1$ turns out to be much more complicated (like Euler in comparison to Navier-Stokes).

The existence of weak solutions to the corotational model (1.12) is due to Lions and Masmoudi [LM00]. The starting point of their result is the energy estimate

$$\begin{aligned}\omega\left\|u\right\|^2_{L^2(\Omega)}(T) + 2\omega(1-\omega)\int_0^T \left\|\nabla u\right\|^2_{L^2(\Omega)} + \frac{\mathrm{We}}{2}\left\|\tau\right\|^2_{L^2(\Omega)}(T) + \int_0^T \left\|\tau\right\|^2_{L^2(\Omega)} \\ \le \omega\left\|u_0\right\|^2_{L^2(\Omega)} + \frac{\mathrm{We}}{2}\left\|\tau_0\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{1.14}$$

They intensively rely on the use of defect measures to pass to the limit in the product $\tau_n W(u_n)$, where (u_n,τ_n) is an approximated smooth solution to (1.12). Note that the regularity provided by (1.14) is barely $\tau\in L^\infty\left((0,\infty);L^2\right)$ and $\nabla u\in L^2\left((0,\infty);L^2\right)$.

For the FENE-P model, an existence result has been achieved by Masmoudi [Mas11], thanks to a relative entropy invented by Hu and Lelièvre [HL07]. The fundamental point is that the decay of the entropy

$$\begin{aligned}&\frac{1}{2}\left\|u\right\|^2_{L^2(\Omega)}(T) + (1-\omega)\int_0^T\left\|\nabla u\right\|^2_{L^2(\Omega)} \\ &\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega\left[-\ln\left(\det A\right) - b\ln\left(1-\frac{\operatorname{Tr}(A)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](T) \\ &\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}^2}\int_0^T\int_\Omega\left[\frac{\operatorname{Tr}A}{\left(1-\frac{\operatorname{Tr}A}{b}\right)^2} - \frac{2d}{1-\frac{\operatorname{Tr}A}{b}} + \operatorname{Tr}\left(A^{-1}\right)\right] \\ &\le \frac{1}{2}\left\|u_0\right\|^2_{L^2(\Omega)} + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega\left[-\ln\left(\det A_0\right) - b\ln\left(1-\frac{\operatorname{Tr}(A_0)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right]\end{aligned} \tag{1.15}$$

controls the $L^2\left((0,\infty);L^2\right)$ norm of τ. We come back to this point in detail in the Chapter 6.

These existence theorems of global weak solutions make it possible to address the convergence $\mathrm{We}\to 0$. Is it easy to see that the solutions of (1.12) converge formally to a solution of the Navier-Stokes system

$$\begin{cases} \partial_t u^0 + u^0\cdot\nabla u^0 - \Delta u^0 + \nabla p^0 &= 0, \quad \Omega, \\ \nabla\cdot u^0 &= 0, \quad \Omega, \\ u^0 &= 0, \quad \partial\Omega. \end{cases} \tag{1.16}$$

Notice that the noslip condition is compatible with the limit, so that no boundary layers are involved in this limit (at least at the leading order in We). That is why, we state our results for Ω a bounded domain, $\Omega = \mathbb{T}^d$ or $\Omega = \mathbb{R}^d$ without making any difference.

As far as we know, the only results concerning the newtonian limit of viscoelastic fluid flows deal with strong solutions. Saut [Sau86] has investigated the case of Maxwell type fluids (no dissipation term in the momentum equation) in the linear régime, and Molinet and Talhouk [MT08] the case of general Johnson-Segalman systems (including the corotational and the Oldroyd-B systems). For these models no energy of the type of (1.14) is available in general, so they rely on a splitting in low and high frequencies at a cut-off frequency depending on We.

The mathematical justification of the formal asymptotics requires a priori bounds uniform in We. Some bounds, like the $L^\infty\left((0,\infty);L^2\right)$ bound on τ for (1.12), are not uniform in We. They were usefull for the Cauchy theory, but are useless for the newtonian limit. In some cases, we have to assume that initial data is well-prepared (in a sense to be made precise later on) in order to get uniform bounds.

Let us briefly describe the main results, we obtain in the Chapter 6. The first theorem is concerned with the strong convergence in the corotational system (1.12).

Theorem F (Theorem 6.3, Chapter 6)**.** *Let $d = 2,\ 3$, $u_0 \in H^{4,\sigma}(\Omega)$ independent of* We *and $\tau_0 \in L^2(\Omega) \cap L^q(\Omega)$, with $2 < q \leq 3$. Notice that τ_0 may depend on* We *in the following sense: $\|\tau_0\|_{L^2(\Omega)} = O(1)$. Let also*

$$u \in L^\infty\left((0,\infty);L^{2,\sigma}\right) \cap L^2\big((0,\infty);\dot{H}^1\big), \quad \textit{and} \quad \tau \in L^\infty\left((0,\infty);L^2\right) \cap L^\infty_{loc}\left((0,\infty);L^q\right)$$

be global weak solutions to (2.51) *in the sense of Lions and Masmoudi [LM00] associated to the initial data u_0 and τ_0.*
Then, there exists $0 < T^* < \infty$ independent of We *and*

$$u^0 \in L^\infty\left((0,\infty);L^{2,\sigma}\right) \cap L^2\big((0,\infty);\dot{H}^1\big)$$

a global weak solution of (1.16) *associated to the initial data u_0, such that, in addition, u^0 belongs to $L^\infty\left((0,T);H^4\right)$ for all $0 < T < T^*$. Moreover, for all $0 < T < T^*$,*

$$\sup_{0<t<T}\left(\omega\left\|u(t,\cdot) - u^0(t,\cdot)\right\|^2_{L^2} + \omega(1-\omega)\int_0^t \left\|\nabla\left(u - u^0\right)\right\|^2_{L^2}\right.$$
$$\left. + \frac{\mathrm{We}}{2}\left\|\tau(t,\cdot) - \tau^0(t,\cdot)\right\|^2_{L^2} + \frac{1}{2}\int_0^t \left\|\tau - \tau^0\right\|^2_{L^2}\right)^{\frac{1}{2}} = O\big(\sqrt{\mathrm{We}}\big).$$

The proof of this theorem relies on the construction of an Ansatz for (u, τ) and on an energy estimate on the remainder. The scheme is analogous to relative entropy methods, used for example by Jourdain, Le Bris, Lelièvre and Otto [JLBLO06] in the study of the long-time behaviour of some polymeric fluid flows. We return in some more details to the relative entropy method in the Chapter 6.

For the FENE-P model, it is hard to get even a formal convergence result. However, it is clear that A converges (at least formally) to $A^0 := \frac{b}{b+d}\,\mathrm{I}$. Hence, we manage to prove a weak convergence result for well-prepared data.

Theorem G (Proposition 6.2, Chapter 6)**.** *Let $d = 2,\ 3$. We consider a global weak solution (u, A, τ) of* (1.13) *in the sense of Masmoudi [Mas11]. Assume that $\|u_0\|_{L^2(\Omega)} = O(1)$. Then:*

- *Assume that initial data is* ill-prepared *in the sense that*

$$\int_\Omega \left[-\ln\left(\det A_0\right) - b\ln\left(1 - \frac{\operatorname{Tr} A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right] = O\left(1\right).$$

Then A tends to A^0 in $L^2\left((0,\infty); L^2\left(\Omega\right)\right)$ and

$$\left\| A - A^0 \right\|_{L^2((0,\infty);L^2(\Omega))} = O\big(\sqrt{\mathrm{We}}\big). \tag{1.17}$$

- *Assume furthermore that initial data is* well-prepared *namely*

$$\int_\Omega \left[-\ln\left(\det A_0\right) - b\ln\left(1 - \frac{\operatorname{Tr} A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right] = O\left(\mathrm{We}\right).$$

Then we have the following improved convergences:

$$\left\| A - A^0 \right\|_{L^\infty((0,\infty);L^2(\Omega))} = O\big(\sqrt{\mathrm{We}}\big), \tag{1.18a}$$

$$\left\| A - A^0 \right\|_{L^2((0,\infty);L^2(\Omega))} = O\left(\mathrm{We}\right). \tag{1.18b}$$

Moreover, τ is bounded uniformly in $L^2\left((0,\infty)\times\Omega\right)$, and u (resp. τ) converges in the sense of distributions toward u^0 (resp. $2\omega D(u^0)$), where u^0 satisfies the Navier-Stokes system (1.16)*.*

To get the strong convergences (1.17), (1.18a) and (1.18b), we use the convexity of the entropy found by Hu and Lelièvre. Our next goal, which is work in progress, is to get the strong convergence of u and τ. An idea to get such results is to build an Ansatz for A, in the same spirit as we have done for the corotational system.

Chapitre 2

Introduction

Sommaire

2.1 Contexte

Cette thèse est consacrée à l'étude asymptotique de quelques équations aux dérivées partielles. Les modèles que l'on étudie sont issus de la science des matériaux et de la mécanique des fluides. De façon schématique, les systèmes étudiés mettent en jeu un paramètre $\varepsilon \ll 1$, une équation

$$\mathcal{L}^\varepsilon u^\varepsilon = 0, \tag{2.1}$$

posée dans un domaine $\Omega^\varepsilon \subset \mathbb{R}^d$ (ou $(0,T) \times \Omega^\varepsilon$), ainsi que des conditions sur le bord $\partial\Omega^\varepsilon$ (et éventuellement des conditions initiales) pour u^ε.

La généralité du système (2.1) masque bien-sûr la diversité des situations qu'il recouvre. Le petit paramètre ε peut avoir une signification physique ou purement mathématique.

Dans le premier cas, il peut traduire des oscillations à l'échelle microscopique, soit dans les propriétés du milieu (hétérogénéités des matériaux composites), soit dans la structure du bord (rugosités en mécanique des fluides), ou provenir d'un adimensionnement. Dans le second cas, sa raison d'être peut être, par exemple, un processus de régularisation utilisé dans une démonstration d'existence de solutions.

L'étude asymptotique de (2.1) dans la limite $\varepsilon \ll 1$ est motivée par des aspects physiques, numériques et mathématiques. Du point de vue de la physique, il s'agit de dériver des modèles limites valables dans différents régimes. Du point de vue numérique, l'enjeu est de capturer l'effet des petites échelles sans avoir à les résoudre explicitement, puisque souvent la taille du paramètre rend inopérant un calcul direct. Du point de vue mathématique, les questions qui se posent sont :

1. Quel est le système limite lorsque $\varepsilon \to 0$?
2. Ce système est-il bien posé ?
3. Peut-on montrer un résultat de convergence pour u^ε ?
4. Peut-on raffiner l'asymptotique de u^ε ?

La présence d'un bord peut compliquer considérablement le déroulement de ce programme. Le fait que le système limite puisse être incompatible avec les conditions aux limites imposées sur u^ε est une des sources majeures de difficultés. À proximité du bord peuvent se développer de nombreuses singularités et instabilités. Il est donc crucial d'analyser finement le comportement des solutions u^ε au voisinage du bord $\partial\Omega^\varepsilon$ dans une couche étroite, la *couche limite*.

Plan de l'introduction

La section 2.2 est dédiée à la présentation des modèles physiques étudiés dans ce travail. Elle est centrée sur la notion d'échelle, qui éclaire les analyses asymptotiques. La section 2.3 est consacrée à quelques asymptotiques formelles. On montre la nécessité de prendre en compte dans l'analyse des termes de couches limites. L'étude mathématique des couches limites est l'objet de la partie 2.4. Enfin, la section 2.5 apporte des éléments d'analyse mathématique mis en œuvre pour justifier les asymptotiques formelles.

On a essayé d'avoir des notations homogènes dans toute cette introduction. Elles diffèrent donc souvent des notations utilisées dans les chapitres correspondants.

2.2 Une histoire d'échelles

Pour nous qui percevons les tremblements de terre affectant la lithosphère, le manteau terrestre a l'apparence d'un solide élastique. À l'échelle des temps géologiques, des mouvements de convection responsables de la tectonique des plaques, rapprochent le manteau terrestre d'un fluide visqueux. Dans un autre domaine, l'extension géographique d'un vaste courant marin permanent comme le Gulf stream sur plusieurs milliers de kilomètres est sans commune mesure avec l'échelle spatiale des courants de marée, qui font subir à une particule d'eau un trajet de l'ordre de la dizaine de kilomètres. Ces deux exemples, cette thèse en contient bien d'autres, montrent que la question des échelles est omniprésente en physique. Gardons à l'esprit qu'elle concerne :

- l'*échelle temporelle*, ou le temps sur lequel les phénomènes sont observés (comportement en temps court et long des fluides viscoélastiques),
- l'*échelle spatiale*, ou la taille du phénomène observé (hétéorogénéités des matériaux composites à l'échelle microscopique, rugosités en microfluidique),

– le *régime physique*, ou l'importance relative des différents paramètres caractérisée par des nombres sans dimension (Reynolds, Rossby, Weissenberg...).

2.2.1 Matériaux composites

Un matériau composite est un assemblage de deux matériaux (un matériau matrice et un renfort) ou plus ayant des propriétés différentes. Le matériau composé a des propriétés, que séparément les matériaux mis en commun n'ont pas (légèreté, rigidité), ce qui explique leur large utilisation dans l'industrie. Les composites peuvent exister à l'état naturel : bois (lignite, renfort en fibre de cellulose), os (collagène, renforts en apatite). Cependant, la plupart des matériaux composites sont produits artificiellement, dans une optique industrielle : fibres de verre, fibres de carbone, GLAss-REinforced et Kevlar utilisés en aéronautique, béton armé.

De multiples structures sont possibles : matériau stratifié, avec des inclusions, tissé, en sandwich, en nid d'abeille. Les matériaux que l'on étudie dans cette thèse ont tous une structure périodique. Cette hypothèse ne couvre qu'une partie de la réalité physique, puisqu'il existe des composites ayant une structure non périodique (bois aggloméré, composites dentaires, epoxy granite), et qu'elle exclut la présence de défauts.

Les matériaux composites présentent une structure multi-échelles :

– l'*échelle macroscopique*, qui est celle du solide considéré dans son intégralité, par exemple un pan de mur,
– l'*échelle mésoscopique*, intermédiaire, par exemple celle d'une maille de l'ossature métallique du béton armé,
– et l'*échelle microscopique*, qui est celle à laquelle on peut observer la structure de base d'assemblage des matériaux, par exemple celle du granulat utilisé comme renfort dans la matrice de ciment.

Pour notre modélisation, on ne va prendre en compte que deux échelles, l'échelle microscopique et l'échelle macroscopique. Appelons $\varepsilon \ll 1$ le rapport de ces deux échelles. Bien-sûr, l'échelle n'est microscopique que relativement à la taille macroscopique de l'objet. Les exemples ci-dessus montrent bien que la taille absolue des microstructures peut être très variable !

Pour un composite en carbone époxy (matrice en résine époxy, renforts en fibre de carbone), utilisé par exemple dans la fabrication des mâts et coques de voiliers de compétition, l'échelle microscopique liée à l'assemblage des fibres dans la matrice est de l'ordre de 5 μm, les échelles macroscopiques sont celles de l'épaisseur d'un panneau stratifié, 1 mm à 2 cm, et de la longueur d'un voilier multicoques, 40 m (cf. [GRMC11]). Ces valeurs donnent un paramètre ε variant de l'ordre de 10^{-3} à 10^{-5}.

L'échelle microscopique est celle à laquelle on peut observer les hétérogénéités. Les variations des propriétés au niveau de la structure de base, donc sur la distance ε, se traduisent par de fortes oscillations dans les propriétés du milieu. Vu de très loin, en revanche, la structure microscopique s'efface et le matériau paraît homogène.

Équations de conservation en milieux hétérogènes

L'étude mathématique des matériaux composites remonte aux années 70, avec la publication, notamment, de l'ouvrage de référence de Bensoussan, Lions et Papanicolaou [BLP78]. Un des modèles qui a été étudié intensivement, et sur lequel une partie de cette thèse est centrée (cf. Chapitres 3 et 4) est le système elliptique linéaire sous forme divergence

$$\left\{\begin{array}{rll} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon} & = f, & x\in\Omega \\ u^{\varepsilon} & = \varphi_0, & x\in\partial\Omega \end{array}\right., \tag{2.2}$$

posée dans un ouvert borné $\Omega \subset \mathbb{R}^d$. La famille de matrices $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ indexée par $1 \leq \alpha, \beta \leq d$ représente les propriétés du milieu, $u^{\varepsilon} = u^{\varepsilon}(x) \in \mathbb{R}^N$ est une inconnue vectorielle, $f = f(x) \in \mathbb{R}^N$ est un terme source et $\varphi_0 = \varphi_0(x) \in \mathbb{R}^N$.

Les échelles microscopiques $y := \frac{x}{\varepsilon}$ et macroscopiques $x \in \Omega$ se lisent directement sur le système (2.2). C'est à l'échelle microscopique que varient les propriétés du matériau. N'étant intéressés que par l'analyse des structures périodiques, une hypothèse majeure est la périodicité de A en la variable microscopique y.

Le système (2.2) représente des phénomènes stationnaires dans des matériaux hétérogènes, en régime linéaire. Il a de nombreuses interprétations physiques :

- Il modélise la répartition de la charge électrique. Dans ce cas, $N = 1$, u^{ε} est le potentiel électrostatique, f la distribution de charges électriques, et $A = \gamma\, \mathrm{I}_d$ la conductivité électrique (isotrope dans cet exemple).
- Il modélise la répartition de la température (équation de la chaleur). Dans ce cas u^{ε} est une quantité scalaire ($N = 1$), la température, f une source de chaleur et $A = A(y) \in M_d(\mathbb{R})$ la matrice de conductivité thermique. La condition de Dirichlet $u^{\varepsilon} = \varphi_0$ impose une température sur le bord.
- Il modélise l'élasticité linéaire. Dans ce cas, $d = 2$ ou 3, $N = d$, u^{ε} est le vecteur déplacement, A est la matrice des coefficients élastiques, f représente les forces volumiques, et φ_0 les forces surfaciques.

De manière générale, le système (2.2) provient d'une équation de conservation sur un tenseur σ^{ε}

$$\nabla\cdot\sigma^{\varepsilon} = f$$

et d'une loi phénoménologique exprimant σ^{ε} comme gradient d'une autre quantité u^{ε}. Dans l'exemple de la chaleur σ^{ε} est le vecteur densité de flux de chaleur, et la loi exprimant σ^{ε} en fonction de u^{ε} est la loi de Fourier

$$\sigma^{\varepsilon} = A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}.$$

Propagation d'ondes basses fréquences en milieux hétérogènes

Intéressons-nous à ces matériaux très hétérogènes à présent plutôt dans une perspective dynamique. L'équation des ondes, hyperbolique et linéaire,

$$\left\{\begin{array}{rll} \partial_t^2 u^{\varepsilon} - \nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon} & = 0, & x\in\Omega \\ u^{\varepsilon} & = 0, & x\in\partial\Omega \end{array}\right., \tag{2.3}$$

est posée dans le domaine $(0,\infty)\times\Omega$. Un ensemble dénombrable de valeurs propres strictement positives $0 < \lambda_0^{\varepsilon} \leq \ldots \lambda_k^{\varepsilon} \leq \ldots$ et une base orthonormée de $L^2(\Omega)$ de vecteurs propres v_k^{ε} sont associés à l'opérateur $-\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla\cdot$ et au domaine Ω :

$$\left\{\begin{array}{rll} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla v_k^{\varepsilon} & = \lambda_k^{\varepsilon} v_k^{\varepsilon}, & x\in\Omega \\ v_k^{\varepsilon} & = 0, & x\in\partial\Omega \end{array}\right.. \tag{2.4}$$

La solution de (2.3) engendrée par les données initiales $u^{\varepsilon}(0,\cdot) = u_0$ et $u^{\varepsilon\prime}(0,\cdot) = u_0'$ indépendantes de ε, peut se développer sur la base de vecteurs propres :

$$u^{\varepsilon}(t,x) = \sum_{k=0}^{\infty} c_k^{\varepsilon,\pm} v_k^{\varepsilon}(x)\exp\left(\pm i\sqrt{\lambda_k^{\varepsilon}}t\right),$$

Pour tout $k \in \mathbb{N}$, les coefficients sont solutions du système

$$\left\{ \begin{array}{rl} c_k^{\varepsilon,+} + c_k^{\varepsilon,-} & = \langle u_0, v_k^\varepsilon \rangle \\ i\sqrt{\lambda_k^\varepsilon} c_k^{\varepsilon,+} - i\sqrt{\lambda_k^\varepsilon} c_k^{\varepsilon,-} & = \langle u_0', v_k^\varepsilon \rangle \end{array} \right. . \tag{2.5}$$

Les nombres $\sqrt{\lambda_k^\varepsilon}$ s'interprètent comme les fréquences propres du matériau.

Supposons que la donnée initiale soit localisée en basses fréquences. Plus précisément, on suppose que les modes $\geq K$ sont négligeables

$$\sum_{k=K}^{\infty} \left| c_k^{\varepsilon,\pm} \right|^2 \lesssim \varepsilon.$$

Comme l'opérateur est linéaire, les fréquences ne sont pas mélangées et la solution tronquée en basses fréquences fournit donc une approximation de u^ε en norme $L^\infty\left((0,\infty); L^2(\Omega)\right)$:

$$\left\| u^\varepsilon - \sum_{k=0}^{K-1} c_k^{\varepsilon,\pm} v_k^\varepsilon(x) \exp\left(\pm i\sqrt{\lambda_k^\varepsilon} t\right) \right\|_{L^\infty((0,\infty);L^2(\Omega))} \lesssim \varepsilon.$$

La question principale peut s'énoncer ainsi : peut-on remplacer la solution tronquée en basses fréquences, qui présente toujours de fortes oscillations dans la limite $\varepsilon \to 0$, par une fonction non oscillante et garder une bonne approximation, au moins pour des temps courts ? L'enjeu est de moyenner en un certain sens, ou d'*homogénéiser*, à la fois les vecteurs propres et les valeurs propres oscillants.

Soit un mode $0 \leq k < K$ fixé. Si l'on sait démontrer

$$v_k^\varepsilon = v_k^0 + O_{L^2}(\varepsilon), \tag{2.6a}$$

$$\lambda_k^\varepsilon = \lambda_k^0 + O(\varepsilon), \tag{2.6b}$$

alors, pour $c_k^{0,\pm}$ solutions de (2.5) avec λ_k^0 (resp. v_k^0) à la place de λ_k^ε (resp. v_k^ε),

$$\left\| c_k^{\varepsilon,\pm} v_k^\varepsilon \exp\left(\pm i\sqrt{\lambda_k^\varepsilon} t\right) - c_k^{0,\pm} v_k^0 \exp\left(\pm i\sqrt{\lambda_k^0} t\right) \right\|_{L^2(\Omega)} \lesssim \varepsilon$$

uniformément en t pour $t \lesssim \frac{1}{\varepsilon}$, de sorte que $c_k v_k^0 \exp\left(\pm i\sqrt{\lambda_k^0} t\right)$ est une bonne approximation du mode oscillant, en temps relativement long. Améliorer le développement (2.6b) de la valeur propre λ_k^ε permet d'étendre la validité de l'approximation sur des temps plus longs. C'est l'un des objectifs du Chapitre 3. Sur ces questions, on peut également se référer aux introductions de [AA99, SV93], ainsi qu'à [SP80].

2.2.2 Fluides géophysiques

> Such motions are present on an enormous range of spatial and temporal scales, from the ephemeral flutter of the softest breeze, to the massive and persistent oceanic and atmospheric current systems. (Pedlosky [Ped87, p. 1])

Des courants océaniques tels que le Gulf Stream dans l'océan Atlantique, ou le Kuroshio sur la côte est du Japon transportent en continu des dizaines de Sverdrup (1 Sverdrup $= 10^6$ $\mathrm{m}^3\mathrm{s}^{-1}$) d'eaux relativement chaudes, des tropiques vers les latitudes élevées de l'hémisphère nord. Les répercussions sur le climat de cette circulation, qui tend à homogénéiser les températures, sont considérables.

D'autres courants se produisent à des échelles plus petites. À la différence des deux exemples précédents qui sont des transports horizontaux, ils mettent en jeu des mouvements d'eau verticaux : phénomènes de pompage ou d'upwelling sur la côte ouest du Pérou et du Chili. La remontée d'eaux froides chargées de nutriments fait de ces régions des zones propices à la pêche.

Un des points communs de ces courants en particulier est qu'ils sont dus à la rotation de la Terre. D'autres fluides géophysiques tels que l'atmosphère ou le noyau liquide de la Terre subissent ses effets. Pour une présentation de la physique des fluides tournants, pas uniquement géophysiques, on renvoie le lecteur à [Gre69]. Pour un exposé centré sur les fluides géophysiques, on peut se référer à [Ped87], en particulier aux chapitres 1 et 4. Pour une introduction aux thématiques de l'océanographie, on pourra lire [Lac71].

Des fluides en rotation rapide

Reprenons la remarque de [CDGG06, p. 1]. Une particule d'eau prise dans le Gulf Stream met environ 50 jours pour traverser l'Atlantique et parcourir 5000 km. Pendant ce temps, la Terre tourne 50 fois. L'effet de la rotation de la Terre est donc sensible pour ce type d'écoulements caractérisés par leur grande extension spatiale L (plusieurs milliers de kilomètres) et leur vitesse relativement faible U (quelques mètres par seconde). Ainsi, la rotation terrestre ne peut être négligée lorsque la durée de l'écoulement (suivant une particule) est grande devant la durée d'une journée, c'est-à-dire

$$\frac{L}{U} \gg \Omega^{-1}$$

où $\Omega = 7,3 \cdot 10^{-5}$ s^{-1} est la fréquence de rotation de la Terre. Le paramètre pertinent apparaît être le nombre de Rossby défini par

$$\varepsilon := \frac{U}{2\Omega L}.$$

Dans l'exemple précédent du Gulf Stream, ε est de l'odre de 10^{-3}, ce qui montre que l'écoulement est fortement influencé par la rotation. Dans le cas de l'anticyclone des Açores pour lequel U (resp. L) est de l'ordre de 10 ms^{-1} (resp. 1000 km), le nombre de Rossby vaut 10^{-1}. Dans le noyau de fer liquide de la Terre, on peut trouver des valeurs de ε bien plus faibles de l'ordre de 10^{-7} (cf. [Dor97, Appendice A]). Des fluides à petit nombre de Rossby sont dits en rotation rapide.

Comme le souligne Ekman [Ekm05] l'influence de la rotation de la Terre sur les écoulements géophysiques a été comprise depuis longtemps, notamment par Hadley (cellules de Hadley pour la circulation atmosphérique tropicale), Coriolis et Ferrel (cellules de Ferrel pour les latitudes moyennes). Bjerknes [Bje02] a montré son importance par rapport à d'autres causes possibles de mouvement telles que des différences de densité. En première approximation, les fluides que l'on considère sont des fluides visqueux incompressibles et en rotation rapide. Tous les autres effets dus à des différences de température, de salinité... sont ignorés.

Sous forme non adimensionnée, la dynamique de l'écoulement est régie par le système de Navier-Stokes-Coriolis (voir [Ped87, Chapitre 1] pour une dérivation complète de ces équations)

$$\left\{ \begin{aligned} \partial_{t^*} u^* + u^* \cdot \nabla u^* + 2\Omega e_3 \times u^* &= -\tfrac{\nabla p^*}{\rho^*} + \nabla \Phi^* + \nu \Delta u^* + f^*, \\ \nabla \cdot u^* &= 0, \end{aligned} \right. \tag{2.7}$$

où $u^* = u^*(t^*, x^*)$ est la vitesse, f^* est un terme de forçage soutenant l'écoulement, ν la viscosité du fluide, ρ^* sa densité et e_3 est le vecteur directeur de l'axe de rotation de la

Terre (pôle sud/pôle nord). Outre les forces inertielles $u^* \cdot \nabla u^*$ et visqueuses $-\nu \Delta u^*$, cette description prend en compte deux forces directement liées à la rotation :

- l'accélération de Coriolis $2\Omega e_3 \times u^*$,
- l'accélération centripète $\nabla \Phi^*$ qui dérive d'un potentiel.

La force de Coriolis et l'accélération centripète sont des forces fictives, dont l'origine est le caractère non inertiel (ou non galiléen) du référentiel lié à la Terre.

Donnons quelques ordres de grandeur. Le terme $u^* \cdot \nabla u^*$ est d'ordre U^2/L ; la force de Coriolis $2\Omega e_3 \times u^*$ est d'ordre $2\Omega U$. Le rapport entre accélération relative et accélération de Coriolis vaut donc

$$\frac{U^2/L}{2\Omega U} = \frac{U}{2\Omega L} = \varepsilon.$$

Le nombre de Rossby est donc une mesure de l'importance relative de ces deux forces. Pour des écoulements géophysiques à large échelle, l'accélération relative est négligeable devant l'accélération de Coriolis. Par ailleurs, la comparaison des forces de friction d'ordre $\nu U/L^2$ à l'accélération de Coriolis conduit à introduire le nombre d'Ekman E

$$\frac{\nu U/L^2}{2\Omega U} = \frac{\nu}{2\Omega L^2} =: E.$$

En introduisant les quantités adimensionnées $t = t^* \frac{U}{L}$, $x = \frac{x^*}{L}$, la vitesse $u(t,x) := u^*(t^*, x^*)$ satisfait

$$\left\{ \begin{array}{rl} \partial_t u + u \cdot \nabla u - \frac{E}{\varepsilon} \Delta u + \frac{1}{\varepsilon} e_3 \times u + \frac{1}{\varepsilon} \nabla p & = f \\ \nabla \cdot u & = 0 \end{array} \right. . \tag{2.8}$$

Le nouveau potentiel p représente la pression du fluide ainsi que l'accélération centripète. Si l'on compare (2.8) au système de Navier-Stokes il apparaît que le seul terme changeant la structure de l'équation est le terme antisymétrique $\frac{1}{\varepsilon} e_3 \times u$. La singularité $\frac{1}{\varepsilon}$ montre la prépondérance de la force de Coriolis dans la balance des forces. Elle est à l'origine de nombreuses particularités des écoulements géophysiques à grande échelle.

Dans tout ce qui suit, nous ne prenons pas en compte la courbure de la Terre dans le choix du domaine. Les conclusions ne seront jamais valables au voisinage de l'équateur, puisqu'à l'équateur la force de Coriolis s'annule. D'autres mécanismes physiques sont à l'œuvre dans les régions équatoriales. *Pour simplifier, nous considérons un domaine voisin des pôles*, bien que les résultats puissent être étendus à des régions subpolaires, notamment aux latitudes moyennes 30–60 degrés. La direction verticale du domaine coïncide donc avec l'axe de rotation de la Terre.

Nos travaux sur ce sujet (cf. section 2.4.4 et Chapitre 5) sont dédiés à l'étude du système (2.8) linéarisé. Bien-sûr le système linéaire ne capture pas toute la physique de l'écoulement. Cependant, la *dynamique des fluides tournants en régime linéaire* est riche, et des phénomènes physiques intéressants sont liés à l'interaction entre les forces de Coriolis, les forces visqueuses et les rugosités des bords horizontaux.

Viscosité turbulente

Comparées aux forces de Coriolis et de pression, les forces de friction agissant sur les écoulements géophysiques à grande échelle sont faibles. La persistance des grands courants marins ou systèmes de vents sur des temps très longs en témoigne. Pourtant, aussi faible soit-elle, l'interaction entre les forces de frottement et la force de Coriolis est à l'origine de phénomènes intéressants.

La dissipation d'énergie se fait à l'échelle moléculaire. Une des tâches majeures de la modélisation en mécanique des fluides géophysiques est de comprendre comment l'écoulement principal nourrit les mouvements turbulents de taille plus petite. En d'autres termes, il s'agit de quantifier la cascade d'énergie des grandes échelles aux petites échelles et in fine de représenter la dissipation en terme de la vitesse macroscopique. Il s'agit donc d'un problème d'homogénéisation très complexe, qui fait toujours l'objet de recherches actives : voir entre autre [DS12] pour la modélisation de la turbulence en général.

Une façon rudimentaire de tenir compte de l'existence de ces phénomènes dissipatifs et de fermer les équations du mouvement est de supposer l'existence d'une viscosité turbulente. Les forces de friction s'expriment alors sous la forme d'une viscosité anisotrope

$$-\nu_h \Delta_h u^* - \nu_v \partial_3^2 u^*$$

où ν_h (resp. ν_v) est une viscosité turbulente horizontale (resp. verticale).

L'hypothèse d'isotropie du modèle (2.7) est discutable (de même que la représentation de la cascade d'énergie par la viscosité turbulente), mais cruciale dans l'analyse mathématique du Chapitre 5. De plus, la détermination de la valeur de ces viscosités turbulentes empiriques est difficile. Pour l'océan et l'atmosphère on estime que $\nu_v \ll \nu_h$ (cf. [Ped87, Chapitre 4] pour des plages de valeurs de ν_h et ν_v). La viscosité isotrope semble plus pertinente pour le noyau liquide de la Terre [Dor97, Appendice A]. Pour ce dernier $E \simeq 10^{-15} = O\left(\varepsilon^2\right)$ et sauf indication contraire, on suppose dans la suite que le nombre d'Ekman est de cet ordre :

$$E \simeq \varepsilon^2.$$

Quelques particularités des fluides tournants (en régime linéaire)

> During the drift of the Fram, Nansen observed that the ice was not carried along in the direction in which the wind blew but that the ice drift deviated, on an average, 28 degrees to the right of the direction towards which the wind was blowing. [...] [Nansen] arrived at the conclusion that the deviation of the ice drift must be ascribed to the effect of the rotation of the earth, which in the northern hemisphere tends to deflect any moving body to the right. Nansen reasoned further that the ice would set the water directly underneath it in motion and that the water would then move to the right in relation to the ice drift [...]. Similar conditions could be expected in the open sea [...]. (Sverdrup [Sve50, p. 181–182])

Les deux termes dominants dans (2.8) sont l'accélération de Coriolis et les forces de pression. À l'ordre principal en ε, ces deux forces sont donc en balance

$$e_3 \times u^0 = -\nabla p^0, \tag{2.9}$$

ce qui constitue l'équilibre géostrophique. On peut lire de nombreuses propriétés de l'écoulement principal sur (2.9) :

direction des vents La direction du vent géostrophique est orthogonale au gradient de pression (ou encore tangente aux lignes isobares).

colonnes de Taylor-Proudman De (2.9) on déduit $0 = \partial_3 p^0$, $u_2^0 = \partial_1 p^0$, $u_1^0 = -\partial_2 p^0$. Il s'ensuit d'une part que p^0 est indépendant de x_3, donc que u_1^0 et u_2^0 le sont aussi, et d'autre part que $\partial_1 u_1^0 + \partial_2 u_2^0 = 0$. Comme l'écoulement est incompressible, $\partial_3 u_3^0 = 0$, donc u_3^0 est indépendant de x_3. Une conclusion (partielle) est que l'écoulement dominant est bidimensionnel et est déterminé par sa valeur sur le bord inférieur ou supérieur du domaine.

La deuxième observation semble contredire certains faits : le fond des océans qui n'est pas plat doit forcer (au moins au voisinage du fond) une variation de la vitesse dans la direction verticale. De plus la circulation dans les océans n'est pas la même à toutes les profondeurs (courants de surface, courants profonds, zones d'upwelling). Ainsi l'équilibre géostrophique n'est valable que loin des bords, et près des bords horizontaux (fond et surface des océans...) l'asymptotique formelle doit être affinée.

Faisons une analyse sommaire de ce qui se passe au voisinage de la surface (supposée plate) d'un océan de profondeur infinie sur lequel souffle le vent. On se focalise sur une fine couche de taille $O(\varepsilon)$ près du bord supérieur, ce qui revient à figer la variable tangentielle x_h :

$$\tilde{u} = \tilde{u}(x_h, z) := u(x_h, \varepsilon z), \qquad \tilde{p} = \tilde{p}(x_h, z) = p(x_h, \varepsilon z).$$

En analysant les ordres des termes dans (2.8), on est amené à chercher une solution particulière $\tilde{u} = \tilde{u}(z)$ et $\tilde{p} = \tilde{p}(z)$ satisfaisant le système d'équations différentielles pour $z < 0$,

$$\left\{ \begin{array}{ll} -\frac{d^2}{dz^2}\tilde{u}_1 - \tilde{u}_2 & = 0 \\ -\frac{d^2}{dz^2}\tilde{u}_2 + \tilde{u}_1 & = 0 \end{array} \right. .$$

Loin du bord, pour $z \to -\infty$, on impose que $\tilde{u}_1,\ \tilde{u}_2$ tendent vers 0. Le forçage par un vent dirigé selon la direction de x_2 se traduit par la condition sur le bord $z = 0$:

$$\frac{d}{dz}\tilde{u}_1(0) = 0, \qquad \frac{d}{dz}\tilde{u}_2(0) = T.$$

Alors,

$$\left\{ \begin{array}{ll} \tilde{u}_1(z) & = \frac{T}{2}\left[\cos\left(\frac{\sqrt{2}}{2}z\right) - \sin\left(\frac{\sqrt{2}}{2}z\right)\right]\exp\left(\frac{\sqrt{2}}{2}z\right) \\ \tilde{u}_2(z) & = \frac{T}{2}\left[\cos\left(\frac{\sqrt{2}}{2}z\right) + \sin\left(\frac{\sqrt{2}}{2}z\right)\right]\exp\left(\frac{\sqrt{2}}{2}z\right) \end{array} \right. .$$

La solution $\tilde{u}$ est la spirale d'Ekman. Deux intégrations par parties donnent

$$\int_{-\infty}^{0} \tilde{u}_1(z) = \frac{T}{\sqrt{2}}, \qquad \int_{-\infty}^{0} \tilde{u}_2(z) = 0.$$

Le transport moyen dans la direction du vent (direction de x_2) est nul. En moyenne, sur toute la couche, le transport se fait dans la direction orthogonale à celle du vent. Juste sous la surface, $\tilde{u}_1(0^-) = \frac{T}{2} = \tilde{u}_2(0^-)$, le transport du fluide se fait dans une direction faisant un angle de 45 degrés avec celle du vent. Ces faits contre-intuitifs font partie des conclusions fondamentales d'Ekman [Ekm05] et donnent une première explication aux observations de Nansen, rapportées par Sverdrup. Ce *transport d'Ekman* par frottement du vent sur la surface est une conséquence de l'interaction des forces visqueuses et de Coriolis. Il s'agit d'un phénomène purement linéaire.

Il apparaît donc que la viscosité est importante dans une fine couche de taille caractéristique ε au voisinage des bords horizontaux du domaine, la *couche d'Ekman*, dans laquelle elle peut équilibrer la force de Coriolis. L'étude de cette couche, dans le régime linéaire, est l'objet du Chapitre 5.

Topographie

> Flow in the benthic boundary layer may be influenced indirectly by bottom features, such as sand waves, on a scale of tens or hundreds of metres [...]. (Bowden [Bow78, p. 262])

L'expérience quotidienne nous apprend que la surface terrestre, loin d'être plate, est composée de structures de tailles caractéristiques variables. Ces *rugosités* peuvent être des vallées et des montagnes, des bâtiments, des forêts... qui ont un impact sur la circulation atmosphérique. La connaissance de la structure du fond des océans, en revanche, n'est pas si ancienne et remonte à la pose des premiers câbles sous-marins. On doit à Heezen, Tharp et Ewing [HTE59] la première carte du fond de l'Atlantique nord (voir aussi Heezen et Hollister [HH71]). L'étude de la bathymétrie fait apparaître un relief très riche (cf. [Lac71] Chapitre 1-IV) : vallées et canyons sous-marins qui entaillent le talus continental, plaines abyssales, dorsales. Ces dernières jouent un rôle hydrologique majeur puisqu'elles gênent les échanges entre les bassins. Enfin, des travaux très récents [NLMSS01, LMNGLH06] mettent en évidence la transition brutale entre le noyau liquide de la Terre et le manteau, ainsi que des structures irrégulières à l'interface entre les deux milieux de taille caractéristique 1 m.

Les rugosités sont susceptibles d'engendrer des instabilités et des phénomènes turbulents. Il est particulièrement important d'étudier leur impact lorsqu'elles sont de même taille caractéristique que la couche limite visqueuse, ou couche d'Ekman. Ceci est notamment le cas à la frontière entre le noyau liquide de la Terre et le manteau. Pour l'océan profond, la couche limite d'Ekman au voisinage du fond a une taille caractéristique de 10 m (cf. [Bow78, Table 1]).

Les premiers travaux mathématiques ont étudié la couche limite visqueuse à proximité soit d'un bord plat (Grenier et Masmoudi [GM97]), soit d'un relief périodique (Gérard-varet [GV03]). Notre objectif dans le Chapitre 5 est de s'affranchir de cette hypothèse pour traiter des profils de rugosités sans structure. L'impact de ce type d'études est important, puisqu'il permet de mieux s'approcher de la réalité physique.

2.2.3 Fluides viscoélastiques

> In essence, Newtonian fluids are fluids whose constituent particles are too small for their dynamics to interact substantially with the macroscopic motion. The Newtonian model becomes inadequate for fluids which have a microstructure involving much larger scales than the atomic scale. (Renardy [Ren00, p. 1])

Les bases de la théorie des fluides viscoélastiques, ayant des propriétés élastiques en temps courts et se comportant comme des fluides visqueux en temps longs, ont été jetées par Maxwell dans l'article [Max67]. Il y affirme que tous les fluides sont viscoélastiques, tout dépendant bien-sûr de l'échelle de temps que l'on considère ! Les fluides pour lesquels les propriétés élastiques sont observables sont omniprésents : les liquides physiologiques (sang, salive), certains fluides géophysiques (glaciers, manteau terrestre), en cuisine (mélange eau/Maïzena, pâte levée), dans l'industrie (gels, mousses, solutions aqueuses de polymères, huiles silicones, peintures).

Le paramètre déterminant le caractère viscoélastique d'un fluide est le *temps de relaxation viscoélastique* séparant les régimes élastiques et visqueux. Pour l'eau ce temps est de l'ordre de 10^{-12} s, ce qui rend l'observation des propriétés élastiques impossible. De ce fait, l'eau est un fluide purement visqueux. Pour des huiles silicones, le temps caractéristique des propriétés élastiques est de l'ordre d'une demi seconde. La glace se comporte élastiquement pendant quelques heures, le manteau terrestre sur des durées de l'ordre de 10^{10} s (soit plusieurs centaines d'années !). Ces valeurs ne sont que des ordres de grandeurs, et la plupart des fluides réels n'ont pas un temps de relaxation aussi bien défini.

L'exemple d'une boule en pâte à modeler Silly Putty, qui peut rebondir plusieurs fois sur le sol comme une balle de ping-pong (comportement élastique) avant de s'étaler pendant plusieurs jours comme le ferait une goutte d'eau (comportement visqueux) met en évidence la dynamique riche des fluides viscoélastiques.

Pour une introduction à la rhéophysique, et aux fluides non-newtoniens en général, on pourra consulter le livre de Oswald [Osw05]. On pourra aussi se référer au livre de Renardy [Ren00], ainsi qu'au texte de Le Bris et Lelièvre [LBL09].

Quelques phénomènes contre-intuitifs

Exposons brièvements quelques effets dus à l'élasticité. Ces effets sont surprenants, puisque nous sommes peu habitués à prêter attention aux propriétés élastiques des fluides. On peut pourtant en faire souvent l'expérience. Le premier effet concerne un fluide (une pâte levée par exemple) dans lequel plonge une tige cylindrique verticale (le pétrin d'un fouet électrique). Si l'on fait tourner cette tige rapidement sur elle-même, on observe la montée du fluide le long de la tige. En l'absence d'élasticité, le fluide s'écarterait de la tige en rotation sous l'effet de l'accélération centrifuge. Dans le fluide viscoélastique, le cisaillement dû à la rotation rapide de la tige étire les molécules longues (par exemple les molécules de gluten), qui en retour exercent une force élastique tendant à les rapprocher de la tige.

Le deuxième effet porte le nom de siphon sans tube. Dès que l'on sort un siphon d'un liquide newtonien (comme de l'eau), la colonne d'eau s'effondre sous son propre poids et le siphonnage s'interrompt. Dans un fluide suffisamment viscoélastique, en revanche, l'effet continue même lorsque le siphon est hors du fluide. L'élasticité est à nouveau fondamentale : les particules de fluide sont étirées et la tension élastique suffit à équilibrer l'effet de la pesanteur. Cet effet se produit aussi dans un pot de peinture, qui remplit à ras bord, s'écoule par un mince filet. L'écoulement persiste, même lorsque le pot de peinture s'est partiellement vidé.

Pour de nombreux autres effets, on peut voir le chapitre introductif de [Ren00]. L'élasticité peut avoir des effets stabilisants, ou conduire à de nouvelles instabilités : pour un lien entre les instabilités magnétohydrodynamiques et viscoélastiques voir [OP03]. La littérature physique est très riche : sur la turbulence des fluides viscoélastiques [GS00, CK00, CBG01, AK02, Kel04], sur la réduction de la traînée [MCY03], et sur la réduction de la traînée en présence de rugosités [Vir70, YD10]. Ces questions ont des répercussions importantes sur certains processus industriels. Injecter quelques particules de polymères par million de particules d'eau suffit à rendre un écoulement turbulent proche du laminaire et donc à diminuer la perte d'énergie de l'écoulement principal par dissipation vers les échelles plus petites. La compréhension mathématique de certains de ces phénomènes est un des objectifs de nos travaux futurs.

Élasticité, viscosité, viscoélasticité

En rhéophysique on exprime la contrainte τ^* (force surfacique ou volumique) en fonction des déformations du solide/liquide. Décrivons l'élasticité et la viscosité de manière schématique.

Un solide élastique (en régime linéaire) soumis à une contrainte se déforme (i.e. ses constituants subissent un déplacement relativement les uns par rapport aux autres) de façon proportionnelle à la force (ou contrainte) appliquée. Ceci constitue la *loi de Hooke*. Pour un matériau isotrope, le rapport entre la contrainte et la déformation est le module élastique noté G. Si la contrainte est supprimée, le matériau retrouve sa configuration initiale, à condition de rester en-deçà de sa limite d'élasticité. Un solide élastique garde donc la mémoire de sa configuration initiale. Les phénomènes purement élastiques sont donc réversibles.

La viscosité correspond à la réponse d'un fluide à un gradient de vitesse. Une couche de fluide rapide frottant sur une autre plus lente crée une contrainte de cisaillement. Cette contrainte (en régime linéaire) est proportionnelle au taux de cisaillement, ou gradient de vitesse, ce qui constitue la *loi de Newton*. Le facteur de proportionnalité est la viscosité dynamique η. Un fluide purement visqueux, en d'autres termes un fluide newtonien, n'a pas de mémoire. La viscosité introduit l'irréversiblité.

Une première approche très simple, mais éclairante, de la viscoélasticité est celle de Maxwell. Il fait l'hypothèse que les taux de déformations élastiques et visqueux du fluide se superposent, conduisant à l'équation

$$\dot{\tau}^* + \frac{G}{\eta}\tau^* = G\left[\nabla u^* + (\nabla u^*)^T\right] \tag{2.10}$$

où $\dot{}$ désigne la dérivée temporelle, u^* est la vitesse du fluide et $D(u^*) := \frac{\nabla u^* + (\nabla u^*)^T}{2}$ est le tenseur des déformations ; toutes les grandeurs étoilées sont dimensionnées. Le quotient $\lambda := \eta/G$ a la dimension d'un temps ; c'est le temps de relaxation viscoélastique. Comme le souligne Renardy [Ren00, p. 14], ce temps est grosso modo le temps sur lequel le fluide se souvient de son histoire. Quand $\lambda \gg 1$, le fluide est purement élastique ; quand $\lambda \ll 1$, le fluide est uniquement visqueux. La solution générale de (2.10) est

$$\tau^*(t^*, x^*) = \exp\left(-\frac{t^*}{\lambda}\right)\tau_0^* + 2\frac{G}{\lambda}\int_0^{t^*} \exp\left(-\frac{t^* - t'}{\lambda}\right) D(u^*)(t')dt'$$

ce qui met en lumière le fait que les contraintes s'exerçant sur un fluide à un instant t^*, tiennent compte de toute l'histoire de l'écoulement.

Enfin, les exemples de fluides plus haut montrent que l'on peut ranger les fluides viscoélastiques en deux catégories :

- les fluides uniquement viscoélastiques, dits *fluides de Maxwell* : glaciers, manteau terrestre, huiles silicones, pâtes levées ;
- les fluides pour lesquels une réponse viscoélastique se superpose à une réponse newtonienne, dits *fluides de Jeffrey* : dans cette catégorie se classent toutes les solutions aqueuses de polymères suffisamment diluées.

Quelques modèles macro-macro

L'étude de ces fluides pose des questions complexes de modélisation multi-échelles. Il s'agit notamment de comprendre comment les microstructures, responsables du comportement élastique, interagissent avec le mouvement macroscopique du fluide (sur cette question voir [LBL12]). Ici on considère des modèles ne mettant en jeu que des grandeurs macroscopiques du fluide : modèles macro-macro.

Présentons quelques modèles que nous étudions dans le Chapitre 6. Ils couplent une équation des moments sur le champ de vitesse macroscopique incompressible u^*,

$$\rho\left(\partial_{t^*} u^* + u^* \cdot \nabla u^*\right) = -\nabla p^* + \nabla \cdot \tau^* + f^*, \qquad \nabla \cdot u^* = 0$$

et une équation sur τ^*. Le tenseur des contraintes macroscopique (matrice symétrique), est la somme d'une partie visqueuse due au solvant $\tau_s^* := 2\eta_s D(u^*)$, nulle dans le cas d'un fluide de Maxwell et d'une partie viscoélastique τ_p^*. Cette dernière est solution d'une équation analogue à (2.10), où la dérivée temporelle notée $\dot{}$ est remplacée par une loi différentielle constitutive plus générale

$$\mathcal{D}_{t^*}\tau_p^* + \frac{1}{\lambda}\tau_p^* = 2\eta_p D(u^*)\,.$$

Pour être physiquement acceptable, la loi doit satisfaire une condition d'indifférence matérielle (principe d'objectivité). Beaucoup de lois sont de la forme

$$\mathcal{D}_{t^*}\tau^* := \partial_{t^*}\tau^* + u^*\cdot\nabla\tau^* + \tau^* W(u^*) - W(u^*)\tau^* - a\left[D(u^*)\tau^* + \tau^* D(u^*)\right] + \beta(\tau^*), \quad (2.11)$$

où $W(u^*) = \frac{\nabla u^* - (\nabla u^*)^T}{2}$, $-1 \leq a \leq 1$ et β est une fonction (non-linéaire).

L'absence de non-linéarité ($\beta = 0$) correspond à la *loi de Johnson-Segalman*. Le cas $a = -1$ (dérivée convectée inférieure) est peu réaliste physiquement. Deux cas particuliers sont très étudiés :

– Le cas $a = 1$ (dérivée convectée supérieure)

$$\tau^* W(u^*) - W(u^*)\tau^* - \left[D(u^*)\tau^* + \tau^* D(u^*)\right] = -\nabla u^* \tau^* - \tau^* (\nabla u^*)^T.$$

Lorsque $\eta_s = 0$ (resp. $\eta_s \neq 0$) il s'agit du *Upper Convected Maxwell fluid* (resp. du modèle d'*Oldroyd-B*). Ce modèle peut-être dérivé d'une description microscopique.

– Le cas $a = 0$ (modèle corotationnel).

Les seules non-linéarités de ces modèles ($\beta = 0$) sont d'origine cinématique. Ils conduisent à des prédictions plus ou moins en accord avec les expériences. Pour être prédits (exemple du comportement rhéofluidifiant, c'est-à-dire diminution de la viscosité lorsque le taux de cisaillement augmente), certains phénomènes nécessitent de prendre en compte des non-linéarités du matériau. Parmi les nombreux modèles citons celui de Phan-Thien et Tanner avec $\beta = \alpha\frac{\lambda}{\eta_p}\tau^* \operatorname{Tr}\tau^*$, ou celui de Giesekus $\beta = -\alpha\frac{\lambda}{\eta_p}\tau^{*2}$. On trouve dans [Ren00, Chapter 3] ainsi que dans [TS95] une discussion de certains modèles.

On pose $\eta := \eta_s + \eta_p$ et $\omega := \frac{\eta_p}{\eta}$. Pour une vitesse (resp. taille) typique U (resp. L) de l'écoulement, on introduit les grandeurs adimensionnées

$$x = \frac{x^*}{L}, \quad t = \frac{U}{L}t^*, \quad u = \frac{u^*}{U}, \quad \tau = \frac{U}{\eta L}\tau^*.$$

Le nombre de Reynolds

$$\mathrm{Re} := \frac{\rho U L}{\eta}$$

est une mesure de l'importance des effets inertiels par rapport aux forces visqueuses. Le *nombre de Weissenberg* (ou nombre de Deborah)

$$\mathrm{We} := \frac{\lambda U}{L}$$

compare le temps de relaxation viscoélastique à l'échelle de temps caractéristique de l'écoulement. Lorsque $\mathrm{We} \ll 1$, les phénomènes élastiques ne sont pas observables. Sous forme adimensionnée, les modèles décrits ci-dessus s'écrivent

$$\left\{\begin{aligned} \mathrm{Re}\left(\partial_t u + u\cdot\nabla u\right) - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau, \\ \nabla\cdot u &= 0, \\ \mathrm{We}\left(\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau - a\left[D(u)\tau + \tau D(u)\right] + \tilde{\beta}(\tau)\right) + \tau &= 2\omega D(u). \end{aligned}\right. \quad (2.12)$$

Notons que cette modélisation n'englobe même pas tous les modèles macro-macro étudiés au Chapitre 6 (cf. modèle de FENE-P). Le but de nos travaux est l'étude de la limite newtonienne de certains fluides viscoélastiques, i.e. la limite $\mathrm{We} \to 0$ et $\mathrm{Re} = O(1)$ fixé.

2.3 Analyse multi-échelles et couches limites

Présentons ici quelques développements formels permettant de dériver les équations satisfaites à la limite $\varepsilon \to 0$. Pour être pertinents, les termes des développements doivent

– tenir compte de la microstructure du problème (hétérogénéités ou rugosités),
– être compatibles avec les conditions au bord.

Pour ces raisons, on est souvent conduit à séparer la question de l'asymptotique en un problème à l'intérieur du domaine, et un problème au voisinage du bord.

2.3.1 Homogénéisation et couches limites

Il y a dans le système elliptique (2.2) à coefficients oscillants deux échelles caractéristiques : l'échelle macroscopique x et l'échelle microscopique représentée par la variable rapide $y = \frac{x}{\varepsilon}$. Pour prendre en compte la périodicité de la microstructure, on peut supposer le développement double-échelle

$$u^\varepsilon = u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^2\left(x, \frac{x}{\varepsilon}\right) + \dots$$

les profils $u^i = u^i(x,y)$ étant périodiques en la seconde variable. En injectant cet Ansatz dans le système (2.3.1) et en identifiant les puissances de ε, on obtient de manière classique la cascade d'équations (cf. [BLP78, CD99])

$$\begin{aligned}
-\nabla_y \cdot A(y)\nabla_y u^0 &= 0,\\
-\nabla_y \cdot A(y)\nabla_y u^1 &= \nabla_x \cdot A(y)\nabla_y u^0 + \nabla_y \cdot A(y)\nabla_x u^0,\\
&\vdots\\
-\nabla_y \cdot A(y)\nabla_y u^{i+2} &= \mathcal{F}\left(u^i, u^{i+1}\right).
\end{aligned}$$

Le cadre périodique permet de calculer explicitement les correcteurs. La première équation entraîne l'indépendance de u^0 par rapport à la variable y

$$u^0(x,y) := \bar{u}^0(x).$$

La deuxième permet par séparation des variables x et y d'exprimer u^1 en fonction de la solution $\chi = \chi^\beta(y) \in M_N(\mathbb{R})$ d'un *problème cellulaire* auxiliaire

$$-\nabla_y \cdot A(y)\nabla_y \chi^\beta = \partial_{y_\alpha} A^{\alpha\beta}, \qquad \int_{\mathbb{T}^d} \chi^\beta = 0. \tag{2.13}$$

Le correcteur à l'ordre 1 est donc

$$u^1(x,y) = \chi^\alpha(y)\partial_{x_\alpha}\bar{u}^0 + \bar{u}^1(x).$$

D'une condition de compatibilité sur la troisième équation découle que $\bar{u}^0$ est solution de

$$\left\{\begin{aligned} -\nabla \cdot \overline{A}\nabla \bar{u}^0 &= f, \quad x \in \Omega\\ \bar{u}^0 &= \varphi_0, \quad x \in \partial\Omega \end{aligned}\right. \tag{2.14}$$

où $\overline{A} := \int_{\mathbb{T}^d} A^{\alpha\beta} + A^{\alpha\gamma}\partial_{y_\gamma}\chi^\beta$. On a ainsi dérivé formellement l'équation *homogénéisée*. Notons que le système (2.14) est elliptique. De plus les coefficients de $\overline{A}$ ne sont pas simplement la moyenne des coefficients de A, ce qui indique (si cette asymptotique formelle est la bonne !) des phénomènes non triviaux dans le passage à la limite.

À première vue, cette asymptotique formelle est très bonne à l'intérieur du domaine Ω : en ajoutant des correcteurs on a

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla\left(u^\varepsilon - \bar{u}^0 - \varepsilon u^1(x,y) - \varepsilon^2 u^2(x,y) \ldots - \varepsilon^i u^i(x,y)\right) = O_{L^2}\left(\varepsilon^{i-1}\right).$$

Dans le cas d'un domaine sans bord (le tore), une simple estimation d'énergie valide ce développement. Mais dans le cas d'un bord, un problème surgit. La trace du développement à l'ordre 1

$$u^\varepsilon - \bar{u}^0 - \varepsilon\left(\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla\bar{u}^0 + \bar{u}^1\right)\Big|_{\partial\Omega} = -\,\varepsilon\left(\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla\bar{u}^0 + \bar{u}^1\right)\Big|_{\partial\Omega} \tag{2.15}$$

est d'amplitude ε. De plus, elle est a priori fortement oscillante, et ces oscillations le long du bord n'ont pas de raison d'être périodiques en général. Tout dépend de la manière dont le bord $\partial\Omega$ intersecte la microstructure périodique. Corriger cette erreur est un enjeu majeur de cette thèse.

L'objectif est de mieux comprendre le comportement de u^ε près du bord, ce qui nécessite d'introduire la couche limite $u_{bl}^{1,\varepsilon}$ corrigeant la partie oscillante de (2.15). La couche limite est solution du système elliptique

$$\begin{cases} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u_{bl}^{1,\varepsilon} &= 0, \qquad\qquad x\in\Omega \\ \qquad\qquad u_{bl}^{1,\varepsilon} &= -\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla\bar{u}^0, \quad x\in\partial\Omega \end{cases} \tag{2.16}$$

à coefficients et donnée sur le bord oscillants. Trouver, même formellement, le système homogénéisé de (2.16) est un problème bien plus difficile que celui de trouver la limite de u^ε. La raison principale est que les oscillations le long du bord, non périodiques en général, forcent les oscillations de la solution $u_{bl}^{1,\varepsilon}$ au voisinage du bord. En faisant l'hypothèse que les fortes oscillations de $u_{bl}^{1,\varepsilon}$ se concentrent dans une couche, voisine du bord, de taille ε, on approche $u_{bl}^{1,\varepsilon}$ par $v_{bl}\left(x,\frac{x}{\varepsilon}\right)$. Comme le bord rompt la microstructure périodique, la périodicité par rapport à la variable rapide ne fait plus sens. Pour $x_0\in\partial\Omega$, $v_{bl} = v_{bl}(x_0,y)$ est alors formellement solution du système

$$\begin{cases} -\nabla_y\cdot A(y)\nabla_y v_{bl} &= 0, \qquad\qquad y\cdot n(x_0) > \frac{c_{x_0}}{\varepsilon} \\ \qquad\qquad v_{bl} &= -\chi(y)\cdot\nabla\bar{u}^0, \quad y\cdot n(x_0) = \frac{c_{x_0}}{\varepsilon} \end{cases}, \tag{2.17}$$

dans lequel x_0 est un paramètre et $x\cdot n(x_0) - c_{x_0} = 0$ est l'équation de la tangente à $\partial\Omega$ en x_0. L'étude de (2.17) est un problème clé dans l'homogénéisation du système de couche limite (2.16).

2.3.2 Une couche limite en géophysique : la couche d'Ekman

Les motivations physiques évoquées plus haut pour l'étude mathématique du système des fluides tournants (2.8) dans un domaine à bords horizontaux rugueux sont particulièrement fortes. Il est notamment important de déterminer l'influence sur l'écoulement des structures de taille comparable à celle de la couche limite visqueuse. On se place donc dans le domaine rugueux, non borné dans la direction tangentielle,

$$\Omega^\varepsilon := \left\{(x_h,x_3),\ x_h\in\mathbb{R}^2,\ \varepsilon\omega\left(\frac{x_h}{\varepsilon}\right) < x_3 < 1\right\}.$$

La fonction ω, bornée, est un profil de rugosité pour le moment complètement arbitraire. On impose une condition de non-glissement sur la vitesse u au niveau du bord rugueux.

Pour simplifier, on ne se préoccupe que du bord inférieur ; les même questions se posent bien-sûr au niveau du bord $x_3 = 1$.

Analysons formellement l'asymptotique du système de Navier-Stokes-Coriolis (2.8) dans le domaine Ω^ε. On a vu qu'à l'ordre principal on a l'équilibre géostrophique

$$e_3 \times u^0 = \nabla p^0.$$

Cette égalité impose la bidimensionnalité du champ de vitesses u^0, qui est incompatible avec la condition de non-glissement. Il faut donc corriger ce développement au moins au près du bord pour faire le lien entre la condition de non-glissement et l'écoulement géostrophique loin du bord. Les remarques physiques, en mettant en évidence l'existence d'une fine couche de fluide dans laquelle les forces visqueuses balancent la force de Coriolis, renforcent cette idée.

Développons suivant l'échelle macroscopique x et l'échelle microscopique $y = \frac{x}{\varepsilon}$, en séparant termes intérieurs et termes de couche limite

$$\begin{aligned} u^\varepsilon &= u^0(t,x) + u^0_{bl}\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon\left(u^1(t,x) + u^1_{bl}\left(t,x,\frac{x}{\varepsilon}\right)\right) + \dots \\ p^\varepsilon &= p^0(t,x) + p^0_{bl}\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon\left(p^1(t,x) + p^1_{bl}\left(t,x,\frac{x}{\varepsilon}\right)\right) + \dots \end{aligned}$$

Alors, à l'intérieur du fluide c'est-à-dire loin des bords, $u^0 = \left(u^0_h(t,x_h), 0\right)$, où $u^0_h = u^0_h(t,x_h) \in \mathbb{R}^2$ est une solution du système d'Euler

$$\left\{ \begin{aligned} \partial_t u^0 + u^0 \cdot \nabla_h u^0 + {u^1}^\perp + \nabla_h p^1 &= 0, \\ \nabla_h \cdot u^0 &= 0, \end{aligned} \right.$$

dans $\mathbb{R}^2$. Dans la couche limite, $u_{bl} = u_{bl}(t,x,y)$ est solution du système d'Ekman

$$\left\{ \begin{aligned} -\Delta_y u + u \cdot \nabla_y u + e_3 \times u + \nabla_y p &= 0, & y_3 &> \omega(y_h) \\ \nabla_y \cdot u &= 0, & y_3 &> \omega(y_h) \\ u(y_h, \omega(y_h)) &= -u^0(t, x_h), & y_h &\in \mathbb{R}^2 \end{aligned} \right. \tag{2.18}$$

posé dans le demi-espace rugueux $y_3 > \omega(y_h)$. Notons que dans ce système, x_h est un paramètre, et que la donnée sur le bord n'a donc pas de propriétés de décroissance. Elle corrige la trace de l'écoulement géostrophique u^0 sur le bord rugueux. Dans le cadre où le profil est périodique, on peut montrer que le terme ${u^1}^\perp$ dans l'équation sur u^0_h est la source d'un phénomène de damping, traduisant la dissipation d'énergie dans la couche limite [GV03, GVD06].

2.4 Couches limites

Cette partie est dédiée à l'analyse mathématique de systèmes de type couches limites. De façon générale, on se pose 3 questions :

1. Le problème est-il bien posé ?
2. Peut-on décrire l'asymptotique loin du bord ?
3. La couche limite est-elle stable ?

On se focalise ici exclusivement sur les deux premières. Pour la troisième, on renvoie le lecteur notamment à [Gre00, Rou01, GR01, DDG04, GR08, Rou04, MR08].

Ces deux premières questions soulèvent de nombreuses difficultés. Par exemple, les problèmes de couches limites sont posés dans des domaines infinis (typiquement des demi-espaces), alors que les données (et donc les solutions) ne décroissent pas à l'infini. Pour construire ces solutions d'énergie infinie, trouver le bon cadre fonctionnel est difficile, surtout en l'absence de propriétés structurales (périodicité, quasi-périodicité...). Pour ce qui concerne l'asymptotique loin du bord, nous verrons qu'un autre problème, de type ergodique, apparaît.

De manière schématique, on aborde les questions d'existence et d'asymptotique par :

des méthodes d'énergie Notons que ces méthodes ne sont pas variationnelles stricto sensu, i.e. associées à la minimisation d'une énergie : en effet, les espaces considérés sont d'énergie infinie. Elles reposent plutôt sur des *estimations de Saint-Venant*. De telles estimations sont possibles lorsque le cadre fonctionnel permet de recourir à des inégalités de type Poincaré,
- soit parce que l'on dispose d'un peu de structure, comme dans les sections 2.4.1 et 2.4.3 (lorsque la rugosité est périodique ou quasi-périodique),
- soit, en l'absence de toute propriété structurale, parce que l'on peut se ramener à travailler dans un domaine borné (au moins dans une direction), par exemple via des conditions aux limites transparentes ; voir la section 2.4.4 et le Chapitre 5.

Lorsque ces estimations échouent, on peut recourir à :

des méthodes potentielles en représentant la solution grâce aux noyaux de Poisson et de Green. On s'appuie alors sur les propriétés asymptotiques des noyaux pour étudier la solution, comme dans la section 2.4.2 et le Chapitre 4.

2.4.1 Couches limites en homogénéisation

On se concentre sur l'analyse du problème de couche limite

$$\left\{ \begin{aligned} -\nabla \cdot A(y)\nabla v_{bl} &= 0, & y\cdot n - a > 0 \\ v_{bl} &= v_0(y), & y \cdot n - a = 0 \end{aligned} \right. , \tag{2.19}$$

posé dans le demi-espace $\{y \cdot n - a > 0\} \subset \mathbb{R}^d$ dirigé par $n \in \mathbb{S}^{d-1}$. Les deux questions qui nous occupent sont d'une part l'existence et l'unicité d'une solution v à (2.19), et d'autre part la convergence de $v_{bl}(y)$ lorsque $y \cdot n \to \infty$. Ce dernier point est décisif. Le comportement de la couche limite loin du bord détermine en effet la donnée homogénéisée sur le bord $\partial\Omega$ dans le système (2.16) (voir la section 2.5.2 et le Chapitre 3).

Le sytème (2.19) est l'analogue du problème cellulaire (2.13), avec des différences majeures rendant son analyse bien plus compliquée. Le domaine est non borné, et la donnée sur le bord ne décroît pas. Le problème mathématique majeur que pose l'analyse de (2.19) est la rupture du réseau périodique du fait du bord. L'enjeu essentiel est de comprendre l'interaction entre le bord et la microstructure périodique et de voir si des propriétés de structure sont préservées. Ces structures sont essentielles pour l'approche par estimations d'énergie. Celle-ci nécessite d'utiliser à un moment une inégalité de type Poincaré. Utiliser la structure du système, s'il y en a une, peut donner de la compacité, au moins dans une direction, et de ce fait fournir cette inégalité.

Pour l'analyse de (2.19), caractère bien posé et asymptotique, on distingue trois cas :

RAT n est proportionnel à un vecteur à coordonnées rationnelles, i.e. $n \in \mathbb{R}\mathbb{Q}^d$. Ce cas à été étudié par de nombreux auteurs, voir entre autres [MV97, AA99, Naz92].

DIV $n \notin \mathbb{R}\mathbb{Q}^d$, et l'hypothèse de petits diviseurs suivante est satisfaite : il existe $C,\ \tau > 0$, tels que pour tous $\xi \in \mathbb{Z}^d \setminus \{0\}$, pour tout $i = 1, \dots\ d-1$,

$$|n_i \cdot \xi| \geq C|\xi|^{-d-\tau}, \tag{2.20}$$

où $(n_1, \dots\ n_{d-1}, n)$ est une base orthogonale de $\mathbb{R}^d$. Ce cas est l'objet de l'article [GVM11].

NON DIV cas général $n \notin \mathbb{R}\mathbb{Q}^d$ sans hypothèse de petits diviseurs. Ce cas est le sujet du Chapitre 4 et de l'article [Pra13].

Pour la suite, faisons un changement de variable dans (2.19) permettant de se ramener à un bord plat. Soit $\mathrm{M} \in M_d(\mathbb{R})$ une matrice orthogonale envoyant e_d (le d-ième vecteur de la base canonique de $\mathbb{R}^d$) sur n. Faire le changement de variable $z = \mathrm{M}^T y$ conduit à $v(z) := v_{bl}\left(\mathrm{M}^T y\right)$ est solution de

$$\left\{ \begin{array}{rll} -\nabla \cdot B(\mathrm{M}z)\nabla v & = 0, & z_d > a \\ v & = v_0(\mathrm{M}z), & z_d = a \end{array} \right. , \tag{2.21}$$

où $B = B^{\alpha\beta} \in M_N(\mathbb{R})$ est une famille de matrices périodiques indexée par $1 \leq \alpha, \beta \leq d$.

Avant d'introduire les principaux résultats pour le cas général **NON DIV** dans la section 2.4.2 ci-dessous, revenons sur les deux premiers **RAT** et **DIV**.

Structure périodique

Dans le cas où $n \in \mathbb{R}\mathbb{Q}^d$, on peut choisir M à coefficients dans $\mathbb{R}\mathbb{Q}^d$. Dès lors

$$z \longmapsto B(\mathrm{M}z), \quad z' \longmapsto v_0(\mathrm{M}(z', a)) \tag{2.22}$$

sont périodiques, et on peut supposer sans perte de généralité que la période est égale à 1. Le cadre périodique simplifie la preuve d'existence, mais encore plus l'analyse asymptotique.

Étudions un cas particulier, où toutes les fonctions peuvent se calculer par développements en séries de Fourier dans la direction tangentielle : $d = 2$ et $B = \mathrm{I}_2$. L'utilisation des séries de Fourier est quelque peu trompeuse, puisque ce n'est pas l'approche à laquelle on recourt lorsque B n'est pas constant. Elle permet cependant de mettre en évidence le type de résultat que l'on attend, et certaines particularités. On suppose de plus $n = e_2$, ou ce qui revient au même $\mathrm{M} = \mathrm{I}_2$, de sorte que $v = v(z_1, z_2) \in \mathbb{R}$ est solution de

$$\left\{ \begin{array}{rll} -\Delta v & = 0, & z_2 > a, \\ v & = v_0, & z_2 = a \end{array} \right. . \tag{2.23}$$

Alors, pour tout $z = (z_1, z_2) \in \mathbb{T} \times (a, \infty)$,

$$v(z_1, z_2) = \sum_{\xi \in \mathbb{Z}} \widehat{v_0}(\xi) e^{-2\pi|\xi|(z_2 - a)} e^{2i\pi\xi z_1}.$$

Il s'ensuit par égalité de Parseval que

$$\left\| v(z_1, z_2) - \int_{\mathbb{T}} v_0(z_1, a) dz_1 \right\|^2_{L^2(\mathbb{T})} \leq \sum_{\xi \in \mathbb{Z}\setminus\{0\}} |\widehat{v_0}(\xi)|^2 \, e^{-4\pi|\xi|(z_2-a)} \leq Ce^{-4\pi(z_2-a)}. \tag{2.24}$$

Il découle de (2.24) la convergence exponentielle de $v(z_1, z_2)$ vers une constante, la *queue de couche limite*, loin du bord, i.e. lorsque $z_2 \to \infty$. De plus, la queue de couche limite $\int_{\mathbb{T}} v_0(z_1, a) dz_1$ dépend de a, ou plutôt de la partie fractionnaire de a.

Revenons brièvement à l'analyse générale du cas rationnel. Pour l'existence, on commence par relever la donnée de Dirichlet $v_0(\mathrm{M}\cdot)$ à l'aide d'une fonction $\varphi \in C_c^\infty(\mathbb{R})$: $\tilde{v} = v - \varphi(z_d) v_0\left(\mathrm{M}(z', 0)\right)$ est solution d'un système analogue à (2.21) avec condition sur le bord homogène et terme source à support compact en z_d. On est alors conduit à résoudre le nouveau système en $\tilde{v}$ dans l'espace de Hilbert des fonctions $H^1_{loc}\left(\mathbb{T}^{d-1}\times(a,\infty)\right)$, nulles sur le bord, telles que

$$\int_a^\infty \int_{\mathbb{T}^{d-1}} |\nabla\tilde{v}|^2 < \infty.$$

La convergence de la couche limite est ensuite déduite d'une estimation de Saint-Venant : pour tout $k\in\mathbb{N}$, $k>a$,

$$\|\nabla v\|_{L^2\left(\mathbb{T}^{d-1}\times(k+1,\infty)\right)} \leq C\,\|\nabla v\|_{L^2\left(\mathbb{T}^{d-1}\times(k,k+1)\right)}. \tag{2.25}$$

L'inégalité (2.25), qui est une sorte d'inégalité de Gronwall discrète, donne immédiatement de la convergence exponentielle vers $V^{\infty,a}\in\mathbb{R}^N$, la queue de couche limite. Pour obtenir (2.25), on combine l'estimation a priori

$$\begin{aligned}\left\|\nabla\left(v-\int_{\mathbb{T}^{d-1}} v(z,k+1)dz'\right)\right\|_{L^2\left(\mathbb{T}^{d-1}\times(k+1,\infty)\right)} &\leq \left\|v-\int_{\mathbb{T}^{d-1}} v(z',k+1)dz'\right\|_{H^{1/2}\left(\mathbb{T}^{d-1}\right)}\\ &\leq \left\|v-\int_{\mathbb{T}^{d-1}} v(z',z_2)dz'\right\|_{H^1\left(\mathbb{T}^{d-1}\times(k,k+1)\right)}\end{aligned}$$

où la constante C est uniforme en k, avec l'inégalité de Poincaré-Wirtinger

$$\left\|v-\int_{\mathbb{T}^{d-1}} v(z',z_2)dz'\right\|_{H^1\left(\mathbb{T}^{d-1}\times(k,k+1)\right)} \leq \left\|\nabla\left(v-\int_{\mathbb{T}^{d-1}} v(z',z_2)dz'\right)\right\|_{L^2\left(\mathbb{T}^{d-1}\times(k,k+1)\right)}.$$

Plus de détails sur le cas **RAT** sont donnés en section 4.2 du Chapitre 4. On renvoie également à [MV97, Appendix].

Notons que les même techniques peuvent permettre de traiter le cas d'une frontière rugueuse périodique. Une remarque importante est que l'estimation de Saint-Venant pour la convergence ci-dessus ne dépend que de la solution loin du bord. Il s'ensuit que ni la rugosité du bord, ni la condition sur le bord rugueux n'interviennent. Cependant, la rugosité impose la structure du problème. On revient sur des problèmes de rugosités aux sections 2.4.3 et 2.4.4 pour des modèles issus de la mécanique des fluides.

Structure quasi-périodique

Lorsque $n\notin\mathbb{R}\mathbb{Q}^d$, la périodicité (2.22) n'est plus vraie. La mise en œuvre d'une démarche analogue à celle du cas rationnel repose sur l'observation suivante. Appelons $\mathrm{N}\in M_{d,d-1}(\mathbb{R})$ la matrice des $d-1$ premières colonnes de M. Il existe une famille de matrices $\mathcal{B}=\mathcal{B}^{\alpha\beta}(\theta,t)\in M_N(\mathbb{R})$, $V_0=V_0(\theta,t)\in\mathbb{R}^N$, $\theta\in\mathbb{T}^d$, $t\geq a$, tels que pour tout $z\in\mathbb{R}^{d-1}\times(a,\infty)$,

$$B(\mathrm{M}z)=B(\mathrm{N}z'+z_dn)=\mathcal{B}(\mathrm{N}z',z_d),\qquad v_0(\mathrm{M}(z',a))=v_0(\mathrm{N}z'+an)=V_0(\mathrm{N}z',a). \tag{2.26}$$

Ainsi, les coefficients $B(\mathrm{M}\cdot)$ et la donnée sur le bord v_0 sont quasi-périodiques en z', c'est-à-dire qu'il existe $\mathrm{Q}\in M_{d,d-1}(\mathbb{R})$ et $F=F(\theta)$ une fonction périodique en $\theta\in\mathbb{T}^d$ telles que $v_0(\mathrm{M}(z',a))=F(\mathrm{Q}z')$. L'idée consiste à utiliser cette structure, et à chercher v sous la forme $v(z)=V(\mathrm{N}z',z_d)$, avec $V=V(\theta,t)\in\mathbb{R}^N$ solution de

$$\left\{\begin{aligned} -\begin{pmatrix}\mathrm{N}^T\nabla_\theta\\ \partial_t\end{pmatrix}\cdot\mathcal{B}(\theta,t)\begin{pmatrix}\mathrm{N}^T\nabla_\theta\\ \partial_t\end{pmatrix}V &=0, & t>a\\ V&=V_0, & t=a\end{aligned}\right.. \tag{2.27}$$

Le gain réalisé en remplaçant (2.21) par (2.27) est la périodicité dans la variable tangentielle $\theta \in \mathbb{T}^d$. On paie ce bénéfice par la perte de l'ellipticité de l'opérateur différentiel. Par exemple en dimension $d = 2$, si $n = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, alors $\mathrm{N} = n^\perp = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ et $\mathrm{N}^T\nabla_\theta = -\frac{1}{2}\partial_{\theta_1} + \frac{\sqrt{3}}{2}\partial_{\theta_2}$. Le gradient "tordu" $\mathrm{N}^T\nabla_\theta$ ne contient donc pas les dérivées dans toutes les directions, ce qui entrave l'ellipticité.

La périodicité en θ fournit un cadre fonctionnel naturel, celui des V tels que

$$\int_a^\infty \int_{\mathbb{T}^d} \left(\left|\mathrm{N}^T\nabla_\theta V\right|^2 + |\partial_t V|^2\right) d\theta dt < \infty.$$

L'existence découle alors d'une régularisation elliptique $\iota\Delta_\theta V$ et d'estimations uniformes en ι. On peut ensuite considérer des dérivées d'ordre supérieur, et établir que pour tout $l \geq 0$, $\alpha \in \mathbb{N}^d$,

$$\int_a^\infty \int_{\mathbb{T}^d} \left(\left|\mathrm{N}^T\nabla_\theta \partial_\theta^\alpha V\right|^2 + \left|\partial_t^{l+1}\partial_\theta^\alpha V\right|^2\right) d\theta dt < \infty. \tag{2.28}$$

L'existence d'une solution peut ainsi être établie pour tout $n \notin \mathbb{R}\mathbb{Q}^d$. En revanche, l'analyse asymptotique reposant sur des estimations d'énergie (comme dans le cas rationnel ci-dessus), ne peut être conduite que sous l'hypothèse de petits diviseurs **DIV**. La convergence vers une queue de couche limite $V^\infty \in \mathbb{R}^N$ est une conséquence d'une estimation de Saint-Venant sur

$$F(T) := \int_T^\infty \int_{\mathbb{T}^d} \left(\left|\mathrm{N}^T\nabla_\theta V\right|^2 + |\partial_t V|^2\right) d\theta dt.$$

L'idée générale est la même que dans le cas rationnel. On doit néanmoins prendre garde au fait que $\mathrm{N}^T\nabla_\theta$ est un gradient dégénéré. Le rôle clé de l'hypothèse de petits diviseurs est d'entraîner un type d'inégalité de Poincaré-Wirtinger avec ce gradient dégénéré : pour tout $1 < p < \infty$, pour tout $t > a$,

$$\int_{\mathbb{T}^d} \left|V(\theta, t) - \int_{\mathbb{T}^d} V(\theta', t) d\theta'\right|^2 d\theta \leq C_p \left(\int_{\mathbb{T}^d} \left|\mathrm{N}^T\nabla_\theta V(\theta, t)\right|^2 d\theta\right)^{\frac{1}{p}}. \tag{2.29}$$

En effet, en posant $\widetilde{V} := V(\theta, t) - \int_{\mathbb{T}^d} V(\theta', t) d\theta'$, on a

$$\int_{\mathbb{T}^d} \left|\widetilde{V}(\theta, t)\right|^2 d\theta \leq \left(\sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \left|\widehat{\widetilde{V}}(\xi, t)\right|^2 \frac{1}{|\xi|^{2(d+\tau)}}\right)^{\frac{1}{p}} \left(\sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \left|\widehat{\widetilde{V}}(\xi, t)\right|^2 |\xi|^{2\frac{d+\tau}{p-1}}\right)^{\frac{p-1}{p}}.$$

Le premier facteur représente la norme de $\widetilde{V}(\cdot, t)$ dans un Sobolev homogène d'exposant négatif et peut être majoré grâce à l'hypothèse de petits diviseurs (2.20)

$$\left\|\mathrm{N}^T\nabla_\theta \widetilde{V}(\theta, t)\right\|^2_{L^2(\mathbb{T}^d)} = \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \left|2i\pi\widehat{\widetilde{V}}(\xi, t)\right|^2 \left|\mathrm{N}^T\xi\right|^2 \geq C \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \left|\widehat{\widetilde{V}}(\xi, t)\right|^2 \frac{1}{|\xi|^{2(d+\tau)}}.$$

Le second facteur peut être borné grâce à l'estimation (2.28) par une constante C_p.

Grâce à l'inégalité (2.29), on peut établir pour tout $\frac{1}{2} < s < 1$ l'inégalité différentielle

$$F(T) \leq C_s \left(-F'(T)\right)^s$$

qui implique la convergence de V vers V^∞ dans $L^\infty\left(\mathbb{T}^d\right)$ lorsque $t \to \infty$, à la vitesse $O\left(t^{-m}\right)$ pour tout $m \in \mathbb{R}$. La queue de couche limite V^∞ est indépendante de a.

L'existence comme la convergence ont leur contrepartie pour v (resp. v_{bl}) solutions de (2.21) (resp. (2.19)). On revient en détail sur le cas **DIV** à la section 4.2 du Chapitre 4. On renvoie à [GVM11] pour la démonstration de ces résultats.

2.4.2 Asymptotique de la couche limite : le cas général (Chapitre 4)

L'objectif de l'article [Pra13] publié dans *SIAM Journal on Mathematical Analysis* et reproduit dans le Chapitre 4, est d'étudier le comportement de $v_{bl}(y)$ pour $y \cdot n \to \infty$ en s'affranchissant de l'hypothèse de petits diviseurs (2.20), i.e. pour $n \notin \mathbb{R}\mathbb{Q}^d$ quelconque. Insistons sur le fait que l'asymptotique de v_{bl} loin du bord détermine la donnée sur le bord homogénéisée dans le système (2.16), ce qui fait de cette question le problème majeur de l'homogénéisation des systèmes de type couche limite (voir la section 2.5.2 et le Chapitre 3).

Difficultés : à quoi l'hypothèse de petits diviseurs sert-elle ?

L'hypothèse de petits diviseurs entraîne une inégalité de type Poincaré-Wirtinger (2.29), sans laquelle l'approche par estimation de Saint-Venant est inopérante. De plus, l'hypothèse de petits diviseurs est utilisée de manière cruciale pour obtenir la convergence rapide vers la queue de couche limite. Cela apparaît nettement sur le système simplifié où $d = 2$, $\mathcal{B} = \mathrm{I}_2$ et $a = 0$. Pour tout $t > 0$, pour tout $m \in \mathbb{N}$, à condition que v_0 soit assez régulière,

$$\begin{aligned}
\left\| V(\theta, t) - \fint_{\mathbb{T}^2} V_0(\theta, 0) d\theta \right\|^2_{L^2(\mathbb{T}^2)} &= t^{-m} \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} \left| \widehat{v_0}(\xi) \right|^2 t^m e^{-4\pi |n^\perp \cdot \xi| t} \\
&\leq C t^{-m} \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} \left| \widehat{v_0}(\xi) \right|^2 |\xi|^{(2+\tau)m} \left(|\mathrm{N} \cdot \xi| t \right)^m e^{-4\pi |n^\perp \cdot \xi| t} \\
&\leq C t^{-m} \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} \left| \widehat{v_0}(\xi) \right|^2 |\xi|^{(2+\tau)m} \leq C_m t^{-m},
\end{aligned}$$

de sorte que $V(\cdot, t)$ converge vers $\widehat{v_0}(0)$ dans $L^2(\mathbb{T}^2)$, plus vite que n'importe quelle puissance de t.

L'explication heuristique de cette convergence rapide est que l'hypothèse de petits diviseurs (2.20) fournit une borne inférieure sur la distance entre le réseau $\mathbb{Z}^d \setminus \{0\}$ et l'hyperplan $y \cdot n = 0$. De même, dans le cas rationnel, le trou spectral au voisinage des basses fréquences, ou encore le fait que le spectre d'une fonction périodique est discret, entraîne la convergence exponentielle (2.24).

Concluons ce point en indiquant que sans aucune hypothèse de structure sur la donnée v_0, il est illusoire de s'attendre à une convergence de v solution de

$$\left\{ \begin{array}{rll} -\Delta v & = 0, & z_2 > 0, \\ v & = v_0, & z_2 = 0 \end{array} \right. .$$

En effet, Gérard-Varet et Masmoudi construisent dans [GVM10, Proposition 11] un exemple de v_0 pour lequel $v(0, z_2)$ n'a pas de limite quand $z_2 \to 0$. Ils soulignent le lien entre la convergence de v loin du bord, et des propriétés d'ergodicité de v_0 sur le bord. Cette dernière propriété est cruciale dans notre travail sur l'asymptotique de la couche limite (2.19) dans le cas général.

Stratégie

Dans le cas où $n \notin \mathbb{R}\mathbb{Q}^d$ est quelconque, on dispose encore du cadre quasi-périodique qui conduit au système (2.27). L'existence de solutions à ce système et donc à (2.19) (pour un théorème précis voir Theorem 4.5 dans le Chapitre 4) est vraie sans l'hypothèse de petits diviseurs. Focalisons-nous sur la convergence. *Quitte à faire une translation de la variable, on se ramène au cas où $a = 0$.*

À quoi peut-on s'attendre ? La quasi-périodicité de la donnée de Dirichlet v_0 sur le bord $y \cdot n = 0$ fournit de l'ergodicité et permet d'envisager l'existence d'une limite pour v_{bl} lorsque $y \cdot n \to \infty$. En revanche l'absence de l'hypothèse de petits diviseurs, et donc le fait que l'hyperplan $y \cdot n = 0$ puisse être arbitrairement proche de $\mathbb{Z}^d \setminus \{0\}$, ne permet pas d'espérer une vitesse de convergence.

L'idée fondamentale de l'article [Pra13] est de représenter la solution variationnelle v_{bl} grâce au noyau de Poisson $P = P(y, \tilde{y}) \in M_N(\mathbb{R})$ associé à l'opérateur $-\nabla \cdot A(y)\nabla\cdot$ et au domaine $\{y \cdot n > 0\}$,

$$v_{bl}(y) = \int_{\tilde{y}\cdot n=0} P(y, \tilde{y})v_0(\tilde{y})d\tilde{y} \tag{2.30}$$

pour tout $y \in \mathbb{R}^d$ tel que $y \cdot n > 0$, puis de développer $P(y, \tilde{y})$ pour $|y - \tilde{y}| \geq |y \cdot n| \gg 1$. Suivant l'observation de [AL91, corollary p. 903], cette dernière question cache un problème d'homogénéisation des noyaux oscillants

$$P^\varepsilon(x, \tilde{x}) = \frac{1}{\varepsilon^{d-1}} P\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right)$$

dans la limite $\varepsilon \to 0$ pour $|x - \tilde{x}|$ proche de 1.

Le noyau de Poisson P^ε est associé à l'opérateur $-\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla\cdot$ à coefficients oscillants et au domaine $\{x \cdot n > 0\}$. Il s'exprime (cf. (4.43)) en terme de $\nabla_{\tilde{x}} G^\varepsilon$, où $G^\varepsilon = G^\varepsilon(x, \tilde{x}) \in M_N(\mathbb{R})$ est la fonction de Green associée aux même opérateur et domaine. Une tâche importante est de démontrer, à x fixé, un développement de $\nabla_{\tilde{x}} G^\varepsilon(x, \tilde{x})$ pour $\varepsilon \to 0$ et $\frac{1}{4} < |x - \tilde{x}| < 4$. La technique consiste à développer G^ε selon la double-échelle $\tilde{x}$ et $\frac{\tilde{x}}{\varepsilon}$. Les oscillations de ce développement sur le bord

$$\Gamma(x, 4) \setminus \overline{\Gamma}\left(x, \frac{1}{4}\right) := \{y \cdot n = 0\} \cap B(x, 4) \setminus \overline{B}\left(x, \frac{1}{4}\right),$$

conduisent à le corriger grâce à des termes de couche limite. Ainsi la différence entre G^ε et son développement, appelons-la $R^{\varepsilon,x}_{bl}$, satisfait un système elliptique du type

$$\begin{cases} -\nabla \cdot A\left(\frac{\tilde{x}}{\varepsilon}\right)\nabla R^{\varepsilon,x}_{bl} &= F^\varepsilon + F^\varepsilon_{bl}, \quad \tilde{x} \in B(x, 4) \setminus \overline{B}\left(x, \frac{1}{4}\right) \\ R^{\varepsilon,x}_{bl} &= \Phi^\varepsilon, \qquad\quad \tilde{x} \in \Gamma(x, 4) \setminus \overline{\Gamma}\left(x, \frac{1}{4}\right) \end{cases},$$

dans lequel les termes sources sont petits.

L'outil crucial permettant d'estimer $R^{\varepsilon,x}_{bl}$ est l'estimation locale sur le gradient due à Avellaneda et Lin [AL87a, lemma 20] (voir aussi Theorem 4.8 dans le Chapitre 4) : pour $0 < \nu < 1$, $0 < \delta$,

$$\begin{aligned} \|\nabla R^{\varepsilon,x}_{bl}\|_{L^\infty\left(B(x,4)\setminus\overline{B}\left(x,\frac{1}{4}\right)\right)} \leq C_{\nu,\delta} \Big[&\|R^{\varepsilon,x}_{bl}\|_{L^\infty\left(B(x,4)\setminus\overline{B}\left(x,\frac{1}{4}\right)\right)} \\ &+ \|F^\varepsilon + F^\varepsilon_{bl}\|_{L^{d+\delta}\left(B(x,4)\setminus\overline{B}\left(x,\frac{1}{4}\right)\right)} + \|\Phi^\varepsilon\|_{C^{1,\nu}\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)} \Big]. \end{aligned}$$

Une des subtilités de l'estimation de $R^{\varepsilon,x}_{bl}$ est que le terme source F^ε_{bl} dépend lui-même de termes de couches limites. Mais l'évaluation de ces termes ne nécessite pas de connaître leur comportement loin du bord, ce qui évite tout problème de circularité dans le raisonnement. On ne peut cependant s'appuyer que sur peu d'informations pour évaluer ce terme.

Résultats de la thèse

Dans l'article [Pra13], on démontre deux types de résultats : un développement de $P(y, \tilde{y})$ dans la limite $|y - \tilde{y}| \to \infty$, et un résultat relatif à l'asymptotique de v_{bl} loin du bord.

Théorème A (Theorem 4.18, Chapter 4). *Il existe une fonction $P_{exp} = P_{exp}(y, \tilde{y}) \in M_N(\mathbb{R})$ dont l'expression est explicite, telle que pour tout $0 < \kappa < \frac{1}{2d}$, pour tout y (resp. $\tilde{y}$) vérifiant $y \cdot n > 0$ (resp. $\tilde{y} \cdot n = 0$),*

$$P(y, \tilde{y}) = P_{exp}(y, \tilde{y}) + O\left(\frac{1}{|y - \tilde{y}|^{1+\kappa}}\right).$$

Ce théorème se situe dans la lignée des résultats de Avellaneda et Lin [AL91, corollaires p. 903 et 905], et sa démonstration emprunte leur méthode. Il est également à rapprocher des travaux de Sevost′yanova [Sev81] et de Kozlov [Koz80]. La différence (importante) réside dans le fait que ces résultats valent pour la fonction de Green associée au domaine $\mathbb{R}^d$. La présence du bord dans notre cas complique les estimations. Notre théorème ne peut être obtenu qu'en étudiant finement le comportement de $G^\varepsilon(x, \tilde{x})$ pour $\tilde{x}$ a proximité du bord $\Gamma(x, 4) \setminus \overline{\Gamma}\left(x, \frac{1}{4}\right)$, ce qui revient à prendre en considération des termes de couche limite dans les développements. Notons que l'on se sert de ces correcteurs de couche limite uniquement près du bord. L'information de l'asymptotique loin du bord (qui est le but de notre étude) n'est donc pas requise.

Signalons également les résultats de Avellaneda et Lin sur l'homogénéisation des fonctions de Poisson [AL89b], et ceux plus récents de Kenig, Lin et Shen [KLS12b] sur les noyaux de Green, Poisson et Neumann d'un opérateur à coefficients oscillants. L'objectif de ces travaux est de fournir un développement du noyau de Green (resp. Poisson, Neumann) dans un domaine borné Ω. Néanmoins, les développements obtenus laissent de côté l'homogénéisation de la couche limite. Elle seule contient toutes les oscillations et singularités de la solution près du bord, ce qui rend son analyse particulièrement ardue.

L'expression de P_{exp} est explicite (cf. (4.64)) et fait apparaître de la quasi-périodicité en la variable $\tilde{y}$. L'exposant $\kappa > 0$ est essentiel pour mener à bien la convergence $y \cdot n \to \infty$ dans (2.30).

Théorème B (Theorems 4.2 et 4.3, Chapter 4). *Supposons $n \notin \mathbb{R}\mathbb{Q}^d$. Soit v_{bl} la solution variationnelle de* (2.19). *Alors,*

1. *Il existe $V^\infty \in \mathbb{R}^N$ tel que*
$$v_{bl}(y) \stackrel{y \cdot n \to \infty}{\longrightarrow} V^\infty$$
localement uniformément en la variable tangentielle.
2. *La queue de couche limite V^∞ est indépendante de a.*
3. *Supposons que n ne satisfait pas l'hypothèse de petits diviseurs* (2.20). *Prenons $d = 2$, $N = 1$ et $A = \mathrm{I}_2$. Pour tout $l > 0$, il existe une donnée de Dirichlet v_0 régulière, telle que la solution v_{bl} associée tende vers V^∞ plus lentement que $O\left((y \cdot n)^{-l}\right)$.*

Le premier point du théorème reste valable pour $n \in \mathbb{R}\mathbb{Q}^d$. Le deuxième est une particularité du cas $n \notin \mathbb{R}\mathbb{Q}^d$, qui a son importance (voir la section 2.5.2 et le Chapitre 3). Le dernier point vient confirmer les remarques faites plus haut sur le rôle de l'hypothèse de petits diviseurs. Plus précisément, dans le cas particulier $d = 2$, $N = 1$ et $A = \mathrm{I}_2$ on montre que cette hypothèse est nécessaire pour avoir une convergence rapide. Dans le cas général, le même phénomène à la source du contre-exemple doit se produire. Signalons enfin que pour $n \notin \mathbb{R}\mathbb{Q}^d$ général tel que **NON DIV**, le fait que l'hypothèse de petits diviseurs n'est pas satisfaite ne permet pas, à notre avis, de construire un exemple de v_0 pour lequel la convergence soit plus lente que $O\left(\frac{1}{\ln n}\right)$, par exemple.

La démonstration du théorème fournit une expression explicite pour V^∞ : voir (4.68) dans le Chapitre 4. Cette expression généralise celle de Moskow et Vogelius [MV97, Proposition 6.6]. Étant donné l'importance des queues de couches limites dans l'homogénéisation de (2.16), l'expression explicite de V^∞ ouvre la voie à des études numériques dans l'esprit de [SV06].

2.4.3 Couches limites en mécanique des fluides

L'homogénéisation de microstructures hétérogènes est à rapprocher d'un autre problème : l'étude des *frontières rugueuses* en mécanique des fluides. Il s'agit par exemple d'approcher le système de Navier-Stokes dans un canal rugueux, par un système de Navier-Stokes dans un canal à frontières plates. La question principale dans ce type de problème est : quelle condition homogénéisée sur le bord plat vient remplacer la condition sur le bord rugueux ? Le but est de déterminer, si possible, la meilleure condition homogénéisée sur le bord. Dans la littérature, la condition homogénéisée porte le nom de *loi de paroi*, ou *condition effective*. Ce sujet intéresse les physiciens et les industriels (problèmes de microfluidique, de réduction de la traînée, d'augmentation du glissement) : voir par exemple [GV08] et les références qui y sont citées, [LBS07] et [VY86].

Dans ce contexte, on est amené à étudier des problèmes de couches limites qui partagent des difficultés avec le système (2.19). Le modèle de ce type de problèmes est le système de Stokes posé dans le demi-espace rugueux $\{y_2 > \omega(y_1)\} \subset \mathbb{R}^2$,

$$\left\{ \begin{aligned} -\Delta u + \nabla p &= 0, & y_2 > \omega(y_1) \\ \nabla \cdot u &= 0, & y_2 > \omega(y_1) \\ u(y_1, \omega(y_1)) &= u_0(y_1), & y_1 \in \mathbb{R} \end{aligned} \right. , \tag{2.31}$$

où ω est typiquement une fonction Lipschitz bornée et u_0 est une donnée sans propriétés de décroissance.

Comme pour les problèmes elliptiques de type couches limites, il y a un lien entre l'asymptotique de (2.31) pour $y_2 \to \infty$ et la condition sur le bord homogénéisée. L'analyse a d'abord été menée pour des structures déterministes périodiques : voir [APV98, AMPV98, JM01, JM03] et l'article de survol [Mik09]. Pour des rugosités aléatoires avec des propriétés d'ergodicité, on renvoie à [BGV08, GV09]. Les résultats pour des structures déterministes non périodiques sont récents [GVM10].

Comme dans le Théorème B, le leitmotiv est de s'affranchir au maximum d'hypothèses de structure. C'est ce que nous faisons au Chapitre 5, où nous étudions la couche limite d'Ekman (linéarisée) pour un bord rugueux arbitraire. L'analyse de ce chapitre repose sur des outils introduits par Gérard-Varet et Masmoudi [GVM10] (et repris dans [DGV11]) que nous présentons brièvement maintenant.

L'étude de Gérard-Varet et Masmoudi [GVM10] envisage le problème (2.31) sous plusieurs angles :

- caractère bien posé pour une rugosité ω quelconque ;
- asymptotique loin du bord pour une rugosité quasi-périodique (même presque-périodique). Ainsi que souligné ci-dessus, sans ergodicité, provenant d'une hypothèse de structure sur ω, on ne peut espérer démontrer la convergence vers une queue de couche limite. La question de l'asymptotique loin du bord est donc sans objet pour ω trop général.

L'idée principale pour traiter le cas d'une frontière ω générale est de diviser le demi-plan $y_2 > \omega(y_1)$ en deux sous domaines : un canal borné dans la direction verticale $0 > y_2 > \omega(y_1)$, dans lequel on dispose d'une inégalité de Poincaré permettant de mettre en œuvre une approche variationnelle, et un demi-plan plat $y_2 > 0$, dans lequel on peut étudier le système grâce à la transformée de Fourier dans la variable tangentielle. La connexion entre les deux sous-domaines se fait au travers d'une *condition aux limites transparente* sur le bord $y_2 = 0$.

Les estimations d'énergie dans le canal, nécessitent de connaître la valeur de la donnée de Neumann $-\partial_2 u + pe_2$ sur le bord $y_2 = 0$. Le calcul explicite en Fourier tangentiel dans le demi-espace $y_2 > 0$ de la solution associée à la donnée de Dirichlet $u|_{y_2=0}$, permet de définir un opérateur de Dirichlet to Neumann DN. L'opérateur DN permet de ramener la démonstration de l'existence d'une solution dans le demi-espace bosselé tout entier au canal bosselé $0 > y_2 > \omega(y_1)$ au travers de la condition transparente

$$-\partial_2 u + pe_2|_{y_2=0} = \mathrm{DN}(u|_{y_2=0}).$$

Une des difficultés principales est que u_0 n'a pas de propriétés de décroissance ; elle est uniformément localement dans $H^{1/2}$, i.e.

$$\|u_0\|_{H^{1/2}_{uloc}} := \sup_{k\in\mathbb{Z}} \|u_0\|_{H^{1/2}(k,k+1)} < \infty.$$

On est donc conduit à travailler dans des espaces d'énergie infinie *uloc*. Suivant Ladyženskaja et Solonnikov [LS80] (canal borné avec conditions de Dirichlet sur les bords), le but dans le canal borné est de montrer, par estimations d'énergie, une inégalité de type Saint-Venant

$$E_k \leq C\left(k^2 + E_{k+1} - E_k\right) \tag{2.32}$$

sur

$$E_k := \int_{-k<y_1<k} \int_{y_2>\omega(y_1)} |\nabla u|^2 \, dy_2 dy_1.$$

En effet, une telle inégalité (2.32) permet d'obtenir par récurrence descendante une borne sur

$$\sup_{k\in\mathbb{Z}} \int_k^{k+1} \int_{y_2>\omega(y_1)} |\nabla u|^2 \, dy_2 dy_1 < \infty.$$

Notons qu'ici, la condition de Dirichlet sur le bord supérieur du canal est remplacée par la condition transparente. Elle est non-locale et l'estimation de Saint-Venant qui en découle est bien plus compliquée.

2.4.4 Caractère bien posé du système d'Ekman linéarisé pour une rugosité quelconque (Chapitre 5)

Ce travail est en collaboration avec Anne-Laure Dalibard.

L'objectif du papier [DP13], reproduit dans le Chapitre 5, est d'étudier le système de Stokes-Coriolis 3d avec donnée au bord u_0 dans un espace de régularité Sobolev *uloc*

$$\left\{\begin{aligned} -\Delta u + e_3 \times u + \nabla p &= 0, \quad y_3 > \omega(y_h) \\ \nabla \cdot u &= 0, \quad y_3 > \omega(y_h) \\ u(y_h, \omega(y_h)) &= u_0, \quad y_3 = \omega(y_h) \end{aligned}\right. \tag{2.33}$$

en s'affranchissant de toute hypothèse de structure sur le profil de rugosité ω. Le système (2.33) est en fait le linéarisé de (2.18). Il s'agit d'une étape dans notre projet d'étudier les couches limites d'Ekman non linéaires pour des fonds rugueux arbitraires.

L'étude du problème linéaire (2.33) pose de nombreuses difficultés mathématiques :

1. Le premier type de difficultés est lié au caractère non borné du domaine dans toutes les directions (absence d'inégalité de Poincaré), à l'énergie infinie des fonctions avec lesquelles on travaille, au bord rugueux (entrave à l'utilisation de la transformée de Fourier dans la direction tangentielle à toutes les hauteurs) et à l'absence de

structure (périodique, quasi-périodique). Ces problèmes sont déjà présents dans le cas du système de Stokes étudié à la section 2.4.3, paragraphe Stokes dans un demi-espace rugueux.

2. Le second type de difficultés est lié à la nature de l'opérateur. Contrairement au cas de l'opérateur de Stokes, nous n'avons pas d'expression explicite, ni d'estimation, du noyau associé à l'opérateur de Stokes-Coriolis. Par ailleurs, l'expression en Fourier de l'opérateur de Dirichlet to Neumann associé à Stokes-Coriolis n'est pas homogène d'ordre 1 comme pour Stokes. Le développement en basses fréquences de $\widehat{\mathrm{DN}}$ fait apparaître des termes homogènes d'ordre -1 et 0 qui sont particulièrement mauvais pour les estimations.
3. Enfin, nous traitons le cas 3d et non 2d comme dans [GVM10], ce qui rend notamment la dérivation d'une estimation de Saint-Venant plus technique.

Les difficultés du premier point s'abordent en recourant à la méthode de [GVM10, section 2] décrite ci-dessus. Nous remplaçons l'étude du caractère bien posé de (2.33) par l'étude du système (après relèvement de la condition de Dirichlet sur le bord oscillant)

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p &= f, & \omega(y_h) < y_3 < 0 \\ \nabla \cdot u &= 0, & \omega(y_h) < y_3 < 0 \\ u(y_h, \omega(y_h)) &= 0, & y_3 = \omega(y_h) \\ -\partial_3 u + p e_3|_{y_3=0} &= \mathrm{DN}(u|_{y_3=0}) + F. \end{cases}$$

L'estimation de l'énergie

$$E_k := \int_{|y_h|<k} \int_{\omega(y_h)}^0 |\nabla u|^2 \, dy_3 dy_h$$

conduit à une inégalité de Saint-Venant sur E_k : pour $m > 0$, pour $k \geq m$,

$$E_k \leq C \left[k^2 + E_{k+m+1} - E_k + \frac{k^4}{m^5} \sup_{j \geq m+k} \frac{E_{j+m} - E_j}{j} \right].$$

En raison du caratère non-local de l'opérateur DN, cette estimation à la différence de (2.32), fait intervenir les énergies E_j de u pour j grand.

Les difficultés du second point liées à la structure de l'opérateur de Stokes-Coriolis compliquent :
- la démonstration du caractère bien posé du problème de Stokes-Coriolis dans le demi-espace plat,
- les estimations de l'opérateur DN.

Donnons une idée des singularités associées aux basses fréquences tangentielles. Pour u solution du système de Stokes-Coriolis dans $y_3 > 0$ avec donnée au bord v_0, $\hat{u}_3 = \hat{u}_3(\xi, y_3)$ est solution de

$$\left(|\xi|^2 - \partial_3^2 \right)^3 \hat{u}_3 - \partial_3^3 \hat{u}_3 = 0.$$

L'équation caractéristique a 3 racines distinctes de parties réelles strictement négatives $-\lambda_1,\ -\lambda_2,\ -\lambda_3$. Au voisinage de 0,

$$\lambda_1 \sim |\xi|^3, \qquad \lambda_2 \sim e^{i\pi/4}, \qquad \lambda_3 \sim e^{-i\pi/4}. \tag{2.34}$$

La solution $\hat{u}_3$ s'écrit alors

$$\hat{u}_3(\xi, y_3) = A_1(\xi) \exp(-\lambda_1 y_3) + A_2(\xi) \exp(-\lambda_2 y_3) + A_3(\xi) \exp(-\lambda_3 y_3)$$

où A_k dépend de $\hat{v}_0$. La dégénerescence de λ_1 au voisinage de 0 entraîne la faible décroissance de $\hat{u}_3$ dans la direction verticale.

En hautes fréquences tangentielles, l'opérateur de Stokes-Coriolis se comporte comme l'opérateur de Stokes. En revanche, les singularités en basses fréquences ne permettent pas de travailler avec des données de Dirichlet $H^{1/2}_{uloc}$ sur le bord plat (resp. sur le bord bosselé) pour l'existence dans le demi-espace plat (resp. bosselé). De plus pour la même raison, l'opérateur DN n'est pas bien défini sur $H^{1/2}_{uloc}$. Pour effacer ces singularités en basses fréquences, on considère des données de Dirichlet $v_0 \in H^{1/2}_{uloc}$ sur le bord plat pour lesquelles il existe un champ V_h tel que

$$v_{0,3} = \nabla_h \cdot V_h. \tag{2.35}$$

Nous appelons cet espace $\mathbb{K}$. Son rôle est clé pour l'existence dans le demi-espace plat et la définition de DN. Si v_0 est la trace de u, solution de (2.33),

$$\begin{aligned} v_{0,3}(y_h) = u_3|_{x_3=0}(y_h) &= \int_{\omega(y_h)}^0 \partial_3 u_3(y_h,t)dt + u_{0,3}(y_h) \\ &= -\nabla_h \cdot \left(\int_{\omega(y_h)}^0 u_h(y_h,t)dt\right) - \nabla_h\omega \cdot u_{0,h} + u_{0,3}. \end{aligned}$$

Pour que (2.35) soit satisfaite, on est conduit à imposer qu'il existe $U_h = U_h(y_h) \in \mathbb{R}^2$ telle que

$$-\nabla_h\omega \cdot u_{0,h} + u_{0,3} = \nabla_h \cdot U_h. \tag{2.36}$$

Notre résultat principal est le caractère bien posé de (2.33) :

Théorème C (Theorem 5.1, Chapter 5). *Soit $\omega \in W^{1,\infty}(\mathbb{R}^2)$. Supposons que $u_0 \in H^2_{uloc}(\mathbb{R}^2)$ et que la condition de compatibilité* (2.36) *est satisfaite.*
Alors, il existe une unique solution faible u de (2.33) *telle que pour tout $a > 0$, et tout $m \geq 4$,*

$$\sup_{l\in\mathbb{Z}^2} \|u\|_{H^1(\{y_h\in l+[0,1]^2,\omega(y_h)<y_3<a\})} < \infty, \tag{2.37a}$$

$$\sup_{l\in\mathbb{Z}^2} \int_1^\infty \int_{l+[0,1]^2} |\nabla^m u|^2 < \infty. \tag{2.37b}$$

Notre théorème s'apparente au résultat [GVM11, Proposition 6 et 10] sur Stokes 2d pour une rugosité arbitraire. Comme sous produit de notre travail, on en obtient une généralisation au cas $d = 3$: *si $u_0 \in H^2_{uloc}(\mathbb{R}^2)$, alors il existe une unique solution faible au système de Stokes posé dans $y_3 > \omega(y_h)$ telle que pour tout $a > 0$ et tout $m \geq 1$, les bornes* (2.37a) *et* (2.37b) *soient satisfaites.* La condition de compatibilité (2.36) n'a pas de lieu d'être pour le système de Stokes.

L'intérêt majeur de notre théorème réside dans l'absence d'hypothèses sur ω, autres que le caractère lipschitzien et borné. La démonstration de notre résultat met en évidence le mauvais comportement de l'opérateur de Stokes-Coriolis au voisinage des basses fréquences tangentielles. Dans le cas périodique, le trou spectral au voisinage des basses fréquences rend celles-ci invisibles. Regarder des profils ω sans structure oblige à comprendre l'action de l'opérateur différentiel sur toutes les fréquences tangentielles. C'est ce qui fait la difficulté de ce type de résultats. En particulier, le fait que (2.37b) ne soit vrai que pour $m \geq 4$ est une différence avec le résultat sur Stokes, et est une conséquence de la dégénerescence de la valeur propre λ_1 au voisinage de 0 (cf. (2.34)). Également, la condition de compatibilité (2.36) de la donnée de Dirichlet et du profil ω est également absente du théorème sur

Stokes. Sa raison d'être est la maîtrise des singularités en basses fréquences, propres à l'opérateur de Stokes-Coriolis. Notons enfin que pour un bord plat $\omega = 0$, (2.36) se ramène à (2.35). Cette condition de compatibilité est automatiquement vérifiée pour la donnée sur le bord du système de la couche limite d'Ekman, puisque dans ce cas, $u_{0,3} = 0$ et $u_{0,h}$ ne dépend pas de y_h.

Le Théorème C va dans le sens d'une généralisation des travaux de Gérard-Varet [GV03] pour le système d'Ekman non-linéaire (2.18) mais limités à une rugosité périodique. Notre objectif est d'étudier le système d'Ekman non-linéaire pour des rugosités arbitraires.

2.5 Asymptotiques

L'objectif de cette section est de présenter certaines techniques utilisées pour justifier des asymptotiques formelles et des passages à la limite dans des EDP. La problématique est très générale. Dans le cadre de cette thèse, il peut s'agir : de démontrer des estimations d'erreur en homogénéisation, d'homogénéiser un système elliptique à coefficients et donnée de Dirichlet oscillants (sur ces points cf. les sections 2.5.1, 2.5.2, 2.5.3 et le Chapitre 3), de passer à la limite dans une formulation variationnelle pour démontrer l'existence de solutions faibles à un modèle mécanique des fluides, de justifier la limite newtonienne pour certains fluides viscoélastiques (sur ces points cf. la section 2.5.4 et le Chapitre 6).

L'ingrédient de base de la convergence est l'obtention de bornes uniformes. Celles-ci permettent en effet

- d'obtenir de la *convergence faible* via le théorème de Banach-Alaoglu [Bre05, Chapitre III],
- de mettre en œuvre des techniques de *convergence double-échelle* [All92] en homogénéisation périodique,
- et d'avoir de la *convergence forte* dans une norme plus faible par injection de Rellich [Bre05, Chapitre IX], [Ada75, Chapter 6] et [BCD11, Theorem 1.68].

Dans la suite, on rencontre assez fréquemment deux problèmes majeurs. Le premier est lié à l'absence de bornes (du moins de bornes évidentes), comme pour le problème de couche limite (2.16)

$$\left\| u_{bl}^{1,\varepsilon} \right\|_{H^1(\Omega)} \leq C \varepsilon^{-\frac{1}{2}}.$$

Lorsque la compacité fait défaut, un espoir est d'obtenir de la compacité dans des normes plus faibles. Ceci est possible pour la couche limite $u_{bl}^{1,\varepsilon}$, grâce à une description fine de $u_{bl}^{1,\varepsilon}$ dans la limite $\varepsilon \to 0$ (cf. section 3.3.2).

Le second problème est lié au mauvais comportement de la convergence faible vis-à-vis des *non-linéarités*. En effet, un produit de suites faiblement convergentes peut être le lieu de résonances dont l'effet est soit d'entraver la convergence faible du produit, soit d'empêcher que la limite faible du produit soit le produit des limites faibles. Pour un exemple simple de cet effet prendre $u_n := \sin\left(\frac{x}{\varepsilon}\right)$: u_n (resp. u_n^2) converge faiblement vers 0 (resp. 1/2) dans $L^2(0,1)$. Dans des contextes moins triviaux, ce phénomène se produit par exemple en homogénéisation pour le passage à la limite dans $A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon$ (cf. section 2.5.1) et en mécanique des fluides pour le passage à la limite dans $\tau_n W(u_n)$ (cf. section 2.5.4).

Le passage à la limite dans un produit de suites faiblement convergentes n'est pas possible en général. Dans certaines situations, on peut néanmoins déterminer la limite, en utilisant :

- la structure du produit ou du terme non-linéaire : techniques de *compacité par compensation* (dont le lemme div-rot) dues à Murat et Tartar [Mur78] (pour des développements récents voir [BCDM09, CDM11]) ;

– la structure de la solution : développements double-échelle, méthode des *fonctions test oscillantes* inspirées par Tartar [Tar78] et [CD99, Chapter 8], *convergence double-échelle* [All92, All93, AC98b].

Mentionnons aussi l'utilisation de *mesures de défaut* de convergence [Eva90] et de techniques de *propagation de la compacité* (utilisation intensive dans les articles [LM00, Mas11], ainsi que dans le Chapitre 6). Pour un exposé des problématiques liées à la convergence faible, le livre de Evans [Eva90] est une bonne référence.

Signalons enfin que pour les applications, l'obtention d'un simple résultat de convergence n'est pas satisfaisant. Un apport plus conséquent consiste à fournir une estimation d'erreur entre la vraie solution et son approximation. C'est ce que nous nous attachons à faire ici dans le contexte de l'homogénéisation.

2.5.1 Estimations d'erreur en homogénéisation

La limite du système (2.2), à coefficients oscillants mais donnée sur le bord non oscillante, est connue depuis les années 70 (voir notamment [BLP78]) : u^ε converge dans $L^2(\Omega)$ vers $\bar{u}^0$ solution du problème elliptique à coefficients homogènes

$$\begin{cases} -\nabla \cdot \overline{A}\nabla \bar{u}^0 &= f, \quad x \in \Omega \\ \bar{u}^0 &= \varphi_0, \quad x \in \partial\Omega \end{cases}. \tag{2.38}$$

Quitte à relever $\varphi_0 \in H^{1/2}(\partial\Omega)$ on peut intégrer la donnée de Dirichlet au terme source f et supposer dans la suite que $\varphi_0 = 0$.

Comment montre-t-on la convergence de u^ε solution de (2.2) lorsque $\varepsilon \to 0$? L'estimation a priori sur le système donne la borne

$$\|u^\varepsilon\|_{H^1_0(\Omega)} \leq C \|f\|_{H^{-1}(\Omega)} \tag{2.39}$$

uniforme en ε. La méthode des fonctions test oscillantes due à Tartar [Tar78] (voir aussi [CD99, Chapter 8] et [Eva90]) permet alors de montrer que

$$u^\varepsilon \rightharpoonup \bar{u}^0, \qquad \text{dans} \quad H^1_0(\Omega)$$

où $\bar{u}^0$ est solution de (2.38).

La notion conceptualisée par Allaire [All92, All93] et Nguetseng [Ngu89] de convergence double-échelle, permet de voir que

$$u^\varepsilon \overset{\text{2-scale}}{\rightharpoonup} \bar{u}^0,$$

i.e. pour tout $\phi = \phi(x,y) \in C^\infty_0\left(\Omega; C^\infty\left(\mathbb{T}^d\right)\right)$ périodique en y,

$$\int_\Omega u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \int_{\mathbb{T}^d} \bar{u}^0(x)\phi(x,y)dxdy.$$

Ce résultat est plus fort que la simple convergence de u^ε vers $\bar{u}^0$ dans $L^2(\Omega)$. De plus, elle permet de montrer

$$u^\varepsilon - \bar{u}^0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla\bar{u}^0 \longrightarrow 0, \qquad \text{dans} \quad H^1(\Omega). \tag{2.40}$$

Ce résultat de convergence forte ne donne en revanche pas d'estimation d'erreur. Pour obtenir une telle estimation, l'idée est plutôt de considérer

$$r^{2,\varepsilon} := u^\varepsilon - \bar{u}^0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla\bar{u}^0 - \varepsilon^2\Gamma\left(\frac{x}{\varepsilon}\right) \cdot \nabla^2\bar{u}^0,$$

qui satisfait le système elliptique

$$\left\{ \begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla r^{2,\varepsilon} & = \varepsilon F\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega \\ r^{2,\varepsilon} & = -\varepsilon \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \bar{u}^0 - \varepsilon^2 \Gamma\left(\frac{x}{\varepsilon}\right) \cdot \nabla^2 \bar{u}^0, & x \in \partial\Omega \end{array} \right. . \tag{2.41}$$

Il découle ensuite de l'estimation a priori sur le problème elliptique (2.41) :

$$\left\| r^{2,\varepsilon} \right\|_{H^1(\Omega)} \leq C \left(\varepsilon \left\| F\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} + \varepsilon \left\| \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \bar{u}^0 + \varepsilon \Gamma\left(\frac{x}{\varepsilon}\right) \cdot \nabla^2 \bar{u}^0 \right\|_{H^{1/2}(\partial\Omega)} \right) \leq C \varepsilon^{\frac{1}{2}},$$

à condition que $\bar{u}^0 \in H^4(\Omega)$ (on ne cherche pas ici à minimiser la régularité en $\bar{u}^0$). On en déduit les estimations classiques de [BLP78] (voir aussi [CD99, Chapter 7])

$$\left\| u^\varepsilon - \bar{u}^0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \bar{u}^0 \right\|_{H^1(\Omega)} = O\left(\varepsilon^{\frac{1}{2}}\right), \tag{2.42a}$$

$$\left\| u^\varepsilon - \bar{u}^0 \right\|_{L^2(\Omega)} = O\left(\varepsilon^{\frac{1}{2}}\right). \tag{2.42b}$$

La perte de $O\left(\varepsilon^{\frac{1}{2}}\right)$ est due à la mauvaise approximation de u^ε par les correcteurs intérieurs $\bar{u}^0$ et u^1, au voisinage du bord $\partial\Omega$.

L'amélioration de (2.42a) nécessite de comprendre le comportement singulier de u^ε au voisinage du bord, capturé par le terme de couche limite $u_{bl}^{1,\varepsilon}$ à l'ordre 1 solution de

$$\left\{ \begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u_{bl}^{1,\varepsilon} & = 0, & x \in \Omega \\ u_{bl}^{1,\varepsilon} & = -\chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \bar{u}^0, & x \in \partial\Omega \end{array} \right. . \tag{2.43}$$

Les oscillations, non nécessairement périodiques, de $\chi\left(\frac{x}{\varepsilon}\right)$ le long du bord créent de forts gradients au voisinage de $\partial\Omega$ responsables de la borne

$$\left\| u_{bl}^{1,\varepsilon} \right\|_{H^1(\Omega)} = O\left(\varepsilon^{-\frac{1}{2}}\right) \tag{2.44}$$

et donc de l'absence de compacité dans $H^1(\Omega)$.

Lorsque $N = 1$ (équation scalaire), l'application du principe du maximum, donne une borne L^∞ sur $u_{bl}^{1,\varepsilon}$:

$$\left\| u_{bl}^{1,\varepsilon} \right\|_{L^\infty(\Omega)} = O(1),$$

de sorte que (2.42a) entraîne la borne (cf. [JKO94])

$$\left\| u^\varepsilon - u^0 \right\|_{L^2(\Omega)} = O(\varepsilon).$$

Des bornes dans $L^p(\Omega)$ sont démontrées dans l'article [AL87b].

Le cas vectoriel $N > 1$ est bien plus compliqué, car le principe du maximum ne s'applique pas. L'obtention d'une borne dans L^p n'a été possible que grâce à l'étude du noyau de Poisson oscillant P^ε associé à l'opérateur $-\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla \cdot$ et au domaine Ω. Les bornes uniformes sur P^ε démontrées par Avellaneda et Lin [AL87a, Theorem 3] conduisent à

$$\left\| u_{bl}^{1,\varepsilon} \right\|_{L^p(\Omega)} \leq \left\| \chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \bar{u}^0 \right\|_{L^p(\Omega)} = O(1)$$

pour tout $1 < p \leq \infty$, à condition que $\partial\Omega$ soit assez régulière, i.e. $C^{1,\alpha}$ pour $0 < \alpha \leq 1$. Ce résultat a récemment été étendu pour $p = 2$ à des ouverts Ω Lipschitz par Kenig et Shen [KS11b], mais sous hypothèse de symétrie de A. Kenig, Lin et Shen [KLS12c, Theorem 2.1], ont même obtenu une borne sur $u_{bl}^{1,\varepsilon}$ uniforme en ε en norme $H^{1/2}(\Omega)$. Dans ce papier, les auteurs établissent également la borne

$$\left\|u^\varepsilon - u^0\right\|_{L^2(\Omega)} \leq \varepsilon |\ln(\varepsilon)|^{\frac{1}{2}+\sigma} \left[\|f\|_{L^2(\Omega)} + \|\varphi_0\|_{H^1(\partial\Omega)}\right]. \tag{2.45}$$

pour $\sigma > 0$. De nombreux autres travaux récents traitent des estimations d'erreur en homogénéisation périodique [KLS10, KS11a, KLS12b, HC10].

Aller plus loin que $O(\varepsilon)$ nécessite de prendre en compte des correcteurs intérieurs d'ordre 2 dans le développement et un terme de couche limite à l'ordre 1 (voir [MV97, Theorem 2.4] pour le cas scalaire, et section 3.2 Theorem 3.13 pour le cas $N > 1$) : pour $\omega > 0$,

$$\left\|u^\varepsilon - \bar{u}^0 - \varepsilon\left(\chi\left(\frac{x}{\varepsilon}\right) \cdot \nabla\bar{u}^0 + u_{bl}^{1,\varepsilon}\right)\right\|_{L^2(\Omega)} \leq C\varepsilon^{1+\frac{\omega}{2}} \left\|\bar{u}^0\right\|_{H^{2+\omega}(\Omega)}. \tag{2.46}$$

En particulier, l'estimation (2.46) met en lumière que l'identification de l'ordre 1 dans l'approximation de u^ε

$$u^\varepsilon \approx \bar{u}^0 + \varepsilon\bar{u}^1$$

nécessite d'étudier la convergence de $u_{bl}^{1,\varepsilon}$, ou encore d'homogénéiser le problème de couche limite et d'obtenir, si possible, des estimations d'erreur.

2.5.2 Homogénéisation de la couche limite (Chapitre 3)

> Of particular importance is the analysis of the behaviour of solutions near boundaries and, possibly, any associated boundary layers. Relatively little seems to be known about this problem. (Bensoussan, Lions et Papanicolaou [BLP78, p. xiii])

Un des objectifs de l'article [Pra11] à paraître dans *Asymptotic Analysis* et reproduit dans le Chapitre 3 est l'homogénéisation du système elliptique, à coefficients et donnée de Dirichlet oscillants

$$\left\{\begin{array}{rll} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u_{bl}^\varepsilon & = 0, & x \in \Omega \\ u_{bl}^\varepsilon & = \varphi\left(x, \frac{x}{\varepsilon}\right), & x \in \partial\Omega \end{array}\right., \tag{2.47}$$

dit problème de couche limite. Le cas particulier crucial où $\varphi := \chi\cdot\nabla\bar{u}^0$, fait le lien avec $u_{bl}^{1,\varepsilon}$ ci-dessus. Cette analyse a été menée dans la cas scalaire par Moskow et Vogelius [MV97] pour des ouverts polygonaux convexes à pentes satisfaisant les hypothèses de rationnalité **RAT** et pour $\bar{u}^0 \in H^{2+\omega}(\Omega)$. Elle a été étendue au cas vectoriel et aux ouverts polygonaux à pentes **DIV** par Gérard-Varet et Masmoudi sous hypothèse $\bar{u}^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$. Notre objectif est de baisser la régularité requise sur $\bar{u}^0$ dans le cas **DIV**, ce qui est décisif pour obtenir un développement des valeurs propres à l'ordre 1 (cf. section 2.5.3).

Homogénéiser le système (2.47) signifie moyenner les coefficients (ce qui est classique au vu de la section 2.5.1) et la donnée de Dirichlet oscillante. Comme le souligne la citation, ce problème a une longue histoire. Les résultats obtenus dans l'article [Pra11] n'ont été possibles que grâce aux progrès récents de Gérard-Varet et Masmoudi [GVM11].

Les difficultés principales sont de deux natures :

1. La borne (2.44) est singulière et entraîne l'absence de compacité sur u_{bl}^ε dans $H^1(\Omega)$. Il en découle que les techniques de fonctions test oscillantes ou de convergence double-échelle sont inopérantes pour déterminer la condition sur le bord homogénéisée.

2. Le bord casse la structure périodique du problème. En effet, les oscillations de $x \in \partial\Omega \mapsto \varphi\left(x, \frac{x}{\varepsilon}\right)$ n'ont pas de raison d'être périodiques en général, même si $\varphi = \varphi(x, y)$ est périodique en y. Il en résulte que l'homogénéisation de (2.47) est un problème d'homogénéisation non périodique.

Par conséquent pour homogénéiser (2.47), on doit faire appel à une description précise de u^ε_{bl}. On établit une asymptotique du type

$$u^\varepsilon_{bl} \approx v_{bl}\left(x, \frac{x}{\varepsilon}\right)$$

où v_{bl} satisfait le système (2.19). L'étude de v_{bl} a été abordée en sections 2.4.1 et 2.4.2.

Prenons pour $\Omega \subset \mathbb{R}^2$ un ouvert convexe et polygonal, et pour φ la fonction à variables lente et rapide séparées

$$\varphi(x, y) = -\chi(y) \cdot \nabla \bar{u}^0.$$

Supposons que les pentes des côtés H^k satisfont **DIV**. L'idée générale pour homogénéiser le système (2.47) est de séparer le problème de l'homogénéisation des coefficients de celui de la donnée sur le bord. On commence par comparer u^ε_{bl} à $\bar{u}^\varepsilon_{bl}$ solution de

$$\left\{\begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla \bar{u}^\varepsilon_{bl} & = 0, & x \in \Omega \\ \bar{u}^\varepsilon_{bl} & = -V^{k,\infty} \cdot \nabla \bar{u}^0, & x \in \partial\Omega \cap H^k \end{array}\right., \tag{2.48}$$

où $V^{k,\infty}$ est la queue de couche limite du correcteur v^k_{bl}, solution de (2.19), associé au côté H^k et à la donnée sur le bord $v_0 = \chi(y)$. À noter que dans le cas **RAT** on n'a pas une queue de couche limite unique mais un continuum de points d'accumulations, ce qui oblige à travailler sur une sous-suite ε_n. L'estimation de $r^\varepsilon_{bl} := u^\varepsilon_{bl} - \bar{u}^\varepsilon_{bl}$ s'appuie sur l'estimation elliptique de Meyers [Mey63, GM99, MS11] et requiert $\bar{u}^0 \in H^{2+\omega}(\Omega)$.

Ce raisonnement montre la nécessité de comprendre l'asymptotique de v_{bl} loin du bord pour identifier la donnée homogénéisée $\overline{\varphi}$. Cette dernière est définie par morceaux, sur chacun des côtés H^k par

$$\overline{\varphi} := -V^{\infty,k} \cdot \nabla \bar{u}^0.$$

Grâce à la régularité sur $\bar{u}^0$, on peut voir que $\overline{\varphi}$ est au moins $H^{1/2}(\partial\Omega)$. La dernière étape consiste à homogénéiser (2.48) ce qui est tout à fait classique, mais requiert aussi de la régularité $H^{2+\omega}(\Omega)$ sur $\bar{u}^0$ si l'on veut avoir un taux de convergence.

Ce faisant, on démontre le théorème suivant :

Théorème D (Theorem 3.16, Chapter 3)**.** *Soit $\Omega \subset \mathbb{R}^2$ un ouvert polygonal convexe, à pentes satisfaisant l'hypothèse de petits diviseurs (2.20). Supposons $\bar{u}^0 \in H^{2+\omega}(\Omega)$. Alors, u^ε_{bl} converge dans $L^2(\Omega)$ vers $\bar{u}^1$ solution du système elliptique*

$$\left\{\begin{array}{rll} -\nabla \cdot \overline{A} \nabla \bar{u}^1 & = 0, & x \in \Omega \\ \bar{u}^1 & = -V^{k,\infty} \cdot \nabla \bar{u}^0, & x \in \partial\Omega \cap H^k \end{array}\right..$$

De plus, il existe $\gamma > 0$ tel que

$$\left\| u^\varepsilon_{bl} - \bar{u}^1 \right\|_{L^2(\Omega)} = O\left(\varepsilon^\gamma\right),$$

et si $\bar{u}^0 \in H^3(\Omega) \cap C^2\left(\overline{\Omega}\right)$ on peut prendre $\gamma = \frac{1}{2}$ dans l'estimation qui précède.

On a bien-sûr un théorème analogue dans le cas des pentes rationnelles, à condition de remplacer la convergence $\varepsilon \to 0$ par la convergence sur une sous-suite ε_n sur laquelle on a unicité de la queue de couche limite. On renvoie à [MV97] pour une description rigoureuse de ces sous-suites. Par ailleurs, on doit pouvoir, en adaptant les idées du papier [GVM11, section 3.4], démontrer un théorème similaire en dimension $d = 3$ pour des polyèdres convexes à normales satisfaisant **RAT** ou **DIV**.

Ce théorème est une extension du résultat de Moskow et Vogelius [MV97, Theorem 4.2]. Il l'améliore dans trois directions :
- Il traite des domaines polygonaux plus généraux, ce qui a été rendu possible par les travaux de Gérard-Varet et Masmoudi [GVM11].
- Il s'affranchit du cadre scalaire et traite les systèmes elliptiques.
- Il donne un taux de convergence, alors que le théorème de Moskow et Vogelius ne contient que la convergence dans $L^2(\Omega)$.

Soulignons le fait que la régularité $\bar{u}^0$ est la régularité minimale que l'on puisse espérer, au vu en tout cas de la démonstration. Cette hypothèse est déjà présente dans [MV97].

Le Théorème D apporte plus généralement une réponse (partielle) à l'homogénéisation de systèmes elliptiques à donnée de Dirichlet oscillante. Il est à rapprocher du théorème de Gérard-Varet et Masmoudi [GVM12, Theorem 1] pour des ouverts réguliers uniformément convexes (typiquement le disque) où les auteurs démontrent l'existence d'une donnée sur le bord homogénéisée $\overline{\varphi}$ telle que

$$\left\| u^\varepsilon_{bl} - \bar{u}^1 \right\|_{L^2} = O\left(\varepsilon^\gamma\right),$$

pour tout $\gamma < \frac{1}{11}$ (en dimension 2). Leur résultat repose sur un argument d'approximation de Ω par des polygônes à pentes satisfaisant des hypothèses de petits diviseurs (2.20), et des versions raffinées des estimations de l'article [GVM11].

Signalons aussi la série de papiers [LS12, ASS12a, ASS12b] qui traite de systèmes elliptiques à donnée de Dirichlet oscillante mais coefficients non oscillants. Dans ce cas, le problème d'homogénéisation est bien plus simple, et peut être abordé par l'analyse de Fourier.

En dépit de ces avancées, l'homogénéisation de (2.47) reste un problème difficile et ouvert en général. En particulier, malgré le Théorème B de cette introduction, il semble que l'absence de vitesse de convergence de v_{bl} vers sa queue de couche limite, inhérente au cas général, entrave la démonstration d'un analogue du Théorème D pour des polygônes arbitraires.

Concluons ce point en indiquant qu'il serait intéressant de généraliser ce type d'études à des systèmes elliptiques généraux non nécessairement sous forme divergence en s'appuyant sur les résultats de Avellaneda et Lin [AL89a]. Des travaux dans ce sens ont été menés par Barles, Da Lio, Lions et Souganidis [BDLLS08], et très récemment Choi et Kim ont étudié le problème de Neumann oscillant pour des opérateurs sous forme non divergence [CK13].

2.5.3 Asymptotique des valeurs propres d'un système elliptique à coefficients oscillants (Chapitre 3)

Cette étude de l'homogénéisation de u^ε_{bl} sous l'hypothèse de faible régularité $\bar{u}^0 \in H^{2+\omega}(\Omega)$ nous permet d'aborder l'autre point de l'article [Pra11], à paraître dans *Asymptotic Analysis*. La problématique est l'homogénéisation du problème aux valeurs propres

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla v^\varepsilon &= \lambda^\varepsilon v^\varepsilon, \quad x \in \Omega \\ \qquad\qquad v^\varepsilon &= 0, \qquad x \in \partial\Omega \end{cases} \tag{2.49}$$

où $\Omega \subset \mathbb{R}^2$ et A est supposée symétrique, i.e. pour tous $1 \leq \alpha,\ \beta \leq 2$, $1 \leq i,\ j \leq N$, $A^{\alpha\beta}_{ij} = A^{\beta\alpha}_{ji}$. Plus précisément, il s'agit de déterminer l'asymptotique de λ^ε_k, quand $\varepsilon \to 0$ et le mode

k est fixé. Ce problème a des applications, par exemple pour la propagation d'ondes basses fréquences en milieux hétérogènes (voir la section 2.2.1). Les problèmes aux valeurs propres sont très étudiés dans le contexte de l'homogénéisation (propagation d'ondes, vibrations de systèmes fluides-structure) : pour l'étude du spectre hautes fréquences voir par exemple [AC98b, AC98a, AF10, CZ00a], pour l'étude du spectre basses fréquences voir [SV93, SV95, MV97, MV98, CZ00b]. L'homogénéisation du problème aux valeurs propres ouvre également la voie à l'étude des effets des rugosités sur les instabilités hydrodynamiques.

La limite de λ_k^ε, que cette valeur propre soit simple ou multliple est connue depuis longtemps (pour une référence, voir [AC98a, section 2]) : $\lambda_k^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \lambda_k^0$ valeur propre associée au problème homogénéisé

$$\left\{ \begin{aligned} -\nabla\cdot\overline{A}\nabla\bar{v}_k^0 &= \lambda_k^0\bar{v}_k^0, \quad x\in\Omega \\ \bar{v}_k^0 &= 0, \qquad\quad x\in\partial\Omega \end{aligned} \right. . \tag{2.50}$$

Le but des travaux présentés ici est de généraliser le papier de Moskow et Vogelius [MV97], et d'améliorer leur développement des valeurs propres à l'ordre 1 en ε :

> It should be extremely interesting to derive a similar representation formula for polygons with sides of irrational slopes or for smooth domains. In particular, it would be interesting to see if this leads to a single first-order correction for any eigenvalue. (Moskow et Vogelius [MV97, p. 1288–1289])

Comme le soulignent Allaire et Amar [AA99], les termes d'ordre 1 du développement des valeurs propres λ_k^ε du spectre basses fréquences, et certaines parties du spectre hautes fréquences sont déterminés par les couches limites. Ce fait apparaît clairement sur la formule de [MV97, Theorem 3.6].

De nombreux travaux sont consacrés à l'asymptotique de λ_k^ε à l'ordre 0 et à des estimations d'erreur entre λ_k^ε et sa limite homogénéisée : Kesavan [Kes79a, Kes79b], Jikov, Kozlov et Oleĭnik [JKO94], Allaire et Conca [AC98a], Kenig, Lin et Shen [KLS12c, KLS12a]. Aller au delà d'une approximation en $O(\varepsilon)$ utilise nécessairement des informations sur le problème de couche limite.

Difficultés et stratégie

L'application du principe min-max [AC98a, Theorem 2.2] conduit à la borne

$$\left|\frac{1}{\lambda_k^\varepsilon}-\frac{1}{\lambda_k^0}\right| \leq \left\|T^\varepsilon - T^0\right\|_{\mathcal{L}(L^2(\Omega))},$$

où T^ε (resp. T^0) associe à la source $f\in L^2(\Omega)$ la solution u^ε (resp. u^0) du problème elliptique à coefficients oscillants (resp. homogénéisé). L'obtention d'un développement à reste en $O\left(\varepsilon^{1+\gamma}\right)$ passe par une formule raffinée due à Osborn : cf. [Osb75, Theorem 3.1] et (3.61) dans le Chapitre 3. Celle-ci met en jeu les opérateurs T^ε et T^0, en restriction au sous-espace propre Vect $\left(\bar{v}_{k+j}^0\right)_{j=0,\dots m}$ associé à la valeur propre $\lambda_k^0 = \dots\ \lambda_{k+m}^0$ (de dimension finie m, mais pas forcément égale à 1 en cas de multiplicité >1). Il faut donc avoir des approximations de termes du type $u_{k+j}^\varepsilon = T^\varepsilon\left(\bar{v}_{k+j}^0\right)$ à l'ordre $O\left(\varepsilon^{1+\gamma}\right)$, ce qui fait appel aux couches limites.

On traite deux types de domaines : des domaines réguliers et des domaines polygonaux convexes à pentes rationnelles ou petits diviseurs. La source principale de difficultés dans ce dernier cas est l'irrégularité du domaine due à la présence des coins. En effet, les estimations d'erreur du Théorème D sur le problème de couche limite font appel à de la régularité

$H^{2+\omega}(\Omega)$ sur $\bar{v}^0_{k+j}$. Celle-ci n'est pas une conséquence de la théorie de régularité elliptique classique de Agmond, Douglis et Nirenberg [ADN59, ADN64].

Comme $\bar{v}^0_{k+j} \in H^1(\Omega)$, il découle des résultats de Grisvard [Gri85, Theorem 3.2.1.2], $\bar{v}^0_{k+j} \in H^{2+\omega}(\Omega)$ dans le cas scalaire $N = 1$. Ce résultat de régularité elliptique devient faux en l'absence de convexité sur Ω. Obtenir cette régularité dans le cas vectoriel $N > 1$ est moins évident. Il faut faire appel à des résultats dus à Dauge [Dau88] d'une part et à Kozlov, Maz′ya et Rossmann [KMR01] d'autre part pour obtenir la régularité $H^{2+\omega}(\Omega)$ sur $\bar{v}^0_{k+j}$. Le Théorème 3.4 du Chapitre 3 contient les résultats de régularité dont on a besoin.

Résultat de la thèse

On démontre le théorème :

Théorème E (Theorems 3.6 et 3.7, Chapter 3)**.** *Soit $\Omega \subset \mathbb{R}^2$ un ouvert qui est soit régulier uniformément convexe, soit polygonal convexe à pentes satisfaisant* **RAT** *ou* **DIV**. *Soit λ^0_k une valeur propre de* (2.50) *de multiplicité m et* $\mathrm{Vect}\left(\bar{v}^0_{k+j}\right)_{j=0,\dots m}$ *le sous-espace propre associé.*
Alors, il existe $\gamma > 0$, il existe $\bar{u}^1_j \in L^2(\Omega)$ pour $j = 1 \dots m$ tels que

$$\left[\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^\varepsilon_{k+j}}\right]^{-1} = \lambda^0_k + \varepsilon\frac{\lambda^0_k}{m}\sum_{j=0}^{m-1}\int_\Omega \bar{u}^1_j(x)\cdot\bar{v}^0_{k+j}(x)dx + O\big(\varepsilon^{1+\gamma}\big)$$

(pour une sous-suite ε_n lorsque le polygône a des pentes rationnelles).

Les corrections des valeurs propres à l'ordre 1 font directement intervenir les solutions $\bar{u}^1_j$ de problèmes de couches limites homogénéisés. De plus (cf. Theorem D) on peut remplacer l'exposant γ par $\frac{1}{2}$ sous hypothèse

$$\mathrm{Vect}\left(\bar{v}^0_{k+j}\right)_{j=0,\dots m} \subset H^3(\Omega) \cap C^2(\overline{\Omega}).$$

Notons enfin que ce théorème, de même que le Théorème D, représente une généralisation des résultats de Moskow et Vogelius dans trois directions : domaines polygonaux plus généraux, systèmes elliptiques et amélioration du reste (la formule de [MV97] est énoncée avecun reste $o(\varepsilon)$ au lieu de $O\left(\varepsilon^{1+\gamma}\right)$).

Perspectives

L'ensemble de ces travaux sur l'homogénéisation de problèmes de couches limites et l'amélioration des estimations d'erreurs qu'ils permettent ouvre la voie à des *études numériques*. Le problème aux valeurs propres à coefficients oscillants (2.49) a déjà fait l'objet d'études ne recourant qu'à des correcteurs intérieurs : voir par exemple les travaux de Kesavan [Kes79a, Kes79b], et ceux de Engquist, Holst et Runborg [EHR11] (homogénéisation numérique de l'équation des ondes (2.3)).

Pour optimiser les calculs d'approximations dans des milieux hétérogènes (valeurs propres…), et capturer les oscillations près du bord, une piste est d'utiliser les correcteurs de couche limite, dans l'esprit des travaux de Versieux et Sarkis [SV06, SV08] (domaine Ω rectangulaire). La formule pour les queues de couches limites démontrée par Moskow et Vogelius [MV97] dans le cas de pentes rationnelles, et généralisée à des pentes quelconques par l'analyse du Chapitre 4 (équation (4.68)), rend possible le calcul des données du problème de couche limite homogénéisé.

Enfin, l'enjeu principal de la suite de nos travaux est de s'affranchir du cadre de l'homogénéisation de microstructures périodiques. Ce sujet de recherche est très actif. Signalons des travaux concernant :

- le cadre faiblement stochastique (périodique avec une petite perturbation stochastique) qui permet de prendre en compte des défauts dans une structure périodique [ALB10, LBT12, ALB12],
- le cadre stochastique continu ou discret [PV81, JKO94, BLBL07, BLBL06] et les travaux de Gloria et Otto [GO11, GO12] sur les estimations d'erreur en homogénéisation stochastique pour un opérateur elliptique discret.

Dans ce cadre des questions nouvelles apparaissent, notamment sur le plan de l'approximation numérique. Les problèmes cellulaires pour les correcteurs intérieurs par exemple, ne sont pas posés dans le tore, mais dans l'espace entier (problématique de la troncature du problème). Sur toutes ces questions, on peut se référer à l'ouvrage récent de Anantharaman, Costaouec, Le Bris, Legoll et Thomines [ACLB+12].

2.5.4 Fluides faiblement viscoélastiques (Chapitre 6)

Ce travail est en collaboration avec Didier Bresch.

Un de nos objectifs est l'étude de la limite $\mathrm{We} \to 0$ de certains modèles fluides viscoélastiques. Les travaux présentés ici constituent une première étape dans notre projet d'étudier mathématiquement les perturbations de la dynamique newtonienne d'un fluide, induites par l'addition d'un peu d'élasticité.

Dans la suite on considère deux modèles macro-macro de fluides viscoélastiques. Le premier est le système (2.12) pour $\mathrm{Re} = O(1)$ fixé

$$\left\{\begin{aligned} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau, \\ \nabla\cdot u &= 0, \\ \mathrm{We}\left(\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau - a\left[D(u)\tau + \tau D(u)\right] + \tilde{\beta}(\tau)\right) + \tau &= 2\omega D(u). \end{aligned}\right. \tag{2.51}$$

Le deuxième modèle est le système de FENE-P

$$\left\{\begin{aligned} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau, \\ \nabla\cdot u &= 0, \\ \tau &= \tfrac{(b+d)\omega}{b}\tfrac{1}{\mathrm{We}}\left(\tfrac{A}{1-\frac{\mathrm{Tr}\,A}{b}} - \mathrm{I}\right), \\ \partial_t A + u\cdot\nabla A - \nabla u A - A\left(\nabla u\right)^T + \tfrac{1}{\mathrm{We}}\tfrac{A}{1-\frac{\mathrm{Tr}\,A}{b}} &= \tfrac{1}{\mathrm{We}}\mathrm{I}, \end{aligned}\right. \tag{2.52}$$

qui est la clôture d'un modèle microscopique approché. Le tenseur de structure A a une interprétation microscopique. Notons que ces systèmes sont posés dans Ω un ouvert borné de $\mathbb{R}^d$, $\Omega = \mathbb{R}^d$ ou $\Omega = \mathbb{T}^d$, que $u = u(t,x) \in \mathbb{R}^d$, $\tau = \tau(t,x) \in M_d(\mathbb{R})$ et $A = A(t,x) \in M_d(\mathbb{R})$. On rappelle que

$$D(u) := \frac{\nabla u + (\nabla u)^T}{2}, \qquad W(u) := \frac{\nabla u - (\nabla u)^T}{2}.$$

On impose une condition de non-glissement sur la vitesse u sur le bord $\partial\Omega$. Il n'y a pas de condition sur le bord ni pour τ, ni pour A. On complète enfin ce système avec des conditions initiales :

$$u(0,\cdot) := u_0, \qquad \tau(0,\cdot) := \tau_0, \qquad A(0,\cdot) := A_0.$$

Pour une introduction à l'analyse mathématique de ces modèles, on pourra consulter l'article de survol récent de [Sau12], qui contient une bibliographie abondante. Dans la suite, on considère ces systèmes pour $0 < \omega < 1$ (fluides de Jeffrey) uniquement dans le cadre des solutions faibles globales en temps.

Existence de solutions faibles

Le point de départ de l'existence de solutions faibles est l'obtention de bornes a priori sur la solution. Ces bornes peuvent provenir d'une estimation d'énergie ou d'une *estimation d'entropie* (ou d'*énergie libre*). Dans cette optique, signalons les progrès que permettent les résultats récents de Hu et Lelièvre [HL07].

Pour le modèle corotationnel ($\tilde{\beta} = 0$ et $a = 0$), une simple estimation d'énergie a priori conduit à

$$\omega \|u\|_{L^2(\Omega)}^2 (T) + 2\omega(1-\omega) \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 + \frac{\mathrm{We}}{2} \|\tau\|_{L^2(\Omega)}^2 (T) + \int_0^T \|\tau\|_{L^2(\Omega)}^2 \\ \leq \omega \|u_0\|_{L^2(\Omega)}^2 + \frac{\mathrm{We}}{2} \|\tau_0\|_{L^2(\Omega)}^2 . \quad (2.53)$$

Cette estimation est à la base du résultat d'existence de solutions faibles pour le modèle corotationnel avec $0 < \omega < 1$ dû à Lions et Masmoudi [LM00]. Dans l'optique d'étudier la limite newtonienne, nous avons repris au Chapitre 6 la preuve dans le cas $d = 2$ en traçant la dépendance en We.

Une des difficultés majeures de la démonstration est le passage à la limite sur une suite de solutions approchées (u_n, τ_n). En effet (2.53) ne donne qu'une borne sur ∇u_n (resp. τ_n) dans $L^2((0,\infty); L^2)$. Le produit $u_n \cdot \nabla u_n$ (resp. $u_n \cdot \nabla \tau_n = \nabla \cdot (u_n \tau_n)$) ne pose pas de problème puisque l'on peut gagner de la compacité par injection de Rellich (ou utiliser un lemme div-rot pour le premier). En revanche $\tau_n W(u_n)$ n'est a priori qu'un produit de suites faiblement convergentes. Pour étudier la convergence de ce terme non-linéaire, on est conduit à introduire des mesures de défaut de convergence, qui quantifient le défaut de convergence forte. En utilisant fortement la structure du système, on peut montrer que si les mesures de défaut sont nulles initialement, elles sont identiquement nulles, ce qui donne la convergence du produit $\tau_n W(u_n)$ dans $\mathcal{D}'$.

Qu'en est-il de l'existence de solutions faibles pour les autres modèles de type (2.51) ? Chercher à obtenir une estimation d'énergie du type (2.53) est voué à l'échec lorsque $a \neq 0$. En effet, le terme corotationnel

$$\int_\Omega (\tau W(u) - W(u)\tau) : \tau = 0,$$

alors que

$$\int_\Omega (\tau D(u) + D(u)\tau) : \tau$$

n'est ni nul, ni positif en général. Pour Oldroyd-B, ni l'estimation d'énergie obtenue en prenant la trace de l'équation sur τ

$$\|u\|_{L^2(\Omega)}^2 (T) + \int_\Omega \operatorname{Tr} \tau(T) + 2(1-\omega) \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{\mathrm{We}} \int_0^T \int_\Omega \operatorname{Tr} \tau \leq \|u_0\|_{L^2(\Omega)}^2 + \int_\Omega \operatorname{Tr} \tau_0,$$

ni l'estimation d'entropie de [HL07] provenant de l'interprétation microscopique du modèle, ne semblent fournir les contrôles suffisants sur τ pour établir l'existence de solutions faibles. Il manque en particulier une borne $L^2((0,\infty); L^2)$ sur τ. Notons que pour certains modèles non-linéaires avec $\tilde{\beta} = \tilde{\alpha}\tau^2$ (Giesekus) ou $\tilde{\beta} = \tilde{\alpha}\tau \operatorname{Tr}\tau$ (Phan-Thien et Tanner), la borne $L^2\left((0,\infty);L^2\right)$ sur τ est facile a obtenir. Ainsi Masmoudi [Mas11] a pu démontrer l'existence de solutions faibles pour ces modèles.

Pour le modèle de FENE-P (2.52), Masmoudi [Mas11] a pu établir l'existence de solutions faibles grâce à la décroissance d'une entropie relative inventée par Hu et Lelièvre [HL07]. Le point fondamental est que l'entropie relative

$$\begin{aligned}
&\frac{1}{2}\left\|u\right\|^2_{L^2(\Omega)}(T) + (1-\omega)\int_0^T \left\|\nabla u\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega \left[-\ln\left(\det A\right) - b\ln\left(1-\frac{\operatorname{Tr}(A)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](T) \\
&\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}^2}\int_0^T\int_\Omega \left[\frac{\operatorname{Tr}A}{\left(1-\frac{\operatorname{Tr}A}{b}\right)^2} - \frac{2d}{1-\frac{\operatorname{Tr}A}{b}} + \operatorname{Tr}\left(A^{-1}\right)\right] \\
&\leq \frac{1}{2}\left\|u_0\right\|^2_{L^2(\Omega)} + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega \left[-\ln\left(\det A_0\right) - b\ln\left(1-\frac{\operatorname{Tr}(A_0)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right]
\end{aligned} \tag{2.54}$$

contrôle la norme $L^2\left((0,\infty);L^2\right)$ de τ. Nous revenons sur cette borne en détail au Chapitre 6.

Limites newtoniennes

Les résultats d'existence de solutions faibles globales en temps ouvrent la voie à l'étude de la limite newtonienne, $\mathrm{We} \to 0$. Formellement, il est facile de voir que les solutions du système corotationnel (2.51) convergent vers des solutions u^0 du système de Navier-Stokes

$$\begin{cases}
\partial_t u^0 + u^0\cdot\nabla u^0 - \Delta u^0 + \nabla p^0 &= 0, \quad \Omega, \\
\nabla\cdot u^0 &= 0, \quad \Omega, \\
u^0 &= 0, \quad \partial\Omega.
\end{cases} \tag{2.55}$$

Notons que la condition de non-glissement sur u est compatible avec le système de Navier-Stokes à la limite, de sorte que dans la limite $\mathrm{We} \to 0$, il n'y a pas de singularités au voisinage de $\partial\Omega$, et donc pas de couches limites. Comme le bord n'introduit pas de complications, on énonce nos résultats indifféremment pour Ω un ouvert borné, $\Omega = \mathbb{T}^d$ ou $\Omega = \mathbb{R}^d$.

À notre connaissance, les seuls résultats s'intéressant à la limite newtonienne de fluides viscoélastiques ont été obtenus dans le cadre de solutions fortes par Saut [Sau86] pour des fluides de Maxwell (pas de diffusion dans l'équation des moments) dans le régime linéaire et par Molinet et Talhouk [MT08] pour le système de Johnson-Segalman (2.51) ($\tilde{\beta} = 0$ et a quelconque). Dans ce dernier cas, pour $a \neq 0$, il n'est pas possible de travailler avec des énergies (cf. ci-dessus). Leur analyse repose sur un découpage en hautes et basses fréquences (la fréquence de coupure dépendant de We).

La justification mathématique des limites formelles pour des solutions faibles passe par l'obtention de bornes a priori, uniformes en We. Certaines bornes, dont la borne $L^\infty\left((0,\infty);L^2\right)$ sur τ pour le système corotationnel (2.51) déduite de (2.53), ne sont pas uniformes en We. Elles ont été utiles pour la théorie de Cauchy, mais ne permettent pas d'aborder la limite newtonienne. Dans certains cas, il est nécessaire de supposer que les données initiales sont bien préparées (en un sens à préciser) pour obtenir l'uniformité des bornes.

Décrivons brièvement les résultats les plus importants que nous obtenons au Chapitre 6. Le premier concerne la convergence forte dans le système corotationnel.

Théorème F (Theorem 6.3, Chapter 6)**.** *Soit* $d = 2,\ 3$, $u_0 \in H^{4,\sigma}(\Omega)$ *indépendant de* We *et* $\tau_0 \in L^2(\Omega) \cap L^q(\Omega)$, *avec* $2 < q \leq 3$. *Remarquons que* τ_0 *peut dépendre de* We *de la façon suivante :* $\|\tau_0\|_{L^2(\Omega)} = O(1)$. *Soient aussi*

$$u \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\big((0,\infty); \dot{H}^1\big), \quad \textit{and} \quad \tau \in L^\infty\left((0,\infty); L^2\right) \cap L^\infty_{loc}\left((0,\infty); L^q\right)$$

des solutions globales faibles à (2.51) *au sens de Lions et Masmoudi [LM00] associées aux données initiales* u_0 *et* τ_0.
Alors, il existe $0 < T^* < \infty$ *indépendant de* We *et*

$$u^0 \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\big((0,\infty); \dot{H}^1\big)$$

une solution globale de (2.55) *associée à la donnée initiale* u_0 *telle que, de plus,* u^0 *appartienne à* $L^\infty\left((0,T); H^4\right)$ *pour tout* $0 < T < T^*$. *De plus, pour tout* $0 < T < T^*$,

$$\sup_{0<t<T} \left(\omega \left\| u(t,\cdot) - u^0(t,\cdot) \right\|_{L^2}^2 + \omega(1-\omega) \int_0^t \left\| \nabla\left(u - u^0\right) \right\|_{L^2}^2 \right.$$
$$\left. + \frac{\mathrm{We}}{2} \left\| \tau(t,\cdot) - \tau^0(t,\cdot) \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left\| \tau - \tau^0 \right\|_{L^2}^2 \right)^{\frac{1}{2}} = O\big(\sqrt{\mathrm{We}}\big). \quad (2.56)$$

La clé pour obtenir un tel résultat de convergence est d'introduire des termes de correcteur pour u et τ puis d'estimer l'entropie relative (2.56). La méthode est décrite au Chapitre 6 et se rapproche des techniques d'entropie relative ou d'énergie modulée introduites par Dafermos [Daf79] et Yau [Yau91]. Ces méthodes sont devenues un outil crucial et largement utilisé pour l'étude asymptotique de modèles cinétiques dans le contexte de l'hydrodynamique [LM01, GSR04] et de la limite quasi-neutre pour le système de Vlasov-Poisson [Bre00, HK11]. Elles ont aussi été utilisées avec succès pour l'approximation des fluides incompressibles par des systèmes hyperboliques [BNP04, NR06] ; voir aussi [Tza05, LT12] pour un exposé de la méthode générale, et l'utilisation des correcteurs. Mentionnons aussi l'utilisation des entropies relatives pour l'étude du comportement en temps long de certains modèles micro-macro de solutions de polymères, et la convergence à l'équilibre par Jourdain, Le Bris, Lelièvre and Otto [JLBLO06].

Pour le système de FENE-P (2.52), la convergence (même formelle) n'est pas évidente. S'il apparaît que A converge (formellement) vers $A^0 := \frac{b}{b+d}\,\mathrm{I}$, la convergence de τ est moins claire en raison de la singularité $1/\mathrm{We}$ dans la définition de τ. Pour conclure, il est nécessaire d'avoir une borne sur τ uniforme en We. Pour le moment, nous n'obtenons qu'un résultat de convergence faible pour des données bien préparées.

Théorème G (Proposition 6.2, Chapter 6)**.** *Soit* $d = 2,\ 3$. *Soit une solution globale faible* (u, A, τ) *de* (2.52) *au sens de Masmoudi [Mas11]. Supposons que* $\|u_0\|_{L^2(\Omega)} = O(1)$. *Alors :*
• *Supposons les conditions initiales sont* mal-préparées *dans le sens*

$$\int_\Omega \left[-\ln\left(\det A_0\right) - b \ln\left(1 - \frac{\operatorname{Tr} A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right] = O\left(1\right).$$

Alors, A *tend vers* A^0 *dans* $L^2\left((0,\infty); L^2(\Omega)\right)$ *et*

$$\left\| A - A^0 \right\|_{L^2((0,\infty); L^2(\Omega))} = O\big(\sqrt{\mathrm{We}}\big). \quad (2.57)$$

• *Supposons de plus que les données initiales sont* bien-préparées *dans le sens*

$$\int_\Omega \left[-\ln\left(\det A_0\right) - b \ln\left(1 - \frac{\operatorname{Tr} A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right] = O\left(\mathrm{We}\right).$$

Alors, le taux de convergence de A est amélioré :

$$\left\|A - A^0\right\|_{L^\infty((0,\infty);L^2(\Omega))} = O\big(\sqrt{\mathrm{We}}\big), \tag{2.58a}$$

$$\left\|A - A^0\right\|_{L^2((0,\infty);L^2(\Omega))} = O\left(\mathrm{We}\right). \tag{2.58b}$$

De plus, τ est borné uniformément dans $L^2\left((0,\infty)\times\Omega\right)$, et u (resp. τ) converge au sens des distributions vers u^0 (resp. $2\omega D(u^0)$), où u^0 satisfait le système de Navier-Stokes (2.55).

Pour obtenir (2.57), (2.58a) et (2.58b) on utilise la convexité de la fonctionnelle entropie et l'inégalité (2.54). Des calculs pour obtenir la convergence forte de u et τ à l'aide d'estimations d'entropies relatives font l'objet d'un travail en cours. Une piste pour obtenir de tels résultats est de construire un Ansatz pour A, sur le principe de ce que nous avons fait sur le système corotationnel.

Chapter 3

First-order expansion for the Dirichlet eigenvalues of an elliptic system with oscillating coefficients

This chapter corresponds to the paper [Pra11], to appear in *Asymptotic Analysis*.

Contents

Abstract This chapter is concerned with the homogenization of the Dirichlet eigenvalue problem, posed in a bounded domain $\Omega \subset \mathbb{R}^2$, for a vectorial elliptic operator $-\nabla \cdot A^\varepsilon(\cdot)\nabla\cdot$ with ε-periodic coefficients. We analyse the asymptotics of the eigenvalues $\lambda^{\varepsilon,k}$ when $\varepsilon \to 0$, the mode k being fixed. A first-order asymptotic expansion is proved for $\lambda^{\varepsilon,k}$ in the case when Ω is either a smooth uniformly convex domain, or a convex polygonal domain with sides of slopes satisfying a small divisors assumption. Our results extend those of Moskow and Vogelius in [MV97] restricted to scalar operators and convex polygonal domains with sides of rational slopes. We take advantage of the recent progress due to Gérard-Varet and Masmoudi [GVM11, GVM12] in the homogenization of boundary layer type systems.

3.1 Introduction

This chapter is devoted to the homogenization of the Dirichlet eigenvalue problem

$$\begin{cases} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon = & \lambda^\varepsilon v^\varepsilon, \quad x\in\Omega \\ \qquad\qquad v^\varepsilon = & 0, \qquad x\in\partial\Omega \end{cases} \tag{3.1}$$

posed in a planar domain Ω with periodic microstructure. Some reasons for the study of the asymptotical behaviour of the eigenvalues when the period $\varepsilon\to 0$ are expounded in [SV93]. Among physical motivations is the analysis of low frequency vibrations in periodic composite media. Significant progress in the direction of a better understanding of the asymptotics of λ^ε when $\varepsilon\to 0$ has been achieved first by Santosa and Vogelius in [SV93] then by Moskow and Vogelius in [MV97] under weaker assumptions. Our work extends the results of [MV97] to the case of elliptic systems and more general domains Ω. Moreover, error estimates have been improved.

Before entering into more details, let us state our mathematical framework. Let $N\in\mathbb{N}$, $N\geq 1$. Throughout this paper, Ω stands for a bounded open subset of $\mathbb{R}^2$, $v^\varepsilon = v^\varepsilon(x)\in\mathbb{R}^N$ and $A = A^{\alpha\beta}(y)\in M_N(\mathbb{R})$ is a family of periodic functions of $y\in\mathbb{T}^2$ indexed by $1\leq\alpha,\beta\leq 2$. Therefore, taking advantage of Einstein's convention for summation:

$$\left(\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon\right)_i = \partial_{x_\alpha}\left(A^{\alpha\beta}_{ij}\left(\frac{x}{\varepsilon}\right)\partial_{x_\beta}v^\varepsilon_j\right).$$

All along these lines, $C>0$ denotes an arbitrary constant independant of ε. The main assumptions on A are:

(A1) ellipticity there exists $\lambda>0$ such that for all $\xi=(\xi^1,\xi^2)\in\mathbb{R}^N\times\mathbb{R}^N$, for all $y\in\mathbb{R}^2$,

$$\lambda\xi^\alpha\cdot\xi^\alpha\leq A^{\alpha\beta}(y)\xi^\alpha\cdot\xi^\beta\leq\lambda^{-1}\xi^\alpha\cdot\xi^\alpha;$$

(A2) periodicity for all $y\in\mathbb{R}^2$, for all $h\in\mathbb{Z}^2$,

$$A(y+h) = A(y);$$

(A3) regularity A is supposed to belong to $C^\infty(\mathbb{R}^2)$;

(A4) symmetry for all $1\leq\alpha,\beta\leq 2$, for all $1\leq i,j\leq N$, $A^{\alpha\beta}_{ij} = A^{\beta\alpha}_{ji}$.

Unless otherwise specified, we always assume **(A1)**, ..., **(A4)**. In a very classical fashion, boundedness of Ω and ellipticity of A imply, through Poincaré inequality and Lax Milgram lemma, that the linear mapping

$$T^\varepsilon:\quad f\in L^2(\Omega)\longmapsto u^\varepsilon\in H^1_0(\Omega),$$

where u^ε is the unique weak solution of (3.1) with r.h.s. equal to f, is well defined, continuous and injective. If one composes T^ε with the compact injection of $H^1_0(\Omega)$ in $L^2(\Omega)$, one gets a compact operator, again denoted by T^ε, from $L^2(\Omega)$ in itself. Assumption **(A4)** tells that T^ε is self-adjoint.

From the previous considerations, we know that our eigenvalue problem (3.1) is well posed. There exists a sequence of eigenvalues $0<\lambda^{\varepsilon,0}\leq\lambda^{\varepsilon,1}\leq\ldots\lambda^{\varepsilon,k}\xrightarrow{k\to\infty}\infty$ and a hilbertian basis $(v^{\varepsilon,k})$ of $L^2(\Omega)$ of corresponding eigenvectors. To tackle the issue of the asymptotical behaviour of $(\lambda^\varepsilon,v^\varepsilon)$ we deeply use the periodic structure of the problem at microscale contained in **(A2)**. We expand, at least formally, v^ε and λ^ε in powers of ε

$$v^\varepsilon(x)\approx v^0\left(x,\frac{x}{\varepsilon}\right)+\varepsilon v^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2v^2\left(x,\frac{x}{\varepsilon}\right)+\ldots \tag{3.2}$$

$$\lambda^\varepsilon\approx\lambda^0+\varepsilon\lambda^1+\varepsilon^2\lambda^2+\ldots \tag{3.3}$$

where for all $i \in \mathbb{N}$, $v^i = v^i(x,y)$ is periodic in the $y \in \mathbb{T}^2$ variable. Plugging (3.2) and (3.3) in (3.1) and identifying the powers of ε yields that v^0 does not depend on y and that (λ^0, v^0) solves the homogenized eigenvalue problem

$$\begin{cases} -\nabla \cdot A^0 \nabla v^0 = & \lambda^0 v^0, \quad x \in \Omega \\ \qquad\quad v^0 = & 0, \qquad x \in \partial\Omega \end{cases}. \tag{3.4}$$

As usual, the constant homogenized tensor $A^0 = A^{0,\alpha\beta} \in M_N(\mathbb{R})$ in (3.4) is given by

$$A^{0,\alpha\beta} := \int_{\mathbb{T}^2} A^{\alpha\beta}(y)dy + \int_{\mathbb{T}^2} A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y)dy,$$

where the family $\chi = \chi^\gamma(y) \in M_N(\mathbb{R})$, $y \in \mathbb{T}^2$, solves the cell problem

$$-\nabla_y \cdot A(y)\nabla_y \chi^\gamma = \partial_{y_\alpha} A^{\alpha\gamma}, \; y \in \mathbb{T}^2 \qquad \text{and} \qquad \int_{\mathbb{T}^2} \chi^\gamma(y)dy = 0. \tag{3.5}$$

Note that A^0 fulfils assumptions **(A1)** and **(A4)**, so there exists a sequence of eigenvalues $0 < \lambda^{0,0} \leq \lambda^{0,1} \leq \dots \lambda^{0,k} \xrightarrow{k\to\infty} \infty$ and a hilbertian basis $(v^{0,k})$ of $L^2(\Omega)$ of corresponding eigenvectors. Let T^0 denote the operator similar to T^ε with A^0 in place of $A^\varepsilon := A\left(\frac{\cdot}{\varepsilon}\right)$.

3.1.1 What is at stake?

Let us now focus on the convergence properties of the eigenvalues $\lambda^{\varepsilon,k}$ when $\varepsilon \to \infty$ and the mode k is fixed. The first thing we know from [AC98a], among other papers, is that for all $k \in \mathbb{N}$,

$$\left| \frac{1}{\lambda^{\varepsilon,k}} - \frac{1}{\lambda^{0,k}} \right| \leq \left\| T^\varepsilon - T^0 \right\|_{\mathcal{L}(L^2(\Omega))} \tag{3.6}$$

and that

$$T^\varepsilon \xrightarrow{\varepsilon\to 0} T^0$$

in $\mathcal{L}\big(L^2(\Omega)\big)$ norm. Therefore, for all $k \in \mathbb{N}$,

$$\lambda^{\varepsilon,k} \xrightarrow{\varepsilon\to 0} \lambda^{0,k}$$

no matter wether $\lambda^{0,k}$ is simple or not. When $N = 1$, i.e. the system (3.1) is a scalar equation, on condition that one has enough regularity on $v^{0,k}$ ($v^{0,k} \in H^2(\Omega)$ is sufficient) so that an estimate like

$$\left\| v^{\varepsilon,k}(x) - v^{0,k}(x) \right\|_{L^2(\Omega)} \leq C\varepsilon \left\| v^{0,k} \right\|_{H^2(\Omega)} \tag{3.7}$$

holds, one has in addition the error estimate

$$\left| \lambda^{\varepsilon,k} - \lambda^{0,k} \right| \leq C_k \varepsilon. \tag{3.8}$$

Estimate (3.8) is the starting point of the work of Moskow and Vogelius. Indeed, it leads to natural questions, such that:

1. What are the limit points of
$$\frac{\lambda^{\varepsilon,k} - \lambda^{0,k}}{\varepsilon}? \tag{3.9}$$
2. Is there possibly one unique limit point?
3. What is the next term in the asymptotic expansion?

When it does not lead to any confusion, we shall now omit the exponent k:

$$\lambda^\varepsilon := \lambda^{\varepsilon,k} \quad (\text{resp. } \lambda^0 := \lambda^{0,k})$$
$$v^\varepsilon := v^{\varepsilon,k} \quad (\text{resp. } v^0 := v^{0,k}).$$

In [MV97], Moskow and Vogelius get an asymptotic formula for the eigenvalue λ^ε of the scalar equation ($N = 1$), valid up to the order 1 in ε, provided that $v^0 \in H^2(\Omega)$, which is actually true for sufficiently smooth domains Ω (convex or C^2 domains). They fully describe the first-order corrections, i.e. the limit points of (3.9), in the case when Ω is a convex polygonal domain with sides of rational slopes. In this case there is a continuum of accumulation points for (3.9); an explicit description of the converging subsequences is to be found in [MV97], section 4.

Theorem 3.1 (Moskow and Vogelius in [MV97])**.** *Assume that Ω is a convex polygonal domain with sides of rational slopes and that $N = 1$. Assume furthermore that λ^0 is a simple eigenvalue.*
Then, for any sequence (ε_n) tending to 0, there exists a subsequence, which we denote again by (ε_n), and a boundary layer corrector $\vartheta^*_{bl} \in L^2(\Omega)$ solving an explicit homogenized elliptic boundary value problem (see (3.42)), such that

$$\lambda^{\varepsilon_n} = \lambda^0 + \varepsilon_n \lambda^0 \int_\Omega \vartheta^*_{bl}(x) v^0(x) dx + o(\varepsilon_n). \tag{3.10}$$

3.1.2 Difficulties and strategy

One faces essentially two kind of difficulties in proving a first-order asymptotic expansion for the eigenvalues like (3.10): the first is linked with the homogenization of boundary layer type systems, the second has to do with the regularity of solutions to elliptic systems in nonsmooth domains like polygons. We sketch how these difficulties are addressed by Moskow and Vogelius and how we extend their results to the case of elliptic systems and more general polygonal or smooth domains Ω.

Homogenization of boundary layer systems

While v^0 solves (3.4) with a homogeneous Dirichlet boundary condition on $\partial\Omega$, $v^1\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ does not cancel in general on $\partial\Omega$. For this reason, the formal asymptotic expansion (3.2) is inadequate to establish a first-order expansion for the eigenvalues. Boundary layers need to be taken into account. Considering

$$v^\varepsilon(x) \approx v^0(x) + \varepsilon \left[v^1\Big(x, \frac{x}{\varepsilon}\Big) + v^{1,\varepsilon}_{bl}(x) \right] + \varepsilon^2 \left[v^2\Big(x, \frac{x}{\varepsilon}\Big) + v^{2,\varepsilon}_{bl}(x) \right] + \dots$$

where $v^{i,\varepsilon}_{bl}$ solves

$$\left\{ \begin{array}{rcll} -\nabla \cdot A\big(\frac{x}{\varepsilon}\big) \nabla v^{i,\varepsilon}_{bl} &=& 0, & x \in \Omega \\ v^{i,\varepsilon}_{bl} &=& -v^i\big(x, \frac{x}{\varepsilon}\big), & x \in \partial\Omega \end{array} \right. \tag{3.11}$$

proves to be more relevant than (3.2). Moreover, ϑ^*_{bl}, appearing in (3.10), comes from the homogenization of an elliptic boundary layer system like (3.11).

The heart of the proof of theorem 3.1 is subsequently the homogenization of boundary layer type systems

$$\left\{ \begin{array}{rcll} -\nabla \cdot A\big(\frac{x}{\varepsilon}\big) \nabla u^{\varepsilon}_{bl} &=& 0, & x \in \Omega \\ u^{\varepsilon}_{bl} &=& \varphi\big(x, \frac{x}{\varepsilon}\big), & x \in \partial\Omega \end{array} \right. \tag{3.12}$$

with $\varphi = \varphi(x, y) := \Phi(y)\varphi_0(x)$, Φ being, unless stated otherwise, a smooth function on $\mathbb{T}^2$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$.

Compared to u^ε solution of

$$\left\{ \begin{array}{rll} -\nabla \cdot A\big(\frac{x}{\varepsilon}\big)\nabla u^\varepsilon = & f, & x \in \Omega \\ u^\varepsilon = & \varphi_0, & x \in \partial\Omega \end{array} \right. \tag{3.13}$$

whose homogenization is now a classical topic, the analysis of the asymptotics of u^ε_{bl} when $\varepsilon \to 0$ is complicated by the oscillating boundary data in (3.12) for at least two reasons:

1. We lack uniform a priori estimates for u^ε_{bl} in $H^1(\Omega)$ norm. This is due to the fact that
$$\left\| \varphi\big(x, \tfrac{x}{\varepsilon}\big) \right\|_{H^{\frac{1}{2}}(\partial\Omega)} = O\big(\varepsilon^{-\frac{1}{2}}\big).$$

2. The behaviour of the boundary layer along the boundary $\partial\Omega$ deeply depends on the interaction between the periodic lattice and the boundary. Thus one can not expect in general periodicity of the boundary layer along the boundary.

The proofs of Moskow and Vogelius intensively rely on a result due to Avellaneda and Lin, which addresses a priori estimates for elliptic equations in domains Ω with quite low regularity.

Theorem 3.2 (Avellaneda and Lin in [AL87b] theorem 3)**.** *Assume that Ω is Lipschitz and satisfies a uniform exterior sphere condition. Assume furthermore that $N = 1$.*
Then, for all $1 < p < \infty$, there exists $C > 0$ such that for all boundary data function $\varphi\big(\cdot, \frac{\cdot}{\varepsilon}\big) \in L^p(\partial\Omega)$, there is a solution $u^\varepsilon_{bl} \in L^p(\Omega)$ of (3.12) satisfying

$$\left\| u^\varepsilon_{bl} \right\|_{L^p(\Omega)} \leq C \left\| \varphi\big(\cdot, \tfrac{\cdot}{\varepsilon}\big) \right\|_{L^p(\partial\Omega)}. \tag{3.14}$$

Its usefulness for our boundary layer system (3.12) comes from the following simple remark: $\varphi\big(\cdot, \frac{\cdot}{\varepsilon}\big)$ is bounded in $L^2(\partial\Omega)$ norm, but not in $H^{\frac{1}{2}}(\partial\Omega)$ norm. Consequently, $\left\| u^\varepsilon_{bl} \right\|_{L^2(\Omega)} = O(1)$. An estimate similar to (3.14) holds also for elliptic systems, yet under stronger regularity assumptions on Ω.

Theorem 3.3 (Avellaneda and Lin in [AL87a] theorem 3)**.** *Let N be any positive integer. Assume that Ω is $C^{1,\alpha}$ with $0 < \alpha \leq 1$.*
Then, the conclusion of theorem 3.2 remains true.

This theorem can be applied when Ω is smooth, but due to its strong regularity assumption on Ω, it is useless in the case when Ω is a polygonal domain. A precise analysis of the boundary layer system is needed in this case. Beyond the results of Avellaneda and Lin, the analysis of (3.12) has been carried out in the context of:

1. convex polygonal domains Ω, first with edges of rational slopes by Moskow and Vogelius in [MV97], Allaire and Amar in [AA99] (scalar case), then with edges of slopes satisfying a generic small divisors assumption by Gérard-Varet and Masmoudi in [GVM11];

2. smooth domains with uniformly convex boundary by Gérard-Varet and Masmoudi in the recent paper [GVM12].

This recent progress in the homogenization of (3.12), due to Gérard-Varet and Masmoudi, opens the way to our generalizations.

Regularity

Besides the issue of the homogenization of boundary layer systems comes the problem of regularity. Regularity is required in order to carry out the energy estimates of the paper. Of course this is only a problem when Ω is a polygonal domain; if Ω is smooth, all functions we deal with belong to $C^\infty(\overline{\Omega})$.

Assume now that Ω is a convex polygon. In the scalar case the results of Grisvard in [Gri85] (theorem 3.2.1.2) and [Gri92] (section 2.7), recalled in [MV97], yield that $v^0 \in H^2(\Omega)$, because of convexity. We even know better. Indeed v^0 solves (3.4) with r.h.s. $\lambda^0 v^0 \in H^1(\Omega)$. Therefore, $v^0 \in H^{2+\omega}(\Omega)$, with $0 < \omega$.

This $H^{2+\omega}(\Omega)$ regularity on v^0 appears to be the minimal regularity one has to assume in order to get a first-order expansion like (3.10). It is a corollary of the work of Dauge [Dau88] on the one hand, and Kozlov, Maz'ya and Rossmann [KMR01] on the other hand, that the results of Grisvard extend to systems with constant coefficients. More precisely:

Theorem 3.4. *Let $N \geq 1$ and $u^0 \in H^1_0(\Omega)$ be the unique solution of* (3.4) *with r.h.s. equal to $f \in H^{-1}(\Omega)$. Assume that Ω is a convex polygonal domain.*

1. *If $f \in H^{-1+\omega}(\Omega)$ with $0 < \omega < 1$ and $\omega \neq \frac{1}{2}$, then $u^0 \in H^{1+\omega}(\Omega)$.*
2. *If $f \in L^2(\Omega)$, then $u^0 \in H^2(\Omega)$.*
3. *If $f \in H^1(\Omega)$, then there exists $0 < \omega \leq 1$ such that $u^0 \in H^{2+\omega}(\Omega)$.*

Let us give a sketch of how to deduce such regularity statements from [Dau88] and [KMR01] (see these references for more details). We know from [Dau88] (see lemma 5.7) that for all $s > 0$, for all vertex $x \in \partial\Omega$

$$\{0 < \mathrm{Re}(\lambda) < s\} \cap \mathrm{Sp}\,\mathcal{L}_x \text{ is finite,}$$

where $\mathrm{Sp}\,\mathcal{L}_x$ denotes the spectrum of a pencil associated to our problem at vertex x. Besides, Kozlov, Maz'ya and Rossmann prove in [KMR01], theorem 8.6.2, that for strongly elliptic systems, with constant coefficients, satisfying the symmetry assumption **(A4)**, posed in the convex polygonal domain Ω,

$$\{0 \leq \mathrm{Re}(\lambda) \leq 1\} \cap \mathrm{Sp}\,\mathcal{L}_x = \varnothing$$

for all vertex x. This fact collapses if Ω has at least one angle $\geq \pi$. Yet Ω being polygonal and convex, it follows from [Dau88], in particular paragraph 7.16, corollary 5.16 and theorem 5.5, that the operator

$$L^{(s)}: \quad u \in H^{s+1}(\Omega) \cap H^1_0(\Omega) \longmapsto -\nabla \cdot A\Big(\frac{x}{\varepsilon}\Big)\nabla u \in H^{s-1}(\Omega)$$

is a Fredholm operator for all $0 \leq s \neq \frac{1}{2}$ satisfying $\{\mathrm{Re}(\lambda) = s\} \cap \mathrm{Sp}\,\mathcal{L}_x = \varnothing$ for all vertex $x \in \partial\Omega$. At this point, one deduces that $L^{(s)}$ is a Fredholm operator for all $0 \leq s \leq 1$, $s \neq \frac{1}{2}$. Moreover, there exists $0 < \omega \leq 1$ such that $L^{(1+\omega)}$ is a Fredholm operator.

Let $0 \leq s$ such that $L^{(s)}$ is a Fredholm operator and let $f \in H^{s-1}(\Omega)$. Two situations are possible. If there is a vertex $x \in \partial\Omega$ and $\lambda \in \{0 < \mathrm{Re}(\lambda) < s\} \cap \mathrm{Sp}\,\mathcal{L}_x$, then theorem 5.11 in [Dau88] yields the existence of $u^0_{reg} \in H^{1+s}(\Omega)$, the regular part, and $u^0_{sing} \in H^{1+\gamma}(\Omega)$, the singular part, with $0 < \gamma < \min_{\lambda \in \{0 < \mathrm{Re}(\lambda) < s\} \cap \bigcup_x \mathrm{Sp}\,\mathcal{L}_x} \mathrm{Re}(\lambda)$, such that

$$u^0 = u^0_{sing} + u^0_{reg} \in H^{1+\gamma}(\Omega).$$

On the contrary, if for all vertex x, $\{0 < \mathrm{Re}(\lambda) < s\} \cap \mathrm{Sp}\,\mathcal{L}_x = \varnothing$, then $u^0 = u^0_{reg}$ is in $H^{1+s}(\Omega)$. The two first points of theorem 3.4 as well as the third now easily follow from the preceding results.

Each point of theorem 3.4 plays a role in our reasoning. One can alternatively invoke weaker regularity results such as:

Theorem 3.5 (Agranovich in [Agr07] theorem 1)**.** *Assume that Ω is a Lipschitz domain. Let $u^\varepsilon \in H^1_0(\Omega)$ be the unique variational solution of* (3.1) *with r.h.s. equal to $f \in H^{-1}(\Omega)$. Assume furthermore that $f \in H^{-1+\omega}(\Omega)$ with $0 \leq \omega < \frac{1}{2}$.*
Then, $u^\varepsilon \in H^{1+\omega}(\Omega)$.

3.1.3 Outline of our results

This article answers relevant questions asked by Moskow and Vogelius. Quoting [MV97] (section 5):

> It should be extremely interesting to derive a similar representation formula for polygons with sides of irrational slopes or for smooth domains. In particular, it would be interesting to see if this leads to a single first-order correction for any eigenvalue.

We manage to free ourselves from the assumption $N = 1$. Our main results then sum up in the upcoming theorems. We treat separately two different classes of domains Ω: on the one hand very smooth domains, on the other hand convex polygonal domains. *All the definitions we use are made rigourous later in the paper.*

Assume that λ^0 is an eigenvalue of order m. Let $\lambda^0 = \lambda^{0,k} = \lambda^{0,k+1} = \ldots = \lambda^{0,k+m-1}$ be the eigenvalues repeated with multiplicity. We call E_{λ^0} the finite-dimensional eigenspace associated to the eigenvalue λ^0. Note that the eigenvectors $v^{0,k}, \ldots, v^{0,k+m-1}$ form an orthogonal basis of E_{λ^0}.

Our first theorem is concerned with smooth domains Ω.

Theorem 3.6. *Assume that Ω is a smooth C^∞ bounded domain with uniformly convex boundary.*
Then, for every $0 \leq j \leq m-1$, there exists a unique $\vartheta^*_{j,bl} \in L^2(\Omega)$ such that for all $0 \leq \gamma < \frac{1}{11}$,

$$\left[\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^{\varepsilon,k+j}}\right]^{-1} = \lambda^0 + \varepsilon\frac{\lambda^0}{m}\sum_{j=0}^{m-1}\int_\Omega \vartheta^*_{j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{1+\gamma}\big). \tag{3.15}$$

The next theorem faces the same problem for convex polygonal domains Ω.

Theorem 3.7. *Assume that Ω is a convex polygonal domain with sides of slopes satisfying a generic small divisors assumption; see section 3.3.2.*

1. *Then for every $0 \leq j \leq m-1$, there exists a unique $\vartheta^*_{j,bl} \in L^2(\Omega)$ and $0 < \gamma$ such that*

$$\left[\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^{\varepsilon,k+j}}\right]^{-1} = \lambda^0 + \varepsilon\frac{\lambda^0}{m}\sum_{j=0}^{m-1}\int_\Omega \vartheta^*_{j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{1+\gamma}\big). \tag{3.16}$$

2. *If $E_{\lambda^0} \subset H^3(\Omega) \cap C^2(\overline{\Omega})$, then for every $0 \leq j \leq m-1$, there exists a unique $\vartheta^*_{j,bl} \in L^2(\Omega)$ such that*

$$\left[\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^{\varepsilon,k+j}}\right]^{-1} = \lambda^0 + \varepsilon\frac{\lambda^0}{m}\sum_{j=0}^{m-1}\int_\Omega \vartheta^*_{j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{\frac{3}{2}}\big). \tag{3.17}$$

We stress that theorem 3.7 has two parts. The first point is a general result: due to the assumptions on A (in particular **(A1)** and **(A4)**) and on Ω (polygonal and convex), the $H^{2+\omega}(\Omega)$ regularity, with $0 < \omega$, needed on the eigenvectors for the proof, happens to be automatically fulfilled. The second part of the theorem states an optimal result in view of our proof, in terms of convergence rate, but needs to assume more regularity on the eigenfunctions.

There is an analog of theorem 3.7 in the case of a convex polygon with sides of rational slopes, which improves theorem 3.1. Estimate (3.16) (resp. (3.17)) still holds however up to the extraction of a subsequence (ε_n). Throughout the paper, we indicate how to adapt the proofs to this case.

When λ^0 is simple, (3.15), (3.16) and (3.17) yield the first-order expansion

$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^0\int_\Omega \vartheta^*_{bl}(x)v^0(x)dx + o\big(\varepsilon^{1+\gamma}\big).$$

valid for appropriate exponents γ.

A consequence of these two theorems 3.6 and 3.7 is that the first-order correction to the eigenvalue λ^0 is identified and unique. Furthermore, it appears in the course of the proof of theorem 3.7 that $\vartheta^*_{j,bl}$ is a solution of an homogenized elliptic boundary value problem (see (3.43)), whose data can be made explicit. It thus opens the door to numerical computations. We expand this point in the section 3.3.3, indicate how to compute the data of the homogenized boundary layer problem, and refer to numerical studies.

3.1.4 Organization of the paper

In section 3.2, we prove some corrector results for u^ε solution of (3.13). Such estimates do exist in the litterature, but we focus on minimal regularity assumptions. In particular, we extend the bounds of [MV97] to elliptic systems (non necessarily symmetric) and get new ones, which are useful in the rest of the paper. Section 3.3 is devoted to the homogenization of boundary layer type systems. We analyse the convergence in $L^2(\Omega)$ of $\vartheta^\varepsilon_{v,bl}$ solution of (3.12) with $\varphi(x,y) := -\chi^\alpha(y)\partial_{x_\alpha}v^0(x)$, in the two different settings: Ω is a smooth domain with uniformly convex boundary or a convex polygonal domain with the additional diophantine condition on the slopes. This work is the central step in the proof of theorems 3.6 and 3.7. The final step is done in section 3.4, where a first-order correction formula for λ^ε, in terms of the limit of $\vartheta^\varepsilon_{v,bl}$ when $\varepsilon \to 0$, is obtained.

3.2 Some error estimates

Let $f \in H^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. The solution u^ε of

$$\begin{cases} -\nabla\cdot A\big(\frac{x}{\varepsilon}\big)\nabla u^\varepsilon = & f, \quad x \in \Omega \\ u^\varepsilon = & \varphi_0, \quad x \in \partial\Omega \end{cases} \tag{3.18}$$

exists, is unique in $H^1(\Omega)$ and converges strongly in $L^2(\Omega)$ towards $u^0 \in H^1(\Omega)$ solving the elliptic system

$$\left\{ \begin{array}{rll} -\nabla \cdot A^0 \nabla u^0 = & f, & x \in \Omega \\ u^0 = & \varphi_0, & x \in \partial\Omega \end{array} \right. . \tag{3.19}$$

We focus here on estimates in norm showing how fast this convergence takes place. We do not need assumption **(A4)**, i.e. the symmetry of A.

3.2.1 Multiscale expansions

Before coming to the estimates, let us recall some basic facts about multiscale expansions. In the same fashion as v^ε (see (3.2)), we expand u^ε:

$$u^\varepsilon(x) \approx u^0\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon u^1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2 u^2\Big(x, \frac{x}{\varepsilon}\Big) + \dots \tag{3.20}$$

Plugging (3.20) in (3.18) and identifying the powers of ε yields, at least formally,

1. that u^0 solves the homogenized system

$$\left\{ \begin{array}{rll} -\nabla \cdot A^0 \nabla u^0 = & f, & x \in \Omega \\ u^0 = & \varphi_0, & x \in \partial\Omega \end{array} \right. ,$$

2. that $u^1 = u^1(x,y) := \chi^\alpha(y)\partial_{x_\alpha} u^0(x) + \bar{u}^1(x)$ where χ^α is the function defined in (3.5),
3. and that $u^2 = u^2(x,y) := \Gamma^{\alpha\beta}(y)\partial_{x_\alpha}\partial_{x_\beta} u^0(x) + \chi^\alpha(y)\partial_{x_\alpha}\bar{u}^1(x) + \bar{u}^2(x)$ where $\Gamma^{\alpha\beta}$ solves

$$-\nabla_y \cdot A(y)\nabla_y \Gamma^{\alpha\beta} = B^{\alpha\beta} - \int_{\mathbb{T}^2} B^{\alpha\beta}(y)dy, \ y \in \mathbb{T}^2 \qquad \text{and} \qquad \int_{\mathbb{T}^2} \Gamma^{\alpha\beta}(y)dy = 0$$

   with

$$B^{\alpha\beta} := A^{\alpha\beta} + A^{\alpha\gamma}\partial_{y_\gamma}\chi^\beta + \partial_{y_\gamma}\big(A^{\gamma\alpha}\chi^\beta\big).$$

We always assume that $\bar{u}^1 = \bar{u}^2 = 0$.

The first-order correction $u^1\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ to u^ε does not satisfy homogeneous Dirichlet boundary conditions on $\partial\Omega$. It is therefore responsible for a $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ term in the $H^1(\Omega)$ estimates involving u^1. In order to correct this, one introduces a boundary layer function $\vartheta^\varepsilon_{u,bl}$ solution of (3.12) with $\varphi(x,y) := -u^1(x,y) = -\chi^\alpha(y)\partial_{x_\alpha}u^0(x)$. Note that $u^1\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \vartheta^\varepsilon_{u,bl}$ belongs to $H^1_0(\Omega)$.

3.2.2 Error estimates

We extend here the estimates of Moskow and Vogelius (cf. [MV97] section 2) to systems.

Proposition 3.8. *Assume that $u^0 \in H^2(\Omega)$.*
Then

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon\vartheta^\varepsilon_{u,bl}(x) \right\|_{H^1(\Omega)} \leq C\varepsilon \left\| u^0 \right\|_{H^2(\Omega)} \tag{3.21}$$

for $C > 0$ independent of ε and u^0.

The proof relies on energy estimates on the error

$$e^\varepsilon := u^\varepsilon(x) - u^0(x) - \varepsilon u^1\Big(x, \frac{x}{\varepsilon}\Big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x).$$

It is a solution of the following system

$$\begin{cases} -\nabla \cdot A\big(\frac{x}{\varepsilon}\big)\nabla e^\varepsilon = & r^\varepsilon, \quad x \in \Omega \\ e^\varepsilon = & 0, \quad x \in \partial\Omega \end{cases} \tag{3.22}$$

where

$$r^\varepsilon := f + \nabla \cdot \Big[A\Big(\frac{x}{\varepsilon}\Big)\nabla u^0\Big] + \varepsilon \nabla \cdot \Big[A\Big(\frac{x}{\varepsilon}\Big)\nabla u^1\Big(x, \frac{x}{\varepsilon}\Big)\Big]. \tag{3.23}$$

We intend to prove that the $H^1(\Omega)$ norm of e^ε is of order ε by showing that the source term r^ε in (3.22) is of order ε in $H^{-1}(\Omega)$. It is not clear, looking at (3.23), that the latter is true. To face this issue, we invoke the classic key lemma, which can be proved using Fourier series expansions:

Lemma 3.9. *Let* $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$.
Assume

$$\nabla \cdot v = 0 \qquad \textit{and} \qquad \int_{\mathbb{T}^2} v = 0.$$

Then there exists $\psi = \psi(y) \in \mathbb{R}$ *such that* $v = \nabla^\perp \psi = \begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix}$.

Proof of proposition 3.8. Expanding the source term r^ε yields

$$\begin{aligned} r^\varepsilon =& \frac{1}{\varepsilon}\Big[\Big[\nabla_y \cdot A(y)\nabla_y u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big) + \Big[\nabla_y \cdot A(y)\nabla_x u^0\Big]\Big(x, \frac{x}{\varepsilon}\Big)\Big] \\ &+ f + \Big[\nabla_x \cdot A(y)\nabla_x u^0\Big]\Big(x, \frac{x}{\varepsilon}\Big) + \Big[\nabla_x \cdot A(y)\nabla_y u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big) + \Big[\nabla_y \cdot A(y)\nabla_x u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big) \\ &+ \varepsilon\Big[\nabla_x \cdot A(y)\nabla_x u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big). \end{aligned} \tag{3.24}$$

The leading idea is to get rid of terms of order 0 and -1 in ε. We call

$$v := A(y)\nabla_y u^1 + A(y)\nabla_x u^0$$

and notice that

$$\begin{aligned} \nabla_y \cdot v &= \nabla_y \cdot A(y)\nabla_x u^0 + \nabla_y \cdot A(y)\nabla_y u^1(x,y) \\ &= \partial_{y_\alpha} A^{\alpha\beta}(y)\partial_{x_\beta} u^0 + \partial y_\alpha\big(A^{\alpha\beta}(y)\partial_{y_\beta}\chi^\gamma(y)\big)\partial_{x_\gamma} u^0 \\ &= 0 \end{aligned} \tag{3.25}$$

because χ^γ solves (3.5). Thus the ε^{-1} order term in (3.24) cancels and it remains to handle the zeroth order term:

$$\begin{aligned} f + \Big[\nabla_x \cdot A(y)\nabla_x u^0\Big]\Big(x, \frac{x}{\varepsilon}\Big) + \Big[\nabla_x \cdot A(y)\nabla_y u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big) + \Big[\nabla_y \cdot A(y)\nabla_x u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big) \\ = f + \Big[\nabla_x \cdot v\Big]\Big(x, \frac{x}{\varepsilon}\Big) + \Big[\nabla_y \cdot A(y)\nabla_x u^1\Big]\Big(x, \frac{x}{\varepsilon}\Big). \end{aligned} \tag{3.26}$$

Here again, we take advantage of (3.25) and the definition of A^0: on the one hand

$$\nabla_y \cdot \big(v - A^0 \nabla u^0\big) = 0$$

and on the other hand

$$\int_{\mathbb{T}^2} \left(v - A^0 \nabla u^0\right) = 0.$$

It follows that one can apply lemma 3.9, component by component, and get a function $\psi = \psi(x,y) \in \mathbb{R}^N$ such that

$$v - A^0 \nabla u^0 = \nabla_y^\perp \psi.$$

Due to the fact that $v - A^0 \nabla u^0$ is a function of separated variables x and y, ψ itself is and factors into

$$\psi(x,y) = \Psi(y) \nabla u^0(x). \tag{3.27}$$

The function $\Psi = \left(\Psi^\alpha(y)\right)_{1 \le \alpha \le 2} \in M_N(\mathbb{R})^2$ is given by the lemma and is therefore of class C^∞. As u^0 is assumed to be in $H^2(\Omega)$, ψ is in $H^1(\Omega)$ with respect to x. We set

$$w := \nabla_x^\perp \psi$$

of regularity $L^2(\Omega)$ towards x, we compute

$$\nabla_y \cdot w = \nabla_y \cdot \nabla_x^\perp \psi = -\nabla_x \cdot \nabla_y^\perp \psi = -\nabla_x \cdot v - f$$

and use this equality to simplify (3.26)

$$\begin{aligned} f + \left[\nabla_x \cdot v\right]\left(x, \frac{x}{\varepsilon}\right) + \left[\nabla_y \cdot A(y) \nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right) = -\left[\nabla_y \cdot w\right]\left(x, \frac{x}{\varepsilon}\right) \\ + \left[\nabla_y \cdot A(y) \nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right). \end{aligned} \tag{3.28}$$

Finally, using $\nabla_x \cdot w = 0$ one obtains

$$\begin{aligned} r^\varepsilon &= -\left[\nabla_y \cdot w\right]\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla \cdot \left[A\left(\frac{x}{\varepsilon}\right) \nabla_x u^1\left(x, \frac{x}{\varepsilon}\right)\right] \\ &= -\left[\nabla_y \cdot w\right]\left(x, \frac{x}{\varepsilon}\right) - \varepsilon\left[\nabla_x \cdot w\right]\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla \cdot \left[A\left(\frac{x}{\varepsilon}\right) \nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right) \\ &= -\varepsilon \nabla \cdot \left[w\left(x, \frac{x}{\varepsilon}\right)\right] + \varepsilon \nabla \cdot \left[A\left(\frac{x}{\varepsilon}\right) \nabla_x u^1\left(x, \frac{x}{\varepsilon}\right)\right]. \end{aligned} \tag{3.29}$$

It remains to estimate the $H^{-1}(\Omega)$ norm of r^ε. The expression (3.29) is convenient for two reasons: it is a sum of two terms of order 1 in ε and is written in divergence form. Moreover, $w\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ as well as $A\left(\frac{\cdot}{\varepsilon}\right) \nabla_x u^1\left(\cdot, \frac{\cdot}{\varepsilon}\right)$ belong to $L^2(\Omega)$ and we have

$$\begin{aligned} \left\| w\left(\cdot, \tfrac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} &\le C \left\|u^0\right\|_{H^2(\Omega)} \\ \left\| A\left(\tfrac{\cdot}{\varepsilon}\right) \nabla_x u^1\left(\cdot, \tfrac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} &\le C \left\|u^0\right\|_{H^2(\Omega)}. \end{aligned}$$

Consequently, for all $\phi \in H_0^1(\Omega)$, by Cauchy-Schwarz inequality

$$\begin{aligned} &\left| \left\langle r^\varepsilon\left(x, \frac{x}{\varepsilon}\right), \phi(x)\right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right| \\ &\qquad = \left| \left\langle -\varepsilon \nabla \cdot \left[w\left(x, \frac{x}{\varepsilon}\right)\right] + \varepsilon \nabla \cdot \left[A\left(\frac{x}{\varepsilon}\right) \nabla_x u^1\left(x, \frac{x}{\varepsilon}\right)\right], \phi(x)\right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right| \\ &\qquad = \left| \varepsilon \int_\Omega w\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla \phi(x) dx - \varepsilon \int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla_x u^1\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla \phi(x) dx \right| \\ &\qquad \le C\varepsilon \left\|u^0\right\|_{H^2(\Omega)} \left\|\phi\right\|_{H_0^1(\Omega)} \end{aligned}$$

which concludes the proof. □

Corollary 3.10. *Assume that* $u^0 \in H^2(\Omega)$.
Then

$$\left\| u^\varepsilon(x) - u^0(x) \right\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{1}{2}} \left\| u^0 \right\|_{H^2(\Omega)}. \tag{3.30}$$

Proof. By the triangular inequality, we get

$$\begin{aligned}
&\left\| u^\varepsilon(x) - u^0(x) \right\|_{L^2(\Omega)} \\
&\leq \left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) \right\|_{H^1(\Omega)} + \varepsilon \left\| u^1\big(x, \tfrac{x}{\varepsilon}\big) \right\|_{L^2(\Omega)} + \varepsilon \left\| \vartheta^\varepsilon_{u,bl}(x) \right\|_{L^2(\Omega)}.
\end{aligned}$$

The estimate (3.30) now follows from (3.21), the uniform boundedness of $u^1\big(\cdot, \frac{\cdot}{\varepsilon}\big)$ in $L^2(\Omega)$ and from the bound

$$\left\| \vartheta^\varepsilon_{u,bl} \right\|_{L^2(\Omega)} \leq \left\| \vartheta^\varepsilon_{u,bl} \right\|_{H^1(\Omega)} \leq C \left\| \chi^\alpha\big(\tfrac{x}{\varepsilon}\big) \partial_{x_\alpha} u^0(x) \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\varepsilon^{-\frac{1}{2}} \left\| u^0 \right\|_{H^2(\Omega)}. \tag{3.31}$$

□

From corollary 3.10, one easily gets a similar $L^2(\Omega)$ estimate under weaker assumptions on u^0.

Corollary 3.11. *Assume that* $u^0 \in H^{1+\omega}(\Omega)$, *with* $0 \leq \omega \leq 1$.
Then

$$\left\| u^\varepsilon(x) - u^0(x) \right\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{\omega}{2}} \left\| u^0 \right\|_{H^{1+\omega}(\Omega)}. \tag{3.32}$$

Proof. A straightforward energy estimate on the elliptic system satisfied by $u^\varepsilon - u^0$ yields

$$\left\| u^\varepsilon(x) - u^0(x) \right\|_{L^2(\Omega)} \leq C \left\| u^\varepsilon(x) - u^0(x) \right\|_{H^1_0(\Omega)} \leq C \left\| u^0 \right\|_{H^1(\Omega)}. \tag{3.33}$$

Inequality (3.32) now comes from interpolating (3.33) and (3.30). The main idea is to think of (3.33) (resp. (3.30)) as the statement that the linear operator

$$u^0 \longmapsto u^\varepsilon(x) - u^0(x)$$

is bounded from $H^1(\Omega)$ to $L^2(\Omega)$ (resp. from $H^2(\Omega)$ to $L^2(\Omega)$) and then to interpolate. □

We call now $\vartheta^{2,\varepsilon}_{u,bl}$ the solution of (3.12) with $\varphi(x,y) := -u^2(x,y)$, whose introduction is motivated by the same reasons as $\vartheta^\varepsilon_{u,bl}$, and state the proposition:

Proposition 3.12. *Assume that* $u^0 \in H^3(\Omega)$.
Then

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) \right\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{3}{2}} \left\| u^0 \right\|_{H^3(\Omega)}. \tag{3.34}$$

Proof. The proof of (3.34) relies on a global energy estimate found in [GVM11], section 3.3:

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) - \varepsilon^2 u^2\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon^2 \vartheta^{2,\varepsilon}_{u,bl}(x) \right\|_{H^1(\Omega)} = O\big(\varepsilon^2\big) \tag{3.35}$$

It requires $u^0 \in H^3(\Omega)$ and it can be showned using the same ideas than those involved in (3.21), the key being again lemma 3.9. Following the lines of [GVM11] it becomes clear that the precised estimate

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) - \varepsilon^2 u^2\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon^2 \vartheta^{2,\varepsilon}_{u,bl}(x) \right\|_{H^1(\Omega)} \leq C\varepsilon^2 \left\| u^0 \right\|_{H^3(\Omega)}$$

holds. It is nothing but a consequence of the way the involved functions factor in the product of a function depending only on y and of ∇u^0 (cf. (3.27)). We conclude, applying the triangular inequality, that

$$\begin{aligned}
&\left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) \right\|_{L^2(\Omega)} \\
&\quad \le \left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) - \varepsilon^2 u^2\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon^2 \vartheta^{2,\varepsilon}_{u,bl}(x) \right\|_{H^1(\Omega)} \\
&\qquad + \varepsilon^2 \left\| u^2\big(x, \tfrac{x}{\varepsilon}\big) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \vartheta^{2,\varepsilon}_{u,bl}(x) \right\|_{L^2(\Omega)}.
\end{aligned}$$

The estimate (3.34) now follows from (3.35), the uniform boundedness of $u^2\big(\cdot, \frac{\cdot}{\varepsilon}\big)$ in $L^2(\Omega)$ and the (3.31)-like bound

$$\left\| \vartheta^{2,\varepsilon}_{u,bl} \right\|_{H^1(\Omega)} \le C\varepsilon^{-\frac{1}{2}} \left\| u^0 \right\|_{H^3(\Omega)}. \qquad \square$$

We conclude this section focusing on a (3.34)-like estimate for u^0 satisfying a weaker assumption.

Theorem 3.13. *Assume $u^0 \in H^{2+\omega}(\Omega)$, with $0 \le \omega \le 1$.*
Then

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon u^1\big(x, \tfrac{x}{\varepsilon}\big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x) \right\|_{L^2(\Omega)} \le C\varepsilon^{1+\frac{\omega}{2}} \left\| u^0 \right\|_{H^{2+\omega}(\Omega)}. \tag{3.36}$$

As in the proof of corollary 3.11, estimate (3.21) (resp. (3.34)) states that the linear operator

$$u^0 \longmapsto u^\varepsilon(x) - u^0(x) - \varepsilon u^1\Big(x, \frac{x}{\varepsilon}\Big) - \varepsilon \vartheta^\varepsilon_{u,bl}(x)$$

is bounded from $H^2(\Omega)$ to $L^2(\Omega)$ (resp. from $H^3(\Omega)$ to $L^2(\Omega)$). By interpolating between the two linear operators, one gets (3.36). All details can be found in [MV97] and apply without any change to the case $N > 1$.

3.3 Homogenization of boundary layer type systems

Throughout this section we are interested in the homogenization of the boundary layer type system

$$\left\{ \begin{array}{rll} -\nabla \cdot A\big(\frac{x}{\varepsilon}\big) \nabla \vartheta^\varepsilon_{u,bl} = & 0, & x \in \Omega \\ \vartheta^\varepsilon_{u,bl} = & -\chi^\alpha\big(\frac{x}{\varepsilon}\big) \partial_{x_\alpha} u^0(x), & x \in \partial\Omega \end{array} \right. \tag{3.37}$$

that is in the study of the asymptotic behaviour of the sequence $\vartheta^\varepsilon_{u,bl}$ when ε tends to 0. This means we both look for a possible limit of the sequence and for estimates in norm of the speed of convergence. For all this section, we assume that u^0 solves (4.46), with $f \in L^2(\Omega)$ and $\varphi_0 = 0$.

This is a crucial step in the proof of theorems 3.6 and 3.7. As explained in the introduction, there is no regularity issue when Ω is smooth. On the contrary, when Ω is a polygon, we concentrate on minimal regularity. That is why we give two convergence rates: the first under the minimal assumption $u^0 \in H^{2+\omega}(\Omega)$ with $0 < \omega < 1$, the second under the stronger regularity assumption $u^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$, where we focus on improving the speed of convergence.

For notational convenience, let us write in this section $\vartheta^\varepsilon_{bl}$ instead of $\vartheta^\varepsilon_{u,bl}$.

3.3.1 Smooth uniformly convex domains

Assume that $\Omega \subset \mathbb{R}^2$ is a smooth (say C^∞) uniformly convex domain i.e. all principal curvatures are bounded from below; see [Bre05] section III.7 for another definition. The regularizing properties of elliptic operators in smooth domains yield that $u^0 \in C^\infty(\overline{\Omega})$ (see [ADN64] theorem 10.5). Therefore, the boundary data function $\varphi(x,y) = -u^1(x,y) = -\chi^\alpha(y)\partial_{x_\alpha} u^0(x)$ is a smooth function. Note that we do not need assumption **(A4)**, i.e. the symmetry of A.

Theorem 3.14 (Gérard-Varet and Masmoudi in [GVM12]). *For all $1 \leq p < \infty$ there exists $\varphi^* \in L^p(\partial\Omega)$ such that $\vartheta_{bl}^\varepsilon$ converges in $L^2(\Omega)$ towards $\vartheta_{bl}^* \in L^p(\Omega)$ solution of*

$$\begin{cases} -\nabla \cdot A^0 \nabla \vartheta_{bl}^* = & 0, \qquad x \in \Omega \\ \qquad\quad \vartheta_{bl}^* = & \varphi^*(x), \quad x \in \partial\Omega \end{cases}.$$

Moreover, for all $0 \leq \gamma < \frac{1}{11}$,

$$\left\| \vartheta_{bl}^\varepsilon - \vartheta_{bl}^* \right\|_{L^2(\Omega)} = O\big(\varepsilon^\gamma\big). \tag{3.38}$$

We do not attempt to weaken the regularity assumption on Ω, which itself implies strong regularity on u^0. For details concerning the proof and relevant remarks, we refer to [GVM12].

3.3.2 Convex polygonal domains

Let us assume Ω to be a bounded convex polygonal domain with M edges, supported by the lines K^k of unitary inward normal $n^k \in S^1$. Thus

$$\Omega = \bigcap_{k=1}^{M} \left\{x,\ n^k \cdot x > c^k\right\}$$

with $c^k \in \mathbb{R}$, and for all $1 \leq k \leq M$,

$$K^k = \left\{x,\ n^k \cdot x = c^k\right\}.$$

Beyond this first assumption on Ω we require either

(RAT) rationality for all $1 \leq k \leq M$,

$$n^k \in \mathbb{R}\mathbb{Q}^2 \tag{3.39}$$

or

(DIV) small divisors there exists $C,\ l > 0$ such that for all $1 \leq k \leq M$,

$$\forall \xi = (\xi_1, \xi_2) \in \mathbb{Z}^2 \setminus \{0\}, \qquad |P_{n^{k\perp}}(\xi)| \geq C|\xi|^{-l} \tag{3.40}$$

where $P_{n^{k\perp}}$ is the orthogonal projector on $n^{k\perp}$.

As $\Omega \subset \mathbb{R}^2$, condition (3.40) boils down to

$$\forall \xi \in \mathbb{Z}^2 \setminus \{0\}, \qquad |n^k \cdot \xi| \geq C|\xi|^{-l} \tag{3.41}$$

where $n^k \cdot \xi := n_1^k \xi_1 + n_2^k \xi_2$. Note that a vector $n \in \mathbb{R}^2$ cannot satisfy both (3.39) and (3.40) or (3.41).

Keeping in mind that $\vartheta_{bl}^\varepsilon$ solves (3.37), one has the following convergence theorems. Note that as soon as we invoke the regularity theorem 3.4 in the proofs, we need the symmetry assumption **(A4)** on A.

Theorem 3.15. *Assume Ω satisfies* **(RAT)**. *Assume furthermore that $u^0 \in H^{2+\omega}(\Omega)$ with $0 < \omega < 1$ (resp. $u^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$).*
Then for any sequence (ε_n) tending to 0, there exists a subsequence, which we denote again by (ε_n), and $(V^{k,\alpha,*})_{\substack{1\le k\le M\\ 1\le\alpha\le 2}} \in M_N(\mathbb{R})^{2\times M}$ such that $\vartheta_{bl}^{\varepsilon_n}$ solution of (3.12) converges in $L^2(\Omega)$ towards ϑ_{bl}^* solution of

$$\left\{\begin{array}{rll} -\nabla\cdot A^0\nabla\vartheta_{bl}^* = & 0, & x\in\Omega \\ \vartheta_{bl}^* = & -V^{k,\alpha,*}\partial_{x_\alpha}u^0(x), & x\in\partial\Omega\cap K^k, \textit{ for all } 1\le k\le M \end{array}\right. . \tag{3.42}$$

Moreover, we have the following convergence rates:

1. *if $u^0 \in H^{2+\omega}(\Omega)$, then there exists $0<\gamma<\frac{\omega}{2}$ such that*
$$\left\|\vartheta_{bl}^{\varepsilon_n} - \vartheta_{bl}^*\right\|_{L^2(\Omega)} = O\big(\varepsilon_n^{\gamma}\big);$$
2. *if $u^0 \in H^3(\Omega)\cap C^2(\overline{\Omega})$, then*
$$\left\|\vartheta_{bl}^{\varepsilon_n} - \vartheta_{bl}^*\right\|_{L^2(\Omega)} = O\big(\varepsilon_n^{\frac{1}{2}}\big).$$

Theorem 3.16. *Assume Ω satisfies* **(DIV)**. *Assume furthermore that $u^0 \in H^{2+\omega}(\Omega)$ with $0 < \omega < 1$ (resp. $u^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$).*
Then there exists $(V^{k,\alpha,*})_{\substack{1\le k\le M\\ 1\le\alpha\le 2}} \in M_N(\mathbb{R})^{2\times M}$ such that $\vartheta_{bl}^{\varepsilon}$ solution of (3.12) converges in $L^2(\Omega)$ towards ϑ_{bl}^* solution of

$$\left\{\begin{array}{rll} -\nabla\cdot A^0\nabla\vartheta_{bl}^* = & 0, & x\in\Omega \\ \vartheta_{bl}^* = & -V^{k,\alpha,*}\partial_{x_\alpha}u^0(x), & x\in\partial\Omega\cap K^k, \textit{ for all } 1\le k\le M \end{array}\right. . \tag{3.43}$$

Moreover, we have the following convergence rates:

1. *if $u^0 \in H^{2+\omega}(\Omega)$, then there exists $0<\gamma<\frac{\omega}{2}$ such that*
$$\left\|\vartheta_{bl}^{\varepsilon} - \vartheta_{bl}^*\right\|_{L^2(\Omega)} = O\big(\varepsilon^{\gamma}\big); \tag{3.44}$$
2. *if $u^0 \in H^3(\Omega)\cap C^2(\overline{\Omega})$, then*
$$\left\|\vartheta_{bl}^{\varepsilon} - \vartheta_{bl}^*\right\|_{L^2(\Omega)} = O\big(\varepsilon^{\frac{1}{2}}\big). \tag{3.45}$$

The only, but major, difference between theorem 3.15 and 3.16 is that, in the small divisors case, convergence holds for the whole sequence, whereas in the rational case, convergence takes place up to the extraction of a subsequence (ε_n), the constant matrices $V^{k,\alpha,*}$ depending on (ε_n).

3.3.3 How to compute the limit?

The homogenized boundary layer corrector ϑ_{bl}^* solving (3.43) can be computed as soon as one is able to make explicit the boundary layer tails $V^{k,\alpha,*} \in M_N(\mathbb{R})$. Let us shortly review the different settings:

RAT The boundary layer tail $V^{k,\alpha,*}$ depends on the subsequence (ε_n). However, given a subsequence (ε_n) such that $\vartheta_{bl}^{\varepsilon_n}$ converges, Moskow and Vogelius show an explicit formula for the boundary layer tail when $N=1$: see formulae (4.6) and (4.7) in [MV97]. Their proofs in appendix 6 (propositions 6.3 and 6.6) rely on the periodic setting and on the fact that the equations are scalar. Numerical investigations have been carried out by Sarkis and Versieux in [SV06, SV08] using these formulae to compute the boundary layer tails in the case when Ω is a square.

DIV The boundary layer tail is shown to be unique (see theorem 3.17 below and the article [GVM11]). The formulae of Moskow and Vogelius have been recently extended by the author to arbitrary polygons and arbitrary $N \geq 1$: see formula (6.4) in [Pra13].

smooth In [GVM12], Gérard-Varet and Masmoudi build the boundary data φ^* as a functional $\varphi^*(x) = \mathcal{A}\left(\varphi(x,\cdot), A(\cdot), n(x)\right)$, where $n(x)$ is the inward normal to the boundary at $x \in \partial\Omega$. In section 3 therein, φ^* is constructed almost everywhere on $\partial\Omega$, at the points where $n(x)$ satisfies the **(DIV)** assumption. This suggests to address the numerics by approximating the boundary by a polygonal with edges satifying the **(DIV)** assumption.

3.3.4 Proof of theorem 3.16

The proofs of theorems 3.15 and 3.16 follow the same steps. They differ mainly in one intermediate result, which explains why in the rational case, the convergence result is true only up to the extraction of a subsequence. Although we focus on the **(DIV)** assumption, we underline the difference with assumption **(RAT)**.

Existence of the boundary layer tails

Let us show the existence of the matrices $V^{k,\alpha,*}$. Let $1 \leq k \leq M$ and $1 \leq \alpha \leq 2$. We are interested in the boundary layer profile in the vicinity of vertex k. Thus, introduce $v_{bl}^{k,\alpha,\varepsilon}$ solution of

$$\left\{\begin{array}{rcll} -\nabla_y \cdot A(y)\nabla_y v_{bl}^{k,\alpha,\varepsilon} &=& 0, & y \in \Omega^{k,\varepsilon} \\ v_{bl}^{k,\alpha,\varepsilon} &=& \chi^\alpha(y), & y \in \partial\Omega^{k,\varepsilon} \end{array}\right. \tag{3.46}$$

where $\Omega^{k,\varepsilon} := \left\{y,\ n^k \cdot y - \frac{c^k}{\varepsilon} > 0\right\}$. Let $M^k \in M_2(\mathbb{R})$ be an orthogonal matrix, mapping $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to n^k.

The following theorem describes the profile of $V^{k,\alpha,\varepsilon} := v_{bl}^{k,\alpha,\varepsilon}(M^k\cdot)$.

Theorem 3.17 (Gérard-Varet and Masmoudi in [GVM11]). *Assume that Ω satisfies* **(DIV)**.
Then,

1. *for all $\varepsilon > 0$, there exists $V^{k,\alpha,\varepsilon} \in C^\infty\Big(\mathbb{R} \times \big[\frac{c^k}{\varepsilon}, \infty\big[\Big)$;*
2. *there exists a matrix*
$$V^{k,\alpha,*} \in M_N(\mathbb{R}) \tag{3.47}$$
such that for all $\beta \in \mathbb{N}^2$, for all $m \in \mathbb{N}$, there is a constant $C_{|\beta|,m} > 0$ satisfying for all $\varepsilon > 0$ and $z_2 > \frac{c^k}{\varepsilon}$,
$$\left(1 + \left|z_2 - \tfrac{c^k}{\varepsilon}\right|^m\right) \sup_{z_1 \in \mathbb{R}} \left|\partial_z^\beta\Big(V^{k,\alpha,\varepsilon}(z_1, z_2) - V^{k,\alpha,*}\Big)\right| \leq C_{|\beta|,m}. \tag{3.48}$$

Remark 3.18. Note that (3.48) is true not only for $m \in \mathbb{N}$ but for $m \in \mathbb{R}$, $m > 0$. Indeed, $[m]$ denoting the integer part of m, we have

$$\left|z_2 - \frac{c^k}{\varepsilon}\right|^m < \left|z_2 - \frac{c^k}{\varepsilon}\right|^{[m]+1}, \text{ if } \left|z_2 - \frac{c^k}{\varepsilon}\right| > 1$$
$$\left|z_2 - \frac{c^k}{\varepsilon}\right|^m \leq 1, \text{ if } \left|z_2 - \frac{c^k}{\varepsilon}\right| \leq 1.$$

Remark 3.19. If we rewrite the second statement of theorem 3.17 in terms of $v_{bl}^{k,\alpha,\varepsilon}$ instead of $V^{k,\alpha,\varepsilon}$ we get: for all $\beta \in \mathbb{N}^2$, for all $m \in \mathbb{N}$, there is a constant $C_{|\beta|,m} > 0$ satisfying for all $\varepsilon > 0$ and $y \in \Omega^{k,\varepsilon}$,

$$\left(1 + \left|y \cdot n^k - \frac{c^k}{\varepsilon}\right|^m\right) \left|\partial_y^\beta \left(v_{bl}^{k,\alpha,\varepsilon}(y) - V^{k,\alpha,*}\right)\right| \leq C_{|\beta|,m}. \tag{3.49}$$

Remark 3.20. If instead of **(DIV)** we assume **(RAT)**, the boundary layer tails $V^{k,\alpha,\varepsilon}$ still exist. Furthermore, an equivalent of theorem 3.17 states: there exists a sequence (ε_n) (here lies the main difference between the two assumptions), a constant matrix $V^{k,\alpha,*} \in M_N(\mathbb{R})$ such that for all $m \in \mathbb{N}$, for all $z_2 > \frac{c^k}{\varepsilon_n}$,

$$\left(1 + \left|z_2 - \frac{c^k}{\varepsilon_n}\right|^m\right) \sup_{z_1 \in \mathbb{R}} \left|\partial_z^\beta \left(V^{k,\alpha,\varepsilon_n}(z_1, z_2) - V^{k,\alpha,*}\right)\right| \leq C_{|\beta|,m}.$$

The latter is sufficient to get our results. However, Moskow and Vogelius in [MV97], as well as Allaire and Amar in [AA99] manage to prove an improved result under assumption **(RAT)**: the convergence of the boundary layer towards its tail is exponential.

Assume from now on that Ω satisfies **(DIV)**. *Let $0 < \omega < 1$ be fixed. The assumptions $u^0 \in H^{2+\omega}(\Omega)$ with $0 < \omega < 1$ and $u^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$ are treated in parallel. In both cases, by Sobolev injection, $u^0 \in C^1(\overline{\Omega})$.*

Well-posedness of (3.43)

It is enough to prove that the boundary function of (3.43) belongs to $H^{\frac{1}{2}}(\partial\Omega)$. One constructs a lifting ϕ_{bl}^* of $\varphi_{bl}^* := -V^{k,\alpha,*}\partial_{x_\alpha} u^0(x)$. There exists $G = (G^1, G^2) \in M_N(\mathbb{R}) \times M_N(\mathbb{R})$ such that

$$\phi_{bl}^* = G^\alpha \partial_{x_\alpha} u^0. \tag{3.50}$$

Following [GVM11], one can show:

Proposition 3.21. *If $u^0 \in H^2(\Omega)$, then $\phi_{bl}^* \in H^1(\Omega)$. If u^0, in addition, belongs to $H^3(\Omega)$, then ϕ_{bl}^* belongs to $H^2(\Omega)$.*

Sketch of the proof of the estimates (3.44) and (3.45)

Our strategy is to split the problem of estimating $\vartheta_{bl}^\varepsilon - \vartheta_{bl}^*$ into three easier ones. For this purpose, we introduce $\vartheta_{bl}^{\varepsilon,*}$ solution of

$$\left\{\begin{array}{rcll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla \vartheta_{bl}^{\varepsilon,*} &=& 0, & x \in \Omega \\ \vartheta_{bl}^{\varepsilon,*} &=& -V^{k,\alpha,*}\partial_{x_\alpha} u^0(x), & x \in \partial\Omega \cap K^k, \text{ for all } 1 \leq k \leq M \end{array}\right. \tag{3.51}$$

to get via the triangular inequality:

$$\left\|\vartheta_{bl}^\varepsilon - \vartheta_{bl}^*\right\|_{L^2(\Omega)} \leq \left\|\vartheta_{bl}^{\varepsilon,*} - \vartheta_{bl}^*\right\|_{L^2(\Omega)} + \left\|\vartheta_{bl}^\varepsilon - \vartheta_{bl}^{\varepsilon,*}\right\|_{L^2(\Omega)}.$$

Note that $\vartheta_{bl}^{\varepsilon,*}$ is well defined because of proposition 3.21.

The study of the first term seems to be more classic as the boundary data function of (3.43) and (3.51) is not oscillating. The second term, on the contrary, requires a deep knowledge about the homogenization of boundary layer systems.

In fact $\vartheta^\varepsilon_{bl} - \vartheta^{\varepsilon,*}_{bl}$ is the solution of (3.12) with $\varphi(x,y) = -\big(\chi^\alpha(y) - V^{k,\alpha,*}\big)\partial_{x_\alpha} u^0(x)$, for all $x \in \partial\Omega \cap K^k$, for all $y \in \mathbb{R}^2$; we call $u^{1,\varepsilon}_{bl}$ the difference $\vartheta^\varepsilon_{bl} - \vartheta^{\varepsilon,*}_{bl}$. It comes from proposition 3.21 that φ defined like this is in $H^{\frac{1}{2}}(\partial\Omega)$. Let

$$v^{k,\varepsilon}_{bl} := -\big(v^{k,\alpha,\varepsilon}_{bl} - V^{k,\alpha,*}\big)\partial_{x_\alpha} u^0, \tag{3.52}$$

where $v^{k,\alpha,\varepsilon}_{bl}$ is defined by (3.46), and its tail $V^{k,\alpha,*}$ by theorem 3.17 (see (3.47)). We expect $u^{1,\varepsilon}_{bl}$ to be close to $\sum_{k=1}^M v^{k,\varepsilon}_{bl}\big(\cdot, \frac{\cdot}{\varepsilon}\big)$:

$$\left\|u^{1,\varepsilon}_{bl}\right\|_{L^2(\Omega)} \leq \sum_{k=1}^{M} \left\|v^{k,\varepsilon}_{bl}\big(x, \tfrac{x}{\varepsilon}\big)\right\|_{L^2(\Omega)} + \left\|u^{1,\varepsilon}_{bl}(x) - \textstyle\sum_{k=1}^M v^{k,\varepsilon}_{bl}\big(x, \tfrac{x}{\varepsilon}\big)\right\|_{L^2(\Omega)}.$$

The rest of the proof is thus devoted to estimate each of the terms in the r.h.s. of:

$$\begin{aligned}\left\|\vartheta^\varepsilon_{bl} - \vartheta^*_{bl}\right\|_{L^2(\Omega)} \leq \left\|\vartheta^{\varepsilon,*}_{bl} - \vartheta^*_{bl}\right\|_{L^2(\Omega)} &+ \sum_{k=1}^{M} \left\|v^{k,\varepsilon}_{bl}\big(x, \tfrac{x}{\varepsilon}\big)\right\|_{L^2(\Omega)} \\ &+ \left\|u^{1,\varepsilon}_{bl}(x) - \textstyle\sum_{k=1}^M v^{k,\varepsilon}_{bl}\big(x, \tfrac{x}{\varepsilon}\big)\right\|_{L^2(\Omega)}.\end{aligned} \tag{3.53}$$

First term in the r.h.s of (3.53)

We resort to corollary 3.11 to estimate this term. In order to get some convergence rate, we need to have a little more regularity on ϑ^*_{bl} than $\vartheta^*_{bl} \in H^1(\Omega)$. According to proposition 3.21, the lifting ϕ^*_{bl} of the boundary data of (3.43) belongs to $H^{1+\omega}(\Omega)$ (resp. $H^2(\Omega)$), provided that u^0 belongs to $H^{2+\omega}(\Omega)$ (resp. $H^3(\Omega)$).

Let us treat the two assumptions on u^0 separately. If $u^0 \in H^{2+\omega}(\Omega)$, it follows from the first point of theorem 3.4, that ϑ^*_{bl} has $H^{1+\gamma}(\Omega)$ regularity for all γ such that $0 \leq \gamma \leq \omega$ and $\gamma \neq \frac{1}{2}$. Therefore,

$$\left\|\vartheta^{\varepsilon,*}_{bl} - \vartheta^*_{bl}\right\|_{L^2(\Omega)} = O\big(\varepsilon^{\frac{\gamma}{2}}\big).$$

If $u^0 \in H^3(\Omega)$, the second point of theorem 3.4 yields that $\vartheta^*_{bl} \in H^2(\Omega)$. Applying corollary 3.10 implies

$$\left\|\vartheta^{\varepsilon,*}_{bl} - \vartheta^*_{bl}\right\|_{L^2(\Omega)} = O\big(\varepsilon^{\frac{1}{2}}\big).$$

Second term in the r.h.s of (3.53)

By linearity of the equations, the boundary layer tail $V^{k,*}(x)$ of $v^{k,\varepsilon}_{bl}(x,\cdot)$ is equal to

$$V^{k,*}(x) = -V^{k,\alpha,*}\partial_{x_\alpha} u^0(x) + V^{k,\alpha,*}\partial_{x_\alpha} u^0(x) = 0.$$

We deduce from theorem 3.17: for all $m \in \mathbb{N}$, there is a constant $C_m > 0$ such that for all $\varepsilon > 0$, for all $x \in \Omega$,

$$\left(1 + \frac{\left|x \cdot n^k - c^k\right|^m}{\varepsilon^m}\right)\left|v^{k,\varepsilon}_{bl}\big(x, \tfrac{x}{\varepsilon}\big)\right| \leq C_m. \tag{3.54}$$

The uniformity in x comes from the fact $u^0 \in C^1(\overline{\Omega})$ and from the boundedness of Ω.

Proposition 3.22. *For all $1 \leq k \leq M$, $\left\|v^{k,\varepsilon}_{bl}\big(x, \frac{x}{\varepsilon}\big)\right\|_{L^2(\Omega)} = O\big(\varepsilon^{\frac{1}{2}}\big)$, where we recall that $v^{k,\varepsilon}_{bl}$ is defined by* (3.52).

Proof. Let $m \in \mathbb{N}$. From (3.54) we get

$$\begin{aligned}\left\| v_{bl}^{k,\varepsilon}\left(x, \tfrac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)}^2 &\leq C \int_\Omega \frac{1}{\left(1 + \frac{|x \cdot n^k - c^k|^m}{\varepsilon^m}\right)^2} dx \\ &\leq C \int_{\widetilde{\Omega}} \frac{1}{\left(1 + \frac{u_2^m}{\varepsilon^m}\right)^2} du\end{aligned}$$

where $\widetilde{\Omega} := {}^t M^k \Omega - \begin{pmatrix} 0 \\ c^k \end{pmatrix}$. Therefore, we have to focus on

$$\int_{[0,\infty[} \frac{1}{\left(1 + \frac{u_2^m}{\varepsilon^m}\right)^2} du_2 = \varepsilon^{2m} \int_{[0,\infty[} \frac{1}{\left(\varepsilon^m + u_2^m\right)^2} du_2.$$

For $2m > 1$ the integral is convergent and

$$\int_{[0,\infty[} \frac{1}{\left(\varepsilon^m + u_2^m\right)^2} du_2 \leq \int_{[0,\varepsilon]} \frac{1}{\varepsilon^{2m}} + \int_{[\varepsilon,\infty[} \frac{1}{u_2^{2m}} du_2 = O\left(\varepsilon^{-2m+1}\right).$$

We immediately deduce that

$$\left\| v_{bl}^{k,\varepsilon}\left(x, \tfrac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)}^2 = O(\varepsilon)$$

which yields the result. □

Remark 3.23. It is easy to adapt the proof of proposition 3.22 to get: for all $1 \leq k \leq M$, for all $1 \leq p \leq \infty$, $\left\| v_{bl}^{k,\varepsilon}\left(x, \tfrac{x}{\varepsilon}\right)\right\|_{L^p(\Omega)} = O\left(\varepsilon^{\frac{1}{p}}\right)$. For $p = \infty$ it is (3.54) with $m = 0$; for $1 \leq p < \infty$ the proof follows the lines of the case $p = 2$, except that one has to replace 2 by p. In the same manner, it is very straightforward to deduce from (3.49) that for all $1 \leq p \leq \infty$, for all $\beta \in \mathbb{N}^d$, for all $m \in \mathbb{N}$,

$$\left\| \partial_y^\beta \left(v_{bl}^{k,\alpha,\varepsilon}\left(\tfrac{x}{\varepsilon}\right) - V^{k,\alpha,*}\right) \frac{|x \cdot n^k - c^k|^m}{\varepsilon^m} \right\|_{L^p(\Omega)} = O\left(\varepsilon^{\frac{1}{p}}\right). \tag{3.55}$$

Third term in the r.h.s of (3.53)

We proceed as usual by carrying out energy estimates on the error

$$e_{bl}^\varepsilon := u_{bl}^{1,\varepsilon}(x) - \sum_{k=1}^M v_{bl}^{k,\varepsilon}\left(x, \frac{x}{\varepsilon}\right),$$

where we recall that $v_{bl}^{k,\varepsilon}$ is defined by (3.52). The error solves the system

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla e_{bl}^\varepsilon = & r_{bl}^\varepsilon, \quad x \in \Omega \\ e_{bl}^\varepsilon = & \varphi_{bl}^\varepsilon, \quad x \in \partial\Omega \end{cases}$$

where the source term

$$r_{bl}^\varepsilon := \sum_{k=1}^M \left\{ \nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right) \nabla_x v_{bl}^{k,\varepsilon}\left(x, \frac{x}{\varepsilon}\right)\right) + \frac{1}{\varepsilon}\left[\nabla_x \cdot A(y) \nabla_y v_{bl}^{k,\varepsilon}\right]\left(x, \frac{x}{\varepsilon}\right)\right\} \tag{3.56}$$

and the piecewise defined boundary function

$$\varphi_{bl}^{\varepsilon}|_{\partial\Omega\cap K^k} := -\Big(\chi^{\alpha}\Big(\frac{x}{\varepsilon}\Big) - V^{k,\alpha,*}\Big)\partial_{x_\alpha}u^0(x) - \sum_{k'=1}^{M} v_{bl}^{k',\varepsilon}\Big(x, \frac{x}{\varepsilon}\Big) = -\sum_{k'\neq k}^{M} v_{bl}^{k',\varepsilon}\Big(x, \frac{x}{\varepsilon}\Big). \quad (3.57)$$

We estimate separately r_{bl}^{ε} (cf. lemma 3.24) and $\varphi_{bl}^{\varepsilon}$ (cf. lemma 3.25).

Lemma 3.24. *The source term r_{bl}^{ε}, defined by* (3.56), *is*

1. *of order $O(\varepsilon^{\gamma})$ in $H^{-1}(\Omega)$ for all $0 < \gamma < \frac{\omega}{2}$ if $u^0 \in H^{2+\omega}(\Omega)$;*
2. *of order $O(\varepsilon^{\frac{1}{2}})$ in $H^{-1}(\Omega)$ if $u^0 \in C^2(\overline{\Omega})$.*

Proof. Assume $u^0 \in H^{2+\omega}(\Omega)$ (resp. $u^0 \in C^2(\overline{\Omega})$). Let $1 \leq k \leq M$ be fixed and consider

$$r_{bl}^{\varepsilon,k} := \nabla\cdot\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla_x v_{bl}^{k,\varepsilon}\Big(x, \frac{x}{\varepsilon}\Big)\Big) + \frac{1}{\varepsilon}\Big[\nabla_x \cdot A(y)\nabla_y v_{bl}^{k,\varepsilon}\Big]\Big(x, \frac{x}{\varepsilon}\Big).$$

We focus on the first term of $r_{bl}^{\varepsilon,k}$. For all $\phi \in H_0^1(\Omega)$,

$$\begin{aligned}
&\left|\Big\langle \nabla\cdot\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla_x v_{bl}^{k,\varepsilon}\Big(x, \frac{x}{\varepsilon}\Big)\Big), \phi(x)\Big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right| \\
&= \left|\int_{\Omega}\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla_x v_{bl}^{k,\varepsilon}\Big(x, \frac{x}{\varepsilon}\Big)\Big)\nabla\phi(x)dx\right| \\
&\leq \Big\|A\big(\tfrac{x}{\varepsilon}\big)\nabla_x v_{bl}^{k,\varepsilon}\big(x, \tfrac{x}{\varepsilon}\big)\Big\|_{L^2(\Omega)}\Big\|\nabla\phi\Big\|_{L^2(\Omega)} \\
&\leq C\Big\|\nabla_x v_{bl}^{k,\varepsilon}\big(x, \tfrac{x}{\varepsilon}\big)\Big\|_{L^2(\Omega)}\Big\|\phi\Big\|_{H_0^1(\Omega)}.
\end{aligned}$$

At this point we need to estimate $\Big\|\nabla_x v_{bl}^{k,\varepsilon}\big(x, \tfrac{x}{\varepsilon}\big)\Big\|_{L^2(\Omega)}$. As, according to (3.52),

$$\nabla_x v_{bl}^{k,\varepsilon}\Big(x, \frac{x}{\varepsilon}\Big) = \Big(v_{bl}^{k,\beta,\varepsilon}\Big(\frac{x}{\varepsilon}\Big) - V^{k,\beta,*}\Big)\partial_{x_\alpha}\partial_{x_\beta}u^0,$$

the idea is to bound the $L^2(\Omega)$ norm of this term using a Hölder inequality and (3.55). Doing so, one has to pay attention to the regularity of u^0, and to carefully choose the $L^p(\Omega)$ spaces involved.

If $u^0 \in C^2(\overline{\Omega})$,

$$\Big\|\Big(v_{bl}^{k,\beta,\varepsilon}\big(\tfrac{x}{\varepsilon}\big) - V^{k,\beta,*}\Big)\partial_{x_\alpha}\partial_{x_\beta}u^0\Big\|_{L^2(\Omega)} \leq \Big\|v_{bl}^{k,\beta,\varepsilon}\big(\tfrac{x}{\varepsilon}\big) - V^{k,\beta,*}\Big\|_{L^2(\Omega)}\Big\|\partial_{x_\alpha}\partial_{x_\beta}u^0\Big\|_{L^\infty(\Omega)}.$$

The assumption $u^0 \in C^2(\overline{\Omega})$ plays here the same role as $u^0 \in C^1(\overline{\Omega})$ for (3.54). Use (3.55) with $p = 2$ to conclude.

If $u^0 \in H^{2+\omega}(\Omega)$, we cannot proceed as above because $\partial_{x_\alpha}\partial_{x_\beta}u^0$ does not belong to $L^\infty(\Omega)$. By the Sobolev injections, $H^{\omega}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $1 \leq q < \frac{2}{1-\omega}$. Yet $\partial_{x_\alpha}\partial_{x_\beta}u^0$ is in $H^{\omega}(\Omega)$. Take $2 \leq q < \frac{2}{1-\omega}$ and $\widehat{q} \geq 2$ such that $\frac{1}{q} + \frac{1}{\widehat{q}} = \frac{1}{2}$. Necessarily $\frac{2}{\omega} < \widehat{q}$. Hölder's inequality yields

$$\Big\|\Big(v_{bl}^{k,\beta,\varepsilon}\big(\tfrac{x}{\varepsilon}\big) - V^{k,\beta,*}\Big)\partial_{x_\alpha}\partial_{x_\beta}u^0\Big\|_{L^2(\Omega)} \leq \Big\|v_{bl}^{k,\beta,\varepsilon}\big(\tfrac{x}{\varepsilon}\big) - V^{k,\beta,*}\Big\|_{L^{\widehat{q}}(\Omega)}\Big\|\partial_{x_\alpha}\partial_{x_\beta}u^0\Big\|_{L^q(\Omega)}.$$

Apply now (3.55) with $p = \widehat{q}$ to get $\Big\|\nabla_x v_{bl}^{k,\varepsilon}\big(x, \tfrac{x}{\varepsilon}\big)\Big\|_{L^2(\Omega)} = O\big(\varepsilon^{\frac{1}{\widehat{q}}}\big)$.

The second term of $r_{bl}^{\varepsilon,k}$ needs to be treated differently. The key ingredient is Hardy's inequality: for all $\phi \in H_0^1(\Omega)$

$$\left\|\frac{\phi(x)}{d(x,\partial\Omega)}\right\|_{L^2(\Omega)} \leq \left\|\nabla\phi\right\|_{L^2(\Omega)}$$

where $d(x,\partial\Omega)$ is the distance from x to $\partial\Omega$. Let $\phi \in H_0^1(\Omega)$. For all $2 \leq q, \widehat{q} \leq \infty$ such that $\frac{1}{q}+\frac{1}{\widehat{q}}=2$,

$$\begin{aligned}
&\left|\int_\Omega \left[\nabla_x \cdot A(y)\nabla_y v_{bl}^{k,\varepsilon}\right]\left(x,\frac{x}{\varepsilon}\right)\phi(x)dx\right| \\
&\qquad \leq C\varepsilon \left\|\partial_{y_\beta} v_{bl}^{k,\gamma,\varepsilon}\left(\tfrac{x}{\varepsilon}\right)\partial_{x_\alpha}\partial_{x_\gamma}u^0(x)\frac{|x\cdot n^k - c^k|}{\varepsilon}\right\|_{L^2(\Omega)} \left\|\frac{\phi(x)}{d(x,\partial\Omega)}\right\|_{L^2(\Omega)} \\
&\qquad \leq C\varepsilon \left\|\partial_{y_\beta} v_{bl}^{k,\gamma,\varepsilon}\left(\tfrac{x}{\varepsilon}\right)\frac{|x\cdot n^k - c^k|}{\varepsilon}\right\|_{L^{\widehat{q}}(\Omega)} \left\|\partial_{x_\alpha}\partial_{x_\gamma}u^0\right\|_{L^q(\Omega)} \left\|\nabla\phi\right\|_{L^2(\Omega)}.
\end{aligned}$$

If $u^0 \in H^{2+\omega}(\Omega)$, then take $2 \leq q < \frac{2}{1-\omega}$ and apply (3.55) with $p=\widehat{q}$ and $m=1$. If $u^0 \in C^2(\overline{\Omega})$, then take $q=\infty$ and $\widehat{q}=2$ and apply (3.55) with $p=2$ and $m=1$. $\square$

Lemma 3.25. *The boundary function φ_{bl}^ε, defined by (3.57), is*

1. *of order $O(\varepsilon^\omega)$ in $W^{1-\frac{1}{p},p}(\partial\Omega)$ for all $1 \leq p < \frac{2}{1+\omega}$, if $u^0 \in H^{2+\omega}(\Omega)$;*
2. *of order $O(\varepsilon)$ in $W^{1-\frac{1}{p},p}(\partial\Omega)$ for all $1 \leq p < 2$, if $u^0 \in H^3(\Omega)\cap C^2(\overline{\Omega})$.*

Proof. First of all, using proposition 3.21 one notices that φ_{bl}^ε belongs to $H^{\frac{1}{2}}(\partial\Omega)$. As φ_{bl}^ε factors into $V\left(\frac{\cdot}{\varepsilon}\right)\nabla u^0$ we immediatly get the very rough estimate

$$\left\|\varphi_{bl}^\varepsilon\right\|_{H^{\frac{1}{2}}(\partial\Omega)} = O\left(\varepsilon^{-\frac{1}{2}}\right)$$

which is far from being enough. We do not try further to get a bound in $H^{\frac{1}{2}}(\partial\Omega)$.

We refer to [GVM11] for the case when $u^0 \in H^3(\Omega)\cap C^2(\overline{\Omega})$. If $u^0 \in H^{2+\omega}(\Omega)$ the proof follows the same scheme, with differences due to the weaker regularity assumption on u^0. *Assume for the rest of the proof that $u^0 \in H^{2+\omega}(\Omega)$.* The edge estimate goes on as in the case $u^0 \in H^3(\Omega)\cap C^2(\overline{\Omega})$ and one gets for all $1 \leq p < 2$, $m \in \mathbb{N}$

$$\left\|\psi\varphi^\varepsilon\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = O\left(\varepsilon^m\right)$$

where ψ is a smooth function on $\partial\Omega$ compactly supported in $\partial\Omega\cap K^k$ outside the vertices.

Let us now focus on the estimate near a vertex O lying at the intersection of K^1 and K^2. We introduce polar coordinates $r=r(x)$ and $\theta=\theta(x)$ centered at O and use a smooth function ψ on $\partial\Omega$ compactly supported in a vicinity of O. Let $1 \leq p$. The tame estimate

$$\left\|fg\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C\left(\left\|f\right\|_{L^\infty(\partial\Omega)}\left\|g\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \left\|g\right\|_{L^\infty(\partial\Omega)}\left\|f\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right) \tag{3.58}$$

holds for all $f,g \in L^\infty(\partial\Omega)\cap W^{1-\frac{1}{p},p}(\partial\Omega)$. Taking advantage of the fact that $H^{2+\omega}(\Omega)$ injects in $C^{1,\omega}(\overline{\Omega})$, one knows $\frac{\nabla u^0}{r^\omega} \in L^\infty(\partial\Omega)$. Besides, $\frac{\nabla u^0}{r^\omega}$ belongs to $W^{1,p}(\Omega)$ for all $1 \leq p < \frac{2}{1+\omega}$. Therefore $\frac{\nabla u^0}{r^\omega} \in L^\infty(\partial\Omega)\cap W^{1-\frac{1}{p},p}(\partial\Omega)$ for all $1 \leq p < \frac{2}{1+\omega}$ and (3.58) yields

$$\begin{aligned}
\left\|\psi^2\varphi_{bl}^\varepsilon\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C\Bigg(&\left\|\psi r^\omega V\left(\tfrac{\cdot}{\varepsilon}\right)\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\left\|\psi\frac{\nabla u^0}{r^\omega}\right\|_{L^\infty(\partial\Omega)} \\
&+ \left\|\psi r^\omega V\left(\tfrac{\cdot}{\varepsilon}\right)\right\|_{L^\infty(\partial\Omega)}\left\|\psi\frac{\nabla u^0}{r^\omega}\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\Bigg).
\end{aligned}$$

By estimating first on $\partial\Omega \cap K^1$ then on $\partial\Omega \cap K^2$ one obtains for all $1 \leq p < 2$

$$\left\|\psi r^\omega V\big(\tfrac{\cdot}{\varepsilon}\big)\right\|_{L^\infty(\partial\Omega)} = O\big(\varepsilon^\omega\big)$$

$$\left\|\psi r^\omega V\big(\tfrac{\cdot}{\varepsilon}\big)\right\|_{L^p(\partial\Omega)} = O\big(\varepsilon^{\omega+\frac{1}{p}}\big) \tag{3.59a}$$

$$\left\|\psi r^\omega V\big(\tfrac{\cdot}{\varepsilon}\big)\right\|_{W^{1,p}(\partial\Omega)} = O\big(\varepsilon^{\omega-1+\frac{1}{p}}\big). \tag{3.59b}$$

Interpolating (3.59a) and (3.59b) gives

$$\left\|\psi r^\omega V\big(\tfrac{\cdot}{\varepsilon}\big)\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C \left\|\psi r^\omega V\big(\tfrac{\cdot}{\varepsilon}\big)\right\|^{1-\frac{1}{p}}_{W^{1,p}(\partial\Omega)} \left\|\psi r^\omega V\big(\tfrac{\cdot}{\varepsilon}\big)\right\|^{\frac{1}{p}}_{L^p(\partial\Omega)} = O\big(\varepsilon^{\omega-1+\frac{2}{p}}\big).$$

Finally, $\left\|\psi^2 \varphi^\varepsilon_{bl}\right\|_{W^{1-\frac{1}{p},p}} = O\big(\varepsilon^\omega\big)$ which concludes our proof. □

We conclude this section by expounding how to deduce a bound on e^ε_{bl} from the lemmas 3.24 and 3.25. We focus on the case when $u^0 \in H^{2+\omega}(\Omega)$, as the reasoning is a little more subtle than in the case $u^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$. It is very straightforward to adapt the proof in the latter case (see also [GVM11]). Lemma 3.25 gives bounds for φ^ε_{bl} in $W^{1-\frac{1}{p},p}(\partial\Omega)$ for $1 \leq p < \frac{2}{1+\omega}$. As we lack an estimate in $H^{\frac{1}{2}}(\partial\Omega)$, we cannot bound e^ε_{bl} in $H^1(\Omega)$. We thus have to use results on elliptic equations in divergence form and with source term in some $W^{-1,p}(\Omega)$ space. Let us state a general theorem that suits to our framework (for references see below).

Theorem 3.26 (Meyers). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ a family of $C^\infty(\overline{\Omega})$ functions. Assume the ellipticity of A.*
There exists a $p_0 < 2$ such that for all $f \in H^{-1}(\Omega)$ if $u \in H^1_0(\Omega)$ is a weak solution of $-\nabla \cdot A\nabla u = f$ in $H^{-1}(\Omega)$ and if for all $p_0 < p < 2$, $f \in W^{-1,p}(\Omega)$, then $u \in W^{1,p}_0(\Omega)$ and there exists $C(p) > 0$,

$$\left\|u\right\|_{W^{1,p}_0(\Omega)} \leq C(p) \left\|f\right\|_{W^{-1,p}(\Omega)}.$$

Such an estimate originally appeared in the work of Meyers [Mey63], where the case of smooth C^2 domains Ω and scalar equations is treated. It has been extended by Gallouet and Monier in [GM99] to domains Ω with Lipschitz boundary. In their recent survey article [MS11], Maz'ya and Shaposhnikova give very general estimates working for Lipschitz domains Ω and systems of elliptic equations. Our theorem 3.26 happens to be a very special case of theorems 1 and 2 in [MS11]. To make the link obvious take $m = 1$, $l = N$, $a = 0$; then $s = 1 - \frac{1}{p}$, $W^{m,a}_p(\Omega) = W^{1,p}(\Omega)$ and $V^a_p(\Omega) = W^{1,p}_0(\Omega)$.

It is important to notice that p_0 (resp. $C(p)$) only depends on the coercivity constant of A (resp. on the coercivity constant of A and p). This makes the theorem applicable to our homogenization problem. We know from the proof of lemma 3.25 that

$$\varphi^\varepsilon_{bl} \in \bigcap_{1 \leq p \leq \frac{2}{1+\omega}} W^{1-\frac{1}{p},p}(\partial\Omega).$$

Thus there exists a lifting ϕ^ε_{bl} of φ^ε_{bl} belonging to $W^{1,p}(\Omega)$ for all $1 \leq p \leq \frac{2}{1+\omega}$ such that

$$\left\|\phi^\varepsilon_{bl}\right\|_{W^{1,p}(\Omega)} \leq C(p) \left\|\varphi^\varepsilon_{bl}\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}$$

with $C(p)$ independent of ε, as usual. The difference $e^\varepsilon_{bl} - \phi^\varepsilon_{bl} \in H^1_0(\Omega)$ solves

$$-\nabla \cdot A\Big(\frac{x}{\varepsilon}\Big)\nabla\left(e^\varepsilon_{bl} - \phi^\varepsilon_{bl}\right) = r^\varepsilon_{bl} + \nabla \cdot A\Big(\frac{x}{\varepsilon}\Big)\nabla\phi^\varepsilon_{bl} =: F^\varepsilon_{bl}.$$

As F^ε_{bl} belongs to $W^{-1,p}(\Omega)$ for $1 \le p \le \frac{2}{1+\omega}$, we have

$$\left\|F^\varepsilon_{bl}\right\|_{W^{-1,p}(\Omega)} \le C(p)\left[\left\|r^\varepsilon_{bl}\right\|_{H^{-1}(\Omega)} + \left\|\varphi^\varepsilon_{bl}\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right]. \tag{3.60}$$

Let $p_0 < 2$ given by theorem 3.26. We can always diminish ω, so as to have $p_0 < \frac{2}{1+\omega}$. Then, for all $p_0 < p \le \frac{2}{1+\omega}$,

$$\left\|e^\varepsilon_{bl} - \phi^\varepsilon_{bl}\right\|_{W^{1,p}_0(\Omega)} \le C(p)\left\|F^\varepsilon_{bl}\right\|_{W^{-1,p}(\Omega)}.$$

Let $0 < \gamma < \frac{\omega}{2}$. Then, it follows from (3.60) and from lemmas 3.24 and 3.25 that for all $p_0 < p < \frac{2}{1+\omega}$,

$$\left\|F^\varepsilon_{bl}\right\|_{W^{-1,p}(\Omega)} = O\big(\varepsilon^\gamma\big).$$

To get an $L^2(\Omega)$ estimate on e^ε_{bl} use the Sobolev injection of $W^{1,p}(\Omega)$ in $L^2(\Omega)$ and, once again, our $W^{1-\frac{1}{p},p}(\partial\Omega)$ bound on φ^ε_{bl}.

3.4 A first-order asymptotic expansion of the eigenvalues

This section is concerned with the final step of the proof of theorems 3.6 and 3.7. Let E_{λ^0} be the finite-dimensional eigenspace associated to λ^0. From the ideas explained in the introduction, and in particular the third part of theorem 3.4, we know, in any case, that $E_{\lambda^0} \subset H^{2+\omega}(\Omega)$, with $0 < \omega$. When Ω is a smooth uniformly convex domain, we take $\omega = 1$.

We have recourse to the ideas involved in [MV97] to prove the asymptotic expansion of the eigenvalues. Moskow and Vogelius use abstract estimates due to Osborn in [Osb75]. We recall the estimate we need in terms of T^ε and T^0. Assume that λ^0 is an eigenvalue of order m. Then, E_{λ^0} is m-dimensional. Let $\lambda^0 = \lambda^{0,k} = \lambda^{0,k+1} = \ldots = \lambda^{0,k+m-1}$. The associated eigenvectors $v^{0,k}, \ldots, v^{0,k+m-1}$ form an orthogonal basis of E_{λ^0}.

Theorem 3.27 (Osborn in [Osb75]). *There exists a constant $C > 0$ such that*

$$\left|\tfrac{1}{\lambda^0} - \tfrac{1}{m}\textstyle\sum_{j=0}^{m-1}\tfrac{1}{\lambda^{\varepsilon,k+j}} - \tfrac{1}{m}\sum_{j=0}^{m-1}\left\langle (T^\varepsilon - T^0)v^{0,k+j}, v^{0,k+j}\right\rangle\right| \quad \le \quad C\left\|(T^\varepsilon - T^0)|_{E_{\lambda^0}}\right\|^2, \tag{3.61}$$

where T^ε and T^0 are seen as operators acting in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega)$.

This theorem is a straightforward corollary of theorem 3.1 in [MV97]. Its proof really uses all properties of the operators T^ε and T^0, among other things selfadjointness and compactness.

The first thing to do is to estimate $\left\|(T^\varepsilon - T^0)|_{E_{\lambda^0}}\right\|$. Let $f \in E_{\lambda^0}$; we call $u^\varepsilon := T^\varepsilon f$ and $u^0 := T^0 f$. We need to estimate $\left\|u^\varepsilon - u^0\right\|_{L^2(\Omega)}$. We can improve the bounds of section 3.2, such as (3.30). Those bounds are not enough to deduce from (3.61) a first-order asymptotic expansion of λ^ε. The loss of $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ in estimate (3.30) is due to the bad bound of $\vartheta^\varepsilon_{u,bl}$ in $H^{\frac{1}{2}}(\partial\Omega)$:

$$\left\|\vartheta^\varepsilon_{u,bl}\right\|_{L^2(\Omega)} \le \left\|\vartheta^\varepsilon_{u,bl}\right\|_{H^1(\Omega)} \le C\left\|\chi^\alpha\big(\tfrac{x}{\varepsilon}\big)\partial_{x_\alpha}u^0(x)\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\varepsilon^{-\frac{1}{2}}\left\|u^0\right\|_{H^2(\Omega)}.$$

If Ω is a smooth domain, (3.7) can be shown thanks to the results of Avellaneda and Lin. Theorem 3.3 yields indeed

$$\left\|\vartheta^{\varepsilon}_{u,bl}\right\|_{L^2(\Omega)} \leq C \left\|\varphi\big(\cdot, \tfrac{\cdot}{\varepsilon}\big)\right\|_{L^2(\partial\Omega)} \leq C \left\|u^0\right\|_{H^2(\Omega)}.$$

The assumption $u^0 \in H^2(\Omega)$ being clearly fulfilled as $u^0 = T^0 f = \frac{1}{\lambda^0} f$, it is easy to adapt the proof of corollary 3.10 to conclude that:

$$\left\|u^{\varepsilon} - u^0\right\|_{L^2(\Omega)} \leq C\varepsilon \left\|u^0\right\|_{H^2(\Omega)} \leq C\varepsilon \left\|u^0\right\|_{H^{2+\omega}(\Omega)}. \tag{3.62}$$

Assume now that Ω is a polygonal domain satisfying either **(RAT)** or **(DIV)**. A uniform bound in ε of $\vartheta^{\varepsilon}_{u,bl}$ does not follow from the results of Avellaneda and Lin. The estimates of section 3.2 are sufficient to get the convergence of the boundary layer in section 3.3, up to the extraction of a subsequence under assumption **(RAT)**. Actually theorem 3.15 (resp. 3.16) implies that: there exists a sequence (ε_n) such that $\left\|\vartheta^{\varepsilon_n}_{u,bl}\right\|_{L^2(\Omega)} \leq C \left\|u^0\right\|_{H^{2+\omega}(\Omega)}$ (resp. for all $0 < \varepsilon$, $\left\|\vartheta^{\varepsilon}_{u,bl}\right\|_{L^2(\Omega)} \leq C \left\|u^0\right\|_{H^{2+\omega}(\Omega)}$). We conclude, as in the case when Ω is smooth, that (3.62) holds. In order to avoid extracting subsequences we now omit the case of polygonal domains under assumption **(RAT)**.

It remains to bound $\left\|u^0\right\|_{H^{2+\omega}(\Omega)}$ by $\left\|f\right\|_{L^2(\Omega)}$. Taking advantage of the equivalence of norms on the finite-dimensional space $E_{\lambda^0} \subset H^{2+\omega}(\Omega) \subset H^2(\Omega)$, there exists $0 < C$ such that for all $w \in E_{\lambda^0}$,

$$\left\|w\right\|_{H^{2+\omega}(\Omega)} \leq C \left\|w\right\|_{L^2(\Omega)}. \tag{3.63}$$

Therefore, combining (3.62) with (3.63), we get

$$\left\|T^{\varepsilon} f - T^0 f\right\|_{L^2(\Omega)} \leq C\varepsilon \left\|u^0\right\|_{H^{2+\omega}(\Omega)} \leq C\varepsilon \left\|f\right\|_{L^2(\Omega)}$$

which shows that

$$\left\|(T^{\varepsilon} - T^0)|_{E_{\lambda^0}}\right\| \leq C\varepsilon.$$

Our final goal is to prove (3.15), (3.16) and (3.17). The reasoning, in every case, follows the lines of [MV97]. Estimate (3.61) now sums up in:

$$\frac{1}{\lambda^0} - \frac{1}{m}\sum_{j=0}^{m-1} \frac{1}{\lambda^{\varepsilon,k+j}} = \frac{1}{m}\sum_{j=0}^{m-1} \left\langle (T^{\varepsilon} - T^0)v^{0,k+j}, v^{0,k+j}\right\rangle + O\big(\varepsilon^2\big). \tag{3.64}$$

Let us focus on $\frac{1}{m}\sum_{j=0}^{m-1} \left\langle (T^{\varepsilon} - T^0)v^{0,k+j}, v^{0,k+j}\right\rangle$ and work on $\left\langle (T^{\varepsilon} - T^0)v^{0,k+j}, v^{0,k+j}\right\rangle$, j being fixed in $\{0, \dots, m-1\}$. We call $u^{\varepsilon,k+j} := T^{\varepsilon} v^{0,k+j}$. This function solves (3.18). According to estimate (3.36) of theorem 3.13, as $v^{0,k+j} \in H^{2+\omega}(\Omega)$,

$$\left\|u^{\varepsilon,k+j}(x) - \tfrac{1}{\lambda^0} v^{0,k+j}(x) - \tfrac{\varepsilon}{\lambda^0}\chi^{\alpha}\left(\tfrac{x}{\varepsilon}\right)\partial_{x_\alpha} v^{0,k+j}(x) + \tfrac{\varepsilon}{\lambda^0}\vartheta^{\varepsilon}_{v,k+j,bl}(x)\right\|_{L^2(\Omega)} = O\big(\varepsilon^{1+\frac{\omega}{2}}\big),$$

where $\vartheta^{\varepsilon}_{v,k+j,bl}$ solves (3.37) with $v^{0,k+j}$ instead of u^0. Cauchy-Schwarz inequality implies

$$\begin{aligned}\left\langle (T^{\varepsilon} - T^0)v^{0,k+j}, v^{0,k+j}\right\rangle &= \int_{\Omega} \left(\frac{1}{\lambda^0} v^{0,k+j}(x) - u^{\varepsilon,k+j}(x)\right) v^{0,k+j}(x)dx \\ = \frac{\varepsilon}{\lambda^0}\int_{\Omega} \chi^{\alpha}\left(\frac{x}{\varepsilon}\right)\partial_{x_\alpha} v^{0,k+j}(x)\cdot v^{0,k+j}(x)dx &+ \frac{\varepsilon}{\lambda^0}\int_{\Omega}\vartheta^{\varepsilon}_{v,k+j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{1+\frac{\omega}{2}}\big).\end{aligned} \tag{3.65}$$

We intend to show that the term involving χ^α in (3.65) is of order $O\big(\varepsilon^{1+\frac{\omega}{2}}\big)$. In order to carry out integrations by parts, we introduce, for each $1 \le \alpha \le 2$, a periodic C^∞ solution $b^\alpha = b^\alpha(y) \in M_N(\mathbb{R})$ to

$$\Delta_y b^\alpha = \chi^\alpha,$$

the Fredholm property being satisfied as $\int_{\mathbb{T}^2} \chi^\alpha(y)dy = 0$. An integration by part gives

$$\begin{aligned}
&\frac{\varepsilon}{\lambda^0}\int_\Omega \chi^\alpha\left(\frac{x}{\varepsilon}\right)\partial_{x_\alpha} v^{0,k+j}(x)\cdot v^{0,k+j}(x)dx\\
&\qquad = \frac{\varepsilon}{\lambda^0}\int_\Omega \varepsilon^2\Delta\left(b^\alpha\left(\frac{x}{\varepsilon}\right)\right)\partial_{x_\alpha} v^{0,k+j}(x)\cdot v^{0,k+j}(x)dx\\
&\qquad = -\frac{\varepsilon^2}{\lambda^0}\int_\Omega \varepsilon\nabla\left(b^\alpha\left(\frac{x}{\varepsilon}\right)\right)\cdot\nabla\left(\partial_{x_\alpha} v^{0,k+j}(x)v^{0,k+j}(x)\right)dx\\
&\qquad \le C\varepsilon^2\left\|\varepsilon\nabla\left(b^\alpha\left(\tfrac{x}{\varepsilon}\right)\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\left(\partial_{x_\alpha} v^{0,k+j}(x)v^{0,k+j}(x)\right)\right\|_{L^1(\Omega)}\\
&\qquad \le C\varepsilon^2.
\end{aligned}$$

We deduce from (3.64) and (3.65) that

$$\begin{aligned}
\frac{1}{\lambda^0}-\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^{\varepsilon,k+j}} &= \frac{1}{m}\sum_{j=0}^{m-1}\left\langle (T^\varepsilon - T^0)v^{0,k+j}, v^{0,k+j}\right\rangle + O\big(\varepsilon^2\big)\\
&= \frac{1}{m}\sum_{j=0}^{m-1}\frac{\varepsilon}{\lambda^0}\int_\Omega \vartheta^\varepsilon_{v,k+j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{1+\frac{\omega}{2}}\big).
\end{aligned}$$

The results of section 3.3 now apply, in particular theorems 3.14, 3.15 (in this case up to the extraction of a subsequence) and 3.16, and yield that

$$\left\|\vartheta^\varepsilon_{v,k+j,bl} - \vartheta^*_{v,k+j,bl}\right\|_{L^2(\Omega)} = O\big(\varepsilon^\gamma\big)$$

for suitable exponents $0 < \gamma$:

1. for all $0 \le \gamma < \frac{1}{11}$, when Ω is a smooth uniformly convex domain;
2. for all $0 \le \gamma < \frac{\omega}{2}$ (resp. for $\gamma = \frac{1}{2}$), when Ω is a convex polygon satisfying either **(RAT)** or **(DIV)** and $E_{\lambda^0} \subset H^{2+\omega}(\Omega)$ (resp. $E_{\lambda^0} \subset H^3(\Omega)\cap C^2(\overline{\Omega})$).

Therefore,

$$\frac{1}{\lambda^0}-\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^{\varepsilon,k+j}} = \varepsilon\frac{1}{m\lambda^0}\sum_{j=0}^{m-1}\int_\Omega \vartheta^*_{v,k+j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{1+\gamma}\big),$$

with γ given above; from the convergence of the eigenvalues $\lambda^{\varepsilon,k+j}$ towards $\lambda^{0,k+j}$, we deduce that

$$\left[\frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{\lambda^{\varepsilon,k+j}}\right]^{-1} = \lambda^0 + \varepsilon\frac{\lambda^0}{m}\sum_{j=0}^{m-1}\int_\Omega \vartheta^*_{v,k+j,bl}(x)\cdot v^{0,k+j}(x)dx + O\big(\varepsilon^{1+\gamma}\big),$$

which achieves the proof of theorems 3.6 and 3.7.

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Chapter 4

Asymptotic analysis of boundary layer correctors in periodic homogenization

This chapter corresponds to the paper [Pra13] published in *SIAM Journal on Mathematical Analysis.*

Contents

Abstract This chapter is devoted to the asymptotic analysis of boundary layers in periodic homogenization. We investigate the behaviour of the boundary layer corrector, defined in the half-space $\Omega_{n,a} := \{y \cdot n - a > 0\}$, far away from the boundary and prove the convergence towards a constant vector field, the boundary layer tail. This problem happens to

depend strongly on the way the boundary $\partial\Omega_{n,a}$ intersects the underlying microstructure. Our study complements the previous results obtained on the one hand for $n \in \mathbb{RQ}^d$, and on the other hand for $n \notin \mathbb{RQ}^d$ satisfying a small divisors assumption. We tackle the case of arbitrary $n \notin \mathbb{RQ}^d$ using ergodicity of the boundary layer along $\partial\Omega_{n,a}$. Moreover, we get an asymptotic expansion of Poisson's kernel $P = P(y, \tilde{y})$, associated to the elliptic operator $-\nabla \cdot A(y)\nabla\cdot$ and $\Omega_{n,a}$, for $|y - \tilde{y}| \to \infty$. Finally, we show that, in general, convergence towards the boundary layer tail can be arbitrarily slow, which makes the general case very different from the rational or the small divisors one.

4.1 Introduction

In this chapter we investigate the behaviour of $v_{bl} = v_{bl}(y) \in \mathbb{R}^N$, solving the elliptic system

$$\left\{ \begin{array}{rll} -\nabla \cdot A(y)\nabla v_{bl} &= 0, & y \cdot n - a > 0 \\ v_{bl} &= v_0(y), & y \cdot n - a = 0 \end{array} \right. \tag{4.1}$$

with periodically oscillating coefficients and Dirichlet data, far away from the boundary of the half-space $\{y \cdot n - a > 0\} \subset \mathbb{R}^d$. Understanding these asymptotics is an important issue in the study of boundary layer correctors in periodic homogenization.

Throughout this chapter, $N \geq 1$, $d \geq 2$, $n \in \mathbb{S}^{d-1}$ is a given unit vector and $a \in \mathbb{R}$. The Dirichlet data $v_0 = v_0(y) \in \mathbb{R}^N$ is defined for $y \in \mathbb{R}^d$; so is the family of matrices $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ indexed by $1 \leq \alpha, \beta \leq d$. Small greek letters like α, β, γ, η usually denote integers belonging to $\{1, \dots\ d\}$, whereas i, j, k stand for integers in $\{1, \dots\ N\}$. Therefore, taking advantage of Einstein's convention:

$$\Big[\nabla \cdot A(y)\nabla v_{bl}\Big]_i = \partial_{y_\alpha}\Big(A^{\alpha\beta}_{ij}(y)\partial_{y_\beta} v_{bl,j}\Big).$$

The main assumptions on A are now

(A1) ellipticity there exists $\lambda > 0$ such that for every family $\xi = \xi^\alpha \in \mathbb{R}^N$ indexed by $1 \leq \alpha \leq d$, for all $y \in \mathbb{R}^d$,

$$\lambda \xi^\alpha \cdot \xi^\alpha \leq A^{\alpha\beta}(y)\xi^\alpha \cdot \xi^\beta \leq \lambda^{-1}\xi^\alpha \cdot \xi^\alpha;$$

(A2) periodicity A is 1-periodic i.e. for all $y \in \mathbb{R}^d$, for all $\xi \in \mathbb{Z}^d$,

$$A(y + \xi) = A(y);$$

(A3) regularity A is supposed to belong to $C^\infty\left(\mathbb{R}^d\right)$.

Moreover, we assume

(B1) periodicity v_0 is a 1-periodic function;

(B2) regularity v_0 is smooth.

Before focusing on system (4.1) itself, let us recall the main steps of the homogenization procedure leading to the study of v_{bl}. We carry out our analysis on the elliptic system with Dirichlet boundary conditions

$$\left\{ \begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon &= f, & x \in \Omega \\ u^\varepsilon &= 0, & x \in \partial\Omega \end{array} \right. \tag{4.2}$$

posed in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ endowed with a periodic microstructure. This system may model, for instance, heat conduction ($N = 1$), or linear elasticity ($d = 2$

or 3 and $N = d$) in a periodic composite medium. We are interested in the asymptotical behaviour of u^ε, when $\varepsilon \to 0$. This problem is of particular importance when it comes to the numerical analysis of (4.2). The oscillations at scale ε are too fast to be captured by classical methods. One therefore looks for an approximation of u^ε, taking into account the small scales, without solving them explicitely. To gain an insight into these numerical issues, we refer to [SV06]; for a general overview of the homogenization theory see [BLP78] or [CD99].

4.1.1 Some error estimates in homogenization

For a given source term $f \in L^2(\Omega)$, boundedness of Ω and ellipticity of A yield the existence and uniqueness of a weak solution u^ε in $H^1_0(\Omega)$ to (4.2). Moreover, we deduce from the a priori bound

$$\|u^\varepsilon\|_{H^1_0(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}\,,$$

where $C > 0$ is a constant independent of ε, that up to the extraction of a subsequence, u^ε converges weakly in $H^1_0(\Omega)$. A classical method to investigate this convergence is to have recourse to multiscale asymptotic expansions. We expand, at least formally, u^ε in powers of ε

$$u^\varepsilon \approx u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^2\left(x, \frac{x}{\varepsilon}\right) + \dots \tag{4.3}$$

assuming that $u^i = u^i(x,y) \in \mathbb{R}^N$ is 1-periodic with respect to the fast variable y. Plugging (4.3) into (4.2), identifying the powers of ε and solving the cascade of equations then gives:

1. that u^0 does not depend on y and that it solves the homogenized system

$$\left\{\begin{array}{rll} -\nabla\cdot A^0\nabla u^0 & = f, & x\in\Omega \\ u^0 & = 0, & x \in \partial\Omega \end{array}\right.$$

   where the constant homogenized tensor $A^0 = A^{0,\alpha\beta} \in M_N(\mathbb{R})$ is given by

$$A^{0,\alpha\beta} := \int_{\mathbb{T}^d} A^{\alpha\beta}(y)dy + \int_{\mathbb{T}^d} A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y)dy,$$

   and $\chi = \chi^\beta(y) \in M_N(\mathbb{R})$ is the family, indexed by $\beta \in \{1,\dots d\}$, of solutions to the cell problem

$$\left\{\begin{array}{rll} -\nabla_y\cdot A(y)\nabla_y\chi^\beta & = \partial_{y_\alpha}A^{\alpha\beta} & ,\ y \in \mathbb{T}^d \\ \int_{\mathbb{T}^d}\chi^\beta(y)dy & = 0 & \end{array}\right. ; \tag{4.4}$$

2. that for all $x \in \Omega$ and $y \in \mathbb{T}^d$, $u^1(x,y) = \chi^\alpha(y)\partial_{x_\alpha}u^0(x) + \bar{u}^1(x)$;
3. and that for all $x \in \Omega$ and $y \in \mathbb{T}^d$, $u^2(x,y) := \Gamma^{\alpha\beta}(y)\partial_{x_\alpha}\partial_{x_\beta}u^0(x) + \chi^\alpha(y)\partial_{x_\alpha}\bar{u}^1(x) + \bar{u}^2(x)$, where $\Gamma = \Gamma^{\alpha\beta}(y) \in M_N(\mathbb{R})$ is the family, indexed by $\alpha,\ \beta \in \{1,\dots d\}$, solving

$$\left\{\begin{array}{rll} -\nabla_y\cdot A(y)\nabla_y\Gamma^{\alpha\beta} & = B^{\alpha\beta} - \int_{\mathbb{T}^d}B^{\alpha\beta}(y)dy, & y\in\mathbb{T}^d \\ \int_{\mathbb{T}^d}\Gamma^{\alpha\beta}(y)dy & = 0 & \end{array}\right. ,$$

   and

$$B^{\alpha\beta}(y) := A^{\alpha\beta}(y) + A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y) + \partial_{y_\gamma}\big(A^{\gamma\alpha}(y)\chi^\beta(y)\big).$$

One can then carry out energy estimates on the error

$$r^{2,\varepsilon}_{bl} := u^\varepsilon(x) - u^0(x) - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \varepsilon^2\Gamma\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2 u^0(x)$$

solving the elliptic system

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla r^{2,\varepsilon} &= f^\varepsilon, \quad x \in \Omega \\ \qquad\qquad r^{2,\varepsilon} &= g^\varepsilon, \quad x \in \partial\Omega \end{cases}.$$

To bound $r^{2,\varepsilon}$ one needs some regularity on u^0, say $u^0 \in H^4(\Omega)$ (for refined estimates, involving lower regularity on u^0, see [MV97, Pra11]). Under this coarse assumption

$$\|f^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \left\|u^0\right\|_{H^4(\Omega)}$$

and

$$\|g^\varepsilon\|_{H^{\frac{1}{2}}(\Omega)} = \varepsilon \left\|\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0 - \varepsilon\Gamma\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2 u^0\right\|_{H^{\frac{1}{2}}(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\left\|u^0\right\|_{H^4(\Omega)}.$$

One can therefore show that $\|r^{2,\varepsilon}\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\|u^0\|_{H^4(\Omega)}$, which implies

$$\left\|u^\varepsilon - u^0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla u^0\right\|_{H^1(\Omega)} \leq \left\|r^{2,\varepsilon}\right\|_{H^1(\Omega)} + \varepsilon^2\left\|\Gamma\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2 u^0\right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\left\|u^0\right\|_{H^4(\Omega)} \tag{4.5}$$

and

$$\left\|u^\varepsilon - u^0\right\|_{L^2(\Omega)} = O\big(\varepsilon^{\frac{1}{2}}\big).$$

The latter estimate shows, that the zeroth-order term u^0 is a correct approximation of u^ε. Estimate (4.5) is however limited by the trace on $\partial\Omega$ of $u^1\left(\cdot, \frac{\cdot}{\varepsilon}\right)$. Actually, the periodicity assumption of the ansatz (4.3) with respect to the microscopic variable y, is not compatible with the homogeneous Dirichlet condition on the boundary. In order to force the Dirichlet condition one introduces a boundary layer term $u_{bl}^{1,\varepsilon} := u_{bl}^{1,\varepsilon}(x) \in \mathbb{R}^N$ at order ε^1 in the expansion (4.3). More precisely, $u_{bl}^{1,\varepsilon}$ solves

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u_{bl}^{1,\varepsilon} &= 0, \qquad\qquad\quad x \in \Omega \\ \qquad\qquad u_{bl}^{1,\varepsilon} &= -u^1\left(x, \frac{x}{\varepsilon}\right), \quad x \in \partial\Omega \end{cases} \tag{4.6}$$

and the corrected Ansatz is

$$u^\varepsilon \approx u^0(x) + \varepsilon\left[u^1\left(x, \frac{x}{\varepsilon}\right) + u_{bl}^{1,\varepsilon}\right] + \varepsilon^2 u^2\left(x, \frac{x}{\varepsilon}\right) + \ldots \tag{4.7}$$

Adding the boundary layer at first-order improves (4.5): if $u^0 \in H^4(\Omega)$,

$$\left\|u^\varepsilon - u^0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla u^0 - \varepsilon u_{bl}^{1,\varepsilon}\right\|_{H^1(\Omega)} \leq C\varepsilon\left\|u^0\right\|_{H^4(\Omega)}. \tag{4.8}$$

However, such a trick remains useless as long as one is not able to describe the asymptotics of $u_{bl}^{1,\varepsilon}$ when $\varepsilon \to 0$.

4.1.2 Homogenization of boundary layer systems

As for (4.2), the problem is to show that (4.6) can be in some sense homogenized. A few remarks are in order:

1. System (4.6) exhibits oscillations in the coefficients as well as on the boundary.
2. The oscillations along $\partial\Omega$ are not periodic in general. This can be expressed by the fact that the boundary breaks the periodic microstructure.

3. The a priori bound on $u_{bl}^{1,\varepsilon}$

$$\left\|u_{bl}^{1,\varepsilon}\right\|_{H^1(\Omega)} \leq C\left\|u^i\left(x,\frac{x}{\varepsilon}\right)\right\| = O\big(\varepsilon^{-\frac{1}{2}}\big)$$

does not provide a uniform bound in $H^1(\Omega)$.

These issues make the homogenization of the boundary layer system (4.6) far more difficult than the homogenization of (4.2). The results obtained in this direction are still partial. The first step towards the asymptotic behaviour of $u_{bl}^{1,\varepsilon}$ is to get a priori bounds uniform in ε. If $N=1$, the maximum principle furnishes a uniform bound in $L^\infty(\Omega)$ on $u_{bl}^{1,\varepsilon}$. A way of investigating (4.6) in the case when $N>1$ is to represent $u_{bl}^{1,\varepsilon}$ in terms of the oscillating Poisson kernel associated to $-\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla\cdot$ and Ω. In the series of papers [AL87a, AL89b, AL91], Avellaneda and Lin manage to get estimates, uniform in ε, on these Green and Poisson kernels, as well as expansions valid in the limit $\varepsilon\to 0$ (for recent progress in this direction, see [KLS12b]). One of the results of [AL87a] (see theorem 3) is the uniform bound $\left\|u_{bl}^{1,\varepsilon}\right\|_{L^p(\Omega)}\leq C$, for $1<p\leq\infty$, valid under the assumption that $\partial\Omega$ is at least $C^{1,\alpha}$, with $0<\alpha\leq 1$. Note that an $L^2(\Omega)$ bound on $u_{bl}^{1,\varepsilon}$ yields an $H^1(\omega)$ bound on the gradient, for $\omega\Subset\Omega$ compactly supported in Ω (see [AA99, GVM11]). The strong oscillations of $\nabla u_{bl}^{1,\varepsilon}$ are filtered out in the interior of the domain and concentrate near the boundary. Hence, the multiscale expansion is right up to the order 1 in ε in the interior:

$$\left\|u^\varepsilon - u^0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\cdot\nabla u^0\right\|_{H^1(\omega)} \leq C\varepsilon\left\|u^0\right\|_{H^4(\Omega)}. \tag{4.9}$$

However, to improve the asymptotics up to the boundary, one needs another approach. Namely, we seek after a 2-scale approximation of the boundary layer corrector

$$u_{bl}^{1,\varepsilon}\approx v\left(x,\frac{x}{\varepsilon}\right).$$

Take $x_0\in\partial\Omega$ a point at which there exists a tangent hyperplane directed by $n:=n(x_0)\in\mathbb{S}^{d-1}$ and assume that Ω is contained in $\{x\cdot n - x_0\cdot n\geq 0\}$. Then plugging formally v into (4.6) gives at order ε^{-2}

$$\left\{\begin{array}{rll} -\nabla_y\cdot A(y)\nabla_y v(x_0,y) & =0, & y\cdot n-\frac{n\cdot x_0}{\varepsilon}>0\\ v(x_0,y) & =-\chi(y)\cdot\nabla u^0(x_0), & y\cdot n-\frac{n\cdot x_0}{\varepsilon}=0\end{array}\right. . \tag{4.10}$$

The variable x_0 is nothing more than a parameter in this system. If Ω is convex, the boundary layer is approximated in the vicinity of each point x_0 of the boundary by a $v_{x_0}=v(x_0,\cdot)$ solving (4.10). This formal idea has been made rigorous for polygonal convex bounded domains $\Omega\subset\mathbb{R}^2$, for which only a finite number of correctors v^k has to be considered, one for each edge K^k. By linearity, v^k factors into

$$v^k(x,y)=v_{bl}^{k,\alpha}(y)\partial_{x_\alpha}u^0(x)$$

where for all $\alpha=1,\dots\ d$, $v_{bl}^{k,\alpha}=v_{bl}^{k,\alpha}(y)\in M_N(\mathbb{R})$ solves

$$\left\{\begin{array}{rll} -\nabla\cdot A(y)\nabla v_{bl}^{k,\alpha} & =0, & y\cdot n^k-a>0\\ v_{bl}^{k,\alpha} & =-\chi^\alpha(y), & y\cdot n^k-a=0\end{array}\right. \tag{4.11}$$

with $a:=\frac{x\cdot n}{\varepsilon}$. Dropping the exponents k and α, we end up with (4.1).

System (4.1) is linear elliptic in divergence form. The main source of difficulties one encounters is the lack of boundedness of the domain $\Omega_{n,a} := \{y \cdot n - a > 0\}$. This complicates the existence theory, but even more the study of the asymptotical behaviour. Moreover, the analysis of (4.1) depends, in a nontrivial manner, on the interaction between $\partial\Omega_{n,a}$ and the underlying lattice. So far, it has been carried out in two different contexts:

(RAT) the rational case, i.e. $n \in \mathbb{R}\mathbb{Q}^d$;

(DIV) the small divisors case, when there exists $C,\ \tau > 0$ such that for all $\xi \in \mathbb{Z}^d \setminus \{0\}$, for all $i = 1, \dots\ d-1$,

$$|n_i \cdot \xi| \geq C\,|\xi|^{-d-\tau} \tag{4.12}$$

where $(n_1, \dots\ n_{d-1}, n)$ forms an orthogonal basis of $\mathbb{R}^d$.

Assumption (4.12) means that the distance from every point, except 0, of the lattice $\mathbb{Z}^d$, to the line $\{\lambda n,\ \lambda \in \mathbb{R}\}$, is in some sense bounded from below. Note that this condition, albeit generic, in the sense that it is satisfied for almost every $n \in \mathbb{S}^{d-1}$ (for more quantitative results, see [GVM12]), is not fulfilled by every vector $n \notin \mathbb{R}\mathbb{Q}^d$.

Let us explain why the description of the asymptotics of v_{bl} far away from the boundary is a crucial step in the homogenization of (4.6). Roughly speaking, one proves in both contexts **(RAT)** and **(DIV)**, that there exits a smooth v_{bl} solving (4.1) and that this solution converges very fast, when $y \cdot n \to \infty$, towards a constant vector field $v_{bl}^{\infty} \in \mathbb{R}^N$, the boundary layer tail. For a polygonal domain Ω with edges satisfying for instance the small divisors assumption, we approximate $u_{bl}^{1,\varepsilon}$ by $\bar{u}^1$ solution of

$$\begin{cases} -\nabla \cdot A^0 \nabla \bar{u}^1 &= 0, \qquad\qquad x \in \Omega \\ \qquad\qquad \bar{u}^1 &= v_{bl}^{k,\infty} \cdot \nabla u^0, \quad x \in \partial\Omega \cap K^k \end{cases}. \tag{4.13}$$

Indeed,

$$\left\| u_{bl}^{1,\varepsilon} - \bar{u}^1 \right\|_{L^2(\Omega)} \leq \left\| u_{bl}^{1,\varepsilon} - \bar{u}^1 - \sum_k \left[v_{bl}^k - v_{bl}^{k,\infty} \right] \cdot \nabla u^0 \right\|_{L^2(\Omega)} + \sum_k \left\| \left[v_{bl}^k - v_{bl}^{k,\infty} \right] \cdot \nabla u^0 \right\|_{L^2(\Omega)}$$

and using the decay of the boundary layer correctors in the interior of Ω, one proves:

Theorem 4.1 ([Pra11], theorem 3.3)**.** *If $u^0 \in H^{2+\omega}(\Omega)$, with $\omega > 0$, then there exists $\kappa > 0$ such that*

$$\left\| u_{bl}^{1,\varepsilon} - \bar{u}^1 \right\|_{L^2(\Omega)} = O\left(\varepsilon^{\kappa}\right).$$

This shows that the oscillating Dirichlet data of (4.6) can be homogenized. The limit system (4.13) involves the tails of the boundary layer correctors. The corrector $\bar{u}^1$ has been used to implement numerical shemes in [SV06]. More recently, the obtention of an 4.1-like theorem for a smooth uniformly convex two-dimensional domain has been achieved in [GVM12]. Again, it relies on the approximation of the domain by polygonals with edges satisfying the small divisors condition, emphasizing the key role of systems (4.1).

We devote section 4.2 of this manuscript to a review of the previous results obtained on (4.1), with an emphasis on the techniques used in the small divisors case. Let us give an insight into these theorems:

(RAT) Among the rich litterature about this case, we refer to [Naz92], [AA99] (lemma 4.4), [MV97] (appendix 6) and the references therein. A precise statement is given in theorem 4.4. It consists of two parts:

existence there exists a variational solution $v_{bl} \in C^{\infty}\left(\overline{\Omega_{n,a}}\right)$ to (4.1) (unique if appropriate decay of ∇v_{bl} is prescribed);

convergence there exists a constant vector, called boundary layer tail, $v_{bl}^{a,\infty}$ depending on a such that $v_{bl}(y) - v_{bl}^{a,\infty}$ and its derivatives tend to 0 exponentially fast when $y \cdot n \to \infty$.

(DIV) This case was treated in the recent paper [GVM11] by Gérard-Varet and Masmoudi. A precise statement is given in theorem 4.5. It consists again of two parts:

existence there exists a variational solution $v_{bl} \in C^\infty\left(\overline{\Omega_{n,a}}\right)$ to (4.1) (unique if appropriate decay of ∇v_{bl} is prescribed);

convergence there exists a boundary layer tail v_{bl}^{∞} independent of a such that $v_{bl}(y) - v_{bl}^{\infty}$ and its derivatives tend to 0 when $y \cdot n \to \infty$, faster than any negative power of $y \cdot n$.

The main difference is that the boundary layer tail depends on a in the rational setting and not in the small divisors one, which implies that the homogenization theorem 4.1 is true up to the extraction of a subsequence ε_n in the former. In both cases, one can come down to some periodic framework to prove the existence of a variational solution. Fast convergence follows from a St-Venant estimate. We come back to these points in detail in section 4.2.

4.1.3 Outline of our results and strategy

Our goal is to analyse (4.1) in the case when $n \notin \mathbb{R}\mathbb{Q}^d$ does not meet the small divisors assumption (4.12). Again, one first wonders if the system has a solution. This question can be investigated by methods analogous to those of [GVM11]. Indeed, their well-posedness result does not rely on the small divisors assumption. The latter hypothesis is however essential to show the convergence towards the boundary layer tail in the work of Gérard-Varet and Masmoudi. We therefore have recourse to another approach based on an integral representation of the variational solution to (4.1) by the mean of Poisson's kernel $P = P(y, \tilde{y}) \in M_N(\mathbb{R})$ associated to $-\nabla \cdot A(y)\nabla\cdot$ and the domain $\Omega_{n,a}$. Basically,

$$v_{bl}(y) = \int_{\partial\Omega_{n,a}} P(y, \tilde{y}) v_0(\tilde{y}) d\tilde{y},$$

for every $y \in \Omega_{n,a}$. At first glance, if $n = e_d$ the d-th vector of the canonical basis of $\mathbb{R}^d$, if $a = 0$ and $y = (0, \varepsilon^{-1})$, $\varepsilon > 0$, then

$$\begin{aligned} v_{bl}\left(0, \frac{1}{\varepsilon}\right) &= \int_{\mathbb{R}^{d-1}} P\left(\frac{1}{\varepsilon}(0,1), (\tilde{y}', 0)\right) v_0(\tilde{y}', 0) d\tilde{y}' \\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{\varepsilon^{d-1}} P\left(\frac{1}{\varepsilon}(0,1), \frac{1}{\varepsilon}(\tilde{x}', 0)\right) v_0\left(\frac{1}{\varepsilon}(\tilde{x}', 0)\right) d\tilde{x}'. \end{aligned}$$

Examining the asymptotics of v_{bl} far away from the boundary, requires subsequently to understand the behaviour of the oscillating kernel $\frac{1}{\varepsilon^{d-1}} P\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right)$ when $\varepsilon \to 0$, or equivalently the asymptotical comportment of $P(y, \tilde{y})$, when $y \cdot n \to \infty$ and $\tilde{y} \in \partial\Omega_{n,0}$. This is done in section 4.5, relying on ideas and results of Avellaneda and Lin [AL87a, AL91]. We prove an expansion for P associated to the domain $\Omega_{n,0}$ for arbitrary $n \in \mathbb{S}^{d-1}$. *To put it in a nutshell, one demonstrates that there exists $\kappa > 0$ such that for all $y \in \Omega_{n,0}$, for all $\tilde{y} \in \partial\Omega_{n,0}$,*

$$|P(y, \tilde{y}) - P_{exp}(y, \tilde{y})| \leq \frac{C}{|y - \tilde{y}|^{d-1+\kappa}},$$

where $P_{exp} = P_{exp}(y, \tilde{y})$ is an explicit kernel, with ergodicity properties tangentially to the boundary. The precise statement of this key result is postponed to section 4.5: see theorem

4.18. We have an explicit expression for the corrector terms. Although this expansion is stated (and proved) for the domain $\Omega_{n,0}$ it extends to $\Omega_{n,a}$ by a simple translation. Studying the tail of v_{bl} now boils down to examining the limit when $y \cdot n \to \infty$ of

$$\int_{\partial\Omega_{n,a}} [P_{exp}(y,\tilde{y})]\, v_0(\tilde{y}) d\tilde{y} + \int_{\partial\Omega_{n,a}} R(y,\tilde{y}) v_0(\tilde{y}) d\tilde{y}$$

with rest $R = R(y,\tilde{y}) \in M_N(\mathbb{R})$ satisfying $|R(y,\tilde{y})| \leq \frac{C}{|y-\tilde{y}|^{d-1+\kappa}}$. The rest integral tends to 0. One takes advantage of the oscillations of v_0 on the boundary to show the convergence of the corrector integrals by the mean of an ergodic theorem. Doing so, we demonstrate the following theorem, which is the core of this paper:

Theorem 4.2. *Assume that $n \notin \mathbb{R}\mathbb{Q}^d$. Then,*

1. *there exists a unique solution $v_{bl} \in C^\infty\left(\overline{\Omega_{n,a}}\right) \cap L^\infty\left(\Omega_{n,a}\right)$ of (4.1) satisfying*

$$\|\nabla v_{bl}\|_{L^\infty(\{y\cdot n - t>0\})} \stackrel{t\to\infty}{\longrightarrow} 0, \tag{4.14a}$$

$$\int_a^\infty \|\partial_n v_{bl}\|^2_{L^\infty(\{y\cdot n-t=0\})}\, dt < \infty, \tag{4.14b}$$

2. *and a boundary layer tail $v_{bl}^\infty \in \mathbb{R}^N$, independent of a, such that*

$$v_{bl}(y) \stackrel{y\cdot n\to\infty}{\longrightarrow} v_{bl}^\infty,$$

locally uniformly in the tangential variable.

Furthermore, one has an explicit expression for v_{bl}^∞ (see (4.68)).

The use of the ergodic theorem to prove the convergence does not yield any rate. One wonders therefore how fast convergence of v_{bl} towards v_{bl}^∞ is. A partial answer is given by the next theorem, whose proof is addressed in section 4.7:

Theorem 4.3. *Assume that $d=2$, $N=1$, $A=\mathrm{I}_2$ and that $n \notin \mathbb{R}\mathbb{Q}^2$ does not satisfy (4.12). Then for every $l>0$, for all $R>0$, there exists a smooth function v_0 and a sequence $(t_M)_M \in]a,\infty[^{\mathbb{N}}$ such that:*

1. *$(t_M)_M$ is strictly increasing and tends to ∞;*
2. *the unique solution v_{bl} of (4.1), in the variational sense, converges towards v_{bl}^∞ as $y\cdot n\to\infty$ and for all $M\in\mathbb{N}$, for all $y' \in \partial\Omega_{n,0}\cap B(0,R)$,*

$$|v_{bl}(y'+t_M n) - v_{bl}^\infty| \geq t_M^{-l}.$$

This result means that convergence can be as slow as we wish in some sense: for fixed $n \notin \mathbb{Q}\mathbb{R}^d$ which does not satisfy the small divisors assumption, for every power function $t \mapsto t^{-l}$ with $l>0$ there exists a 1-periodic $v_0 = v_0(y) \in \mathbb{R}$, such that the solution v_{bl} of

$$\begin{cases} -\Delta v_{bl} &= 0, \qquad y\cdot n - a > 0 \\ \quad v_{bl} &= v_0(y), \quad y\cdot n - a = 0 \end{cases}$$

converges slower to its tail than $(y\cdot n)^{-l}$ when $y\cdot n\to\infty$. The main obstruction preventing v_{bl} from converging faster in general lies indeed in the fact that the distance between a given point $\xi \in \mathbb{Z}^d\setminus\{0\}$ and the line $\{\lambda n,\ \lambda\in\mathbb{R}\}$ is not bounded from below. This is in big contrast with the small divisors case, where (4.12) asserts the existence of a lower bound. It underlines the strong dependence of the boundary layer on the interaction between $\partial\Omega_{n,a}$ and $\mathbb{Z}^d$.

4.1.4 Organization of the paper

In this paper we address the proofs of theorems 4.18 (expansion of Poisson's kernel), 4.2 (convergence of the boundary layer corrector) and 4.3 (slow convergence). Section 4.2 is devoted to a review of the rational and small divisors settings. We insist on the existence proof in the non rational case and underline the role of the small divisors assumption in the asymptotic analysis. In section 4.3, some essential properties and estimates on Green and Poisson kernels associated to elliptic operators with periodic coefficients are recalled. We prove a uniqueness theorem for (4.1), which makes it possible to rely on Poisson's formula to represent the variational solution of (4.1) and explain to what extent the description of the large scale asymptotics of Poisson's kernel P boils down to an homogenization problem. The latter is the focus of sections 4.4, where we study a dual problem, and 4.5 in which an asymptotic expansion of Poisson's kernel, for $y \cdot n \to \infty$, is established (see theorem 4.18). This work is the central step in our proof of theorem 4.2. The last step is done in section 4.6, where we prove on the one hand the convergence towards the boundary layer tail using theorem 4.18 and on the other hand the independence of v_{bl} of a. Section 4.7 is concerned with the proof of theorem 4.3.

4.1.5 Notations

The following notations apply for the rest of the paper. The half-space $\{y \cdot n - a > 0\}$ is denoted by $\Omega_{n,a}$ and in the sequel, for any $y \in \overline{\Omega_{n,a}}$ and $r > 0$, $D(y,r) := B(y,r) \cap \Omega_{n,a}$ and $\Gamma(y,r) := B(y,r) \cap \partial\Omega_{n,a}$. The case when $a = 0$ is frequently used: $\Omega_{n,0} =: \Omega_n$. For a function $H = H(y,\tilde{y})$ depending on y, $\tilde{y} \in \mathbb{R}^d$ we may use the following notation: $\partial_{1,\alpha} H := \partial_{y_\alpha} H$ (resp. $\partial_{2,\alpha} H := \partial_{\tilde{y}_\alpha} H$) for all $\alpha \in \{1, \dots d\}$. The vectors $e_1, \dots e_d$ represent the canonical basis of $\mathbb{R}^d$. The matrix $\mathrm{M} \in M_d(\mathbb{R})$ is an orthogonal matrix such that $\mathrm{M}e_d = n$; $\mathrm{N} \in M_{d,d-1}(\mathbb{R})$ is the matrix of the $d-1$ first columns of M. All along these lines, $C > 0$ denotes an arbitrary constant independent of ε.

4.2 Review of the rational and small divisors settings

In this section we make a short review of the mathematical results on system (4.1). We concentrate on giving precise statements for the theorems announced in the introduction (cf. section 4.1.2) and insist much more on the small divisors case, whose existence part is useful to us. To determine the role of n, we make the change of variable $z = \mathrm{M}^T y$ in (4.1). One obtains that $v(z) := v_{bl}(\mathrm{M}z)$ solves

$$\left\{ \begin{array}{rll} -\nabla \cdot B(\mathrm{M}z)\nabla v & = 0, & z_d > a \\ v & = v_0(\mathrm{M}z), & z_d = a \end{array} \right. . \tag{4.15}$$

The family of matrices $B = B^{\alpha\beta}(y) \in M_N(\mathbb{R})$, indexed by $1 \leq \alpha, \beta \leq d$, satisfies for every $i,\ j \in \{1, \dots N\}$,

$$B_{ij} = \mathrm{M}A_{ij}\mathrm{M}^T.$$

and is hence 1-periodic, elliptic and smooth.

4.2.1 Rational case

This case has been studied by many authors (see [Naz92, AA99, MV97]). The assumption $n \in \mathbb{Q}\mathbb{R}^d$ simplifies much the existence and convergence proof. Indeed, as n has rational coordinates, one can choose an orthogonal matrix M with columns in $\mathbb{RQ}^d$ sending e_d on n. Subsequently, there exists a d-uplet of periods $(L_1, \dots L_d)$ such that

$$z \mapsto B(\mathrm{M}z) \qquad \text{and} \qquad z \mapsto v_0(\mathrm{M}z)$$

are $(L_1, \dots L_d)$-periodic functions. Without loss of generality, let us assume that $L_1 = \dots = L_d = 1$. We can also fix $a = 0$ for the moment. We come back later to this hypothesis. Then, lifting the Dirichlet data v_0 using $\varphi \in C_c^\infty(\mathbb{R})$, compactly supported in $[-1, 1]$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, yields that $\tilde{v} := v - \varphi(z_d)v_0(\mathrm{M}(z', 0))$ solves

$$\begin{cases} -\nabla \cdot B(\mathrm{M}z)\nabla\tilde{v} &= \nabla \cdot B(\mathrm{M}z)\nabla\left(\varphi(z_d)v_0(\mathrm{M}(z', 0))\right), \quad z_d > 0 \\ \tilde{v} &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; z_d = 0 \end{cases}. \tag{4.16}$$

An appropriate framework to write a variational formulation for (4.16) is the completion of the space $C_c^\infty\left(\mathbb{T}^{d-1} \times \mathbb{R}_+\right)$ with respect to the $L^2\left(\mathbb{T}^{d-1} \times \mathbb{R}_+\right)$ of the gradient. The fact that the source term $\nabla \cdot B(\mathrm{M}z)\nabla\left(\varphi(z_d)v_0(\mathrm{M}(z', 0))\right)$ in (4.16) is compactly supported in the normal direction, allows to resort to a Poincaré inequality. We can prove the existence of a weak solution $\tilde{v}$ to (4.16) by the mean of the Lax-Milgram lemma.

The existence part asserts that $\nabla v \in L^2\left(\mathbb{T}^{d-1} \times \mathbb{R}_+\right)$. We aim at showing that ∇v actually decays faster. The key observation is that for any $k \in \mathbb{N}$, $v^{(k)} := v(z', z_d + k)$ defined for $z_d > 0$ solves

$$\begin{cases} -\nabla \cdot B(\mathrm{M}z)\nabla v^{(k)} &= 0, \qquad\qquad z_d > 0 \\ v^{(k)} &= v_0(\mathrm{M}z), \quad z_d = 0 \end{cases}.$$

Moreover, one has for all $k \in \mathbb{N}$

$$\|\nabla v\|_{L^2\left(\mathbb{T}^d \times]k, \infty[\right)} = \left\|\nabla v^{(k)}\right\|_{L^2\left(\mathbb{T}^d \times]0, \infty[\right)}.$$

Hence, the St-Venant estimate

$$\|\nabla v\|_{L^2\left(\mathbb{T}^{d-1} \times]k+1, \infty[\right)} \leq C\left[\|\nabla v\|_{L^2\left(\mathbb{T}^{d-1} \times]k, \infty[\right)} - \|\nabla v\|_{L^2\left(\mathbb{T}^{d-1} \times]k+1, \infty[\right)}\right], \tag{4.17}$$

yields the exponential decay of $\|\nabla v\|_{L^2\left(\mathbb{T}^{d-1} \times]t, \infty[\right)}$, when $t \to \infty$. The existence of the boundary layer tail $v_{bl}^{0,\infty} \in \mathbb{R}^N$ comes from the fact that $\int_{\mathbb{T}^{d-1} \times]k, k+1[} v(t)dt$ is a Cauchy sequence. The decay of higher order derivatives follows from elliptic regularity (see [ADN64]). This implies, through Sobolev injections, pointwise convergence of $v(z', z_d)$ towards $v_{bl}^{0,\infty}$ at an exponential rate.

Let us come back to the assumption $a = 0$. Let a be any real number, v the associated solution of (4.15) and denote by $\bar{a} = a - [a]$ its fractional part. Then, $v^a = v(\cdot + ae_d)$

$$\begin{cases} -\nabla \cdot B\left(\mathrm{M}(z + \bar{a}e_d\right)\nabla v^a &= 0, \qquad\qquad\qquad\quad z_d > 0 \\ v^a &= v_0\left(\mathrm{M}(z + \bar{a}e_d)\right), \quad z_d = 0 \end{cases}.$$

If we now assume that $N = 1$, $d = 2$ and that $B(\mathrm{M}\cdot)$ is the constant identity matrix I_2, we can carry out Fourier analysis to compute the tail. We get that the tail $v_{bl}^{a,\infty}$ of v and v^a is equal to

$$v_{bl}^{a,\infty} = \int_0^1 v_0\left(\mathrm{M}(z', \bar{a})\right) dz',$$

which depends on $\bar{a}$. Thus, the boundary layer tail is not independent of a in this rational setting. We now summarize the preceding results (forgetting about the assumptions $L_1 = \dots = L_d = 1$ and $a = 0$) in the:

Theorem 4.4 (lemma 2 p. 84 in [Naz92], lemma 4.4 in [AA99], appendix 6 in [MV97]). *Assume that* $n \in \mathbb{R}\mathbb{Q}^d$. *Then,*

1. *there exists a solution* $v \in C^\infty\left(\mathbb{R}^{d-1} \times [a, \infty[\right)$ *of* (4.15), *unique under the condition that for all* $R > 0$
$$\nabla v \in L^2\Big((-R,R)^{d-1} \times]a, \infty[\Big),$$

2. *and* $\kappa > 0$, $v_{bl}^{a,\infty} \in \mathbb{R}^N$ *depending on* a *such that for all* $\alpha \in \mathbb{N}^d$, *for all* $z = (z', z_d) \in \mathbb{R}^{d-1} \times]a, \infty[$,
$$e^{\kappa(z_d - a)} \left|\partial_z^\alpha \left(v(z) - v_{bl}^{a,\infty}\right)\right| \leq C_\alpha. \tag{4.18}$$

We conclude this section by a remark. It is concerned with the exponent $\kappa = \kappa_n$ in (4.18). In [NR00], the case of a two-dimensional layered media $\Omega_{n,0}$ is considered. It is shown that exponential convergence of ∇v_{bl} to 0, uniform in $n \in \mathbb{R}\mathbb{Q}^d$, cannot be expected. Indeed, depending on n, κ_n can be arbitrarily small.

4.2.2 Small divisors case

All the results we recall here stem from the original article [GVM11] by Gérard-Varet and Masmoudi. The periodic framework in the rational case makes the study of (4.15) more simple for the main reason that it yields compactness in the horizontal direction. One can thus rely on Poincaré-Wirtinger inequalities, which are essential to prove the St-Venant estimate (4.17). In the non rational case, i.e. when $n \notin \mathbb{Q}\mathbb{R}^d$, one also attempts to recover a periodic setting. Note that for all $z = (z', a) \in \mathbb{R}^{d-1} \times \{a\}$, $v_0\left(\mathrm{M}(z', a)\right) = v_0\left(\mathrm{N}z' + \mathrm{M}(0, a)\right)$, where $\mathrm{N} \in M_{d,d-1}(\mathbb{R})$ is the matrix of the $d-1$ first columns of M. Therefore, $v_0(\mathrm{M}\cdot)$ is quasiperiodic in z', i.e. there exists $V_0 = V_0(\theta, t) \in \mathbb{R}^N$ defined for $\theta \in \mathbb{T}^d$ and $t \in \mathbb{R}$ such that for all $z = (z', a) \in \mathbb{R}^{d-1} \times \{a\}$, $v_0\left(\mathrm{M}(z', a)\right) = V_0(\mathrm{N}z', a)$. Similarly, there exists $\mathcal{B} = \mathcal{B}(\theta, t)$ such that for all $z = (z', z_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, $B\left(\mathrm{M}(z', z_d)\right) = \mathcal{B}(\mathrm{N}z', z_d)$. Hence, one looks for a quasiperiodic solution of (4.15); for details concerning quasiperiodic functions the reader is referred to [JKO94], section 7.1. Assume that there exists $V = V(\theta, t) \in \mathbb{R}^N$ defined for $\theta \in \mathbb{T}^d$ and $t > a$ such that for all $z = (z', z_d) \in \mathbb{R}^{d-1} \times]a, \infty[$,
$$v(z', z_d) = v_{bl}\left(\mathrm{M}(z', z_d)\right) = V(\mathrm{N}z', z_d).$$

Now, if V solves
$$\left\{\begin{array}{rll} -\begin{pmatrix}\mathrm{N}^T\nabla_\theta \\ \partial_t\end{pmatrix} \cdot \mathcal{B}(\theta,t) \begin{pmatrix}\mathrm{N}^T\nabla_\theta \\ \partial_t\end{pmatrix} V & = 0, & t > a \\ V & = V_0, & t = a \end{array}\right., \tag{4.19}$$

then $v = v(z', z_d) = V(\mathrm{N}z', z_d)$ solves (4.15).

We focus now on system (4.19). We examine successively the existence of a solution and the convergence towards a constant vector field. By making the change of unknown function, we have gained compactness in the horizontal direction, but lost ellipticity of the differential operator. System (4.19) is however well-posed (see [GVM11] proposition 2): *there exists a unique variational solution* $V \in C^\infty\left(\mathbb{T}^d \times]a, \infty[\right)$ *such that for all* $l \in \mathbb{N}$, *for all* $\alpha \in \mathbb{N}^d$,

$$\left\| \partial_t^l \partial_\theta^\alpha \begin{pmatrix}\mathrm{N}^T\nabla_\theta \\ \partial_t\end{pmatrix} \right\|_{L^2\left(\mathbb{T}^d \times]a,\infty[\right)} = \int_{\mathbb{T}^d}\int_a^\infty \left(\left|N^T\nabla_\theta\partial_t^l\partial_\theta^\alpha V\right|^2 + \left|\partial_t^{l+1}\partial_\theta^\alpha V\right|^2\right) d\theta dt < \infty. \tag{4.20}$$

To compensate for the lack of ellipticity, one uses a regularization method. We add a corrective term $-\iota\Delta_\theta$ to the operator. The regularized system is

$$\begin{cases} -\begin{pmatrix} \mathrm{N}^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta,t) \begin{pmatrix} \mathrm{N}^T\nabla_\theta \\ \partial_t \end{pmatrix} V^\iota - \iota\Delta_\theta V^\iota &= 0, \quad t > a \\ V^\iota &= V_0, \quad t = a \end{cases}. \tag{4.21}$$

The regularizing term yields ellipticity of the differential operator. Lifting the boundary data V_0 into a function compactly supported in the direction normal to the boundary, makes it possible, as in the rational case, to prove the existence of a variational solution V^ι to (4.21). Carrying out energy estimates on (4.21) gives (4.20)-like a priori bounds on V^ι, uniform in ι. Thanks to this compactness on the sequence $(V^\iota)_{\iota>0}$, one can extract a subsequence, which converges weakly to V variational solution of (4.19) when $\iota \to 0$. Uniqueness follows from the a priori bounds. An important point is that the justification of this well-posedness result does not resort to the small divisors assumption.

We come to the asymptotical analysis far away from the boundary. Sobolev injections and the a priori bounds (4.20) yield pointwise convergence to 0 of $\mathrm{N}^T\nabla_\theta V(\theta,t)$, $\partial_t V(\theta,t)$ and their derivatives, when $t \to \infty$, uniformly in $\theta \in \mathbb{T}^d$. Let us investigate this convergence in more detail. Without loss of generality, we assume temporarily that $a = 0$. The idea is to establish a St-Venant estimate on $\begin{pmatrix} \mathrm{N}^T\nabla_\theta \\ \partial_t \end{pmatrix} V$. In the same spirit as in the rational case, we look at the quantity

$$K(T) := \int_{\mathbb{T}^d}\int_T^\infty \left(\left|\mathrm{N}^T\nabla_\theta V\right|^2 + \left|\partial_t V\right|^2\right) d\theta dt$$

defined for $T > 0$. One proves that

$$K(T) \leq C\left(-K'(T)\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d}\left|\widetilde{V}(\theta,T)\right|^2 d\theta\right)^{\frac{1}{2}}, \tag{4.22}$$

with $\widetilde{V}(\theta,T) := V(\theta,T) - \int_{\mathbb{T}^d} V(\cdot,T)$. The stake is to control the second factor in the r.h.s. of (4.22). The key argument of [GVM11] is a type of Poincaré-Wirtinger inequality implied by the small divisors assumption. *Assume that $n \notin \mathbb{R}\mathbb{Q}^d$ satisfies* (4.12). *Then, for all $1 < p < \infty$, there exists $C_p > 0$ such that*

$$\int_{\mathbb{T}^d}\left|\widetilde{V}(\theta,T)\right|^2 d\theta \leq C_p \left(\int_{\mathbb{T}^d}\left|\mathrm{N}^T\nabla_\theta V(\theta,T)\right|^2 d\theta\right)^{\frac{1}{p}}. \tag{4.23}$$

Note that the r.h.s. does not involve the $L^2(\mathbb{T}^d)$ norm of $\nabla_\theta V$, but the norm of the incomplete gradient $\mathrm{N}^T\nabla_\theta V$. We give a sketch of the reasoning leading to (4.23). The proof relies on Fourier analysis in the tangential variable. In order to emphasize the role of the small divisors assumption, let us give an equivalent statement of (4.12): taking for $n_1, \dots n_{d-1}$ the $d-1$ first columns $\mathrm{N} \in M_{d,d-1}(\mathbb{R})$ of M, (4.12) becomes

$$\left|\mathrm{N}^T\xi\right| \geq C\,|\xi|^{-d-\tau}. \tag{4.24}$$

Let $T > 0$ be fixed, $1 < p < \infty$ and p' its conjugate Hölder exponent: $\frac{1}{p} + \frac{1}{p'} = 1$. By Parceval's equality we can compute the $L^2\left(\mathbb{T}^d\right)$ norm on the l.h.s. of (4.23) and then apply Hölder's inequality

$$\int_{\mathbb{T}^d}\left|\widetilde{V}(\theta,T)\right|^2 d\theta \leq \left(\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}\left|\widehat{\widetilde{V}}(\xi,T)\right|^2 \frac{1}{|\xi|^{2(d+\tau)}}\right)^{\frac{1}{p}} \left(\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}\left|\widehat{\widetilde{V}}(\xi,T)\right|^2 |\xi|^{2\frac{d+\tau}{p-1}}\right)^{\frac{1}{p'}}.$$

For the first factor in the r.h.s., which represents the norm of $\widetilde{V}(\cdot,T)$ in an homogeneous Sobolev space with negative exponent, we have, using (4.24):

$$\left\|\mathrm{N}^T\nabla_\theta\widetilde{V}(\theta,T)\right\|^2_{L^2\left(\mathbb{T}^d\right)} = \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}\left|2i\pi\widehat{\widetilde{V}}(\xi,T)\right|^2\left|\mathrm{N}^T\xi\right|^2 \geq C\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}\left|\widehat{\widetilde{V}}(\xi,T)\right|^2\frac{1}{|\xi|^{2(d+\tau)}}.$$

This bound and (4.20) imply in particular a control on $\left\|\widetilde{V}(\cdot,T)\right\|_{H^s\left(\mathbb{T}^d\right)}$ for all $s\geq 0$. Hence, one bounds the second factor by a constant independent of T.

The bound (4.24), in combination with (4.22), yields for every $1<p<\infty$ the differential inequality on K, $K(T)\leq C_p\left(-K'(T)\right)^{\frac{p+1}{2p}}$ from which we get the decay: for all $T>0$, $0\leq K(T)\leq C_pT^{\frac{p+1}{1-p}}$. We do the same on higher order derivatives considering for $s\in\mathbb{N}$,

$$K_s(T) := \sum_{0\leq|\alpha|+l\leq s}\int_{\mathbb{T}^d}\int_T^\infty\left(\left|\mathrm{N}^T\partial_t^l\partial_\theta^\alpha\nabla_\theta V\right|^2+\left|\partial_t^{l+1}\partial_\theta^\alpha V\right|^2\right)d\theta dt.$$

The existence of a boundary layer tail follows from a procedure similar to the rational case. For arbitrary $a\in\mathbb{R}$, one can prove (see proposition 5 in [GVM11]) that the boundary layer tail does not depend on a. This fact is a characteristic of the non rational setting: Gérard-Varet and Masmoudi prove it under the small divisors assumption. Note that we manage to free ourselves from this assumption in section 4.6.2. To summarize (see [GVM11] propositions 4 and 5): *if $n\notin\mathbb{R}\mathbb{Q}^d$ satisfies the small divisors assumption* (4.24)*, then there exists a constant vector field $v_{bl}^\infty\in\mathbb{R}^N$, independent of a, such that for all $m\in\mathbb{N}$, for all $\alpha\in\mathbb{N}^d$, $l\in\mathbb{N}$, for all $t>a$,*

$$\sup_{\theta\in\mathbb{T}^d}\left|(1+|t-a|^m)\partial_\theta^\alpha\partial_t^l\left(V(\theta,t)-v_{bl}^\infty\right)\right|\leq C_{m,\alpha}.$$

We end this section by a translation of the existence and convergence statement for V into a statement for v solution of (4.15). We recall that for all $z=(z',z_d)\in\mathbb{R}^{d-1}\times]a,\infty[$, $v(z',z_d)=V(\mathrm{N}z',z_d)$. A solution v of (4.15) (or v_{bl} of (4.1)) built like this is called a *variational solution.*

Theorem 4.5 ([GVM11]). *Assume that $n\notin\mathbb{R}\mathbb{Q}^d$.*

1. *Then, there exists a solution $v\in C^\infty\left(\mathbb{R}^{d-1}\times[a,\infty[\right)$ of* (4.15) *satisfying*

$$\|\nabla v\|_{L^\infty\left(\mathbb{R}^{d-1}\times]t,\infty[\right)}\overset{t\to\infty}{\longrightarrow}0 \tag{4.25a}$$

$$\int_a^\infty\|\partial_{z_d}v(\cdot,t)\|^2_{L^\infty\left(\mathbb{R}^{d-1}\right)}dt<\infty. \tag{4.25b}$$

2. *Moreover, if n satisfies the small divisors assumption* (4.12)*, then for all $m\in\mathbb{N}$, for all $\alpha\in\mathbb{N}^d$, for all $z=(z',z_d)\in\mathbb{R}^{d-1}\times]a,\infty[$,*

$$(1+|z_d-a|^m)\left|\partial_z^\alpha\left(v(z)-v_{bl}^\infty\right)\right|\leq C_{\alpha,m}.$$

Estimates (4.25a) and (4.25b) rely on (4.20), but not on the small divisors assumption. Indeed, for every $k\in\mathbb{N}$,

$$\left\|\mathrm{N}^T\nabla_\theta V\right\|_{H^k\left(\mathbb{T}^d\times]t,\infty[\right)}+\|\partial_t V\|_{H^k\left(\mathbb{T}^d\times]t,\infty[\right)}\overset{t\to\infty}{\longrightarrow}0$$

which together with Sobolev's injection theorem yields (4.25a) for k sufficiently big. As

$$\|\partial_t V\|_{H^k(\mathbb{T}^d\times]a,\infty[)} \geq \left\| \|\partial_t V\|_{H^k_\theta(\mathbb{T}^d)} \right\|_{L^2_t(]0,\infty[)},$$

a similar reasoning leads to (4.25b). The a priori bounds (4.20), contain further information: for $k \geq 0$ sufficiently large and $k' \geq 1$,

$$\begin{aligned}
\left\| \begin{pmatrix} \mathrm{N}^T\nabla_\theta \\ \partial_t \end{pmatrix} V \right\|_{H^{k'+k-1}(\mathbb{T}^d\times]a,\infty[)} &\geq \left\| \left\| \begin{pmatrix} \nabla_\theta \\ \partial_t \end{pmatrix}^{k'-1} \begin{pmatrix} \mathrm{N}^T\nabla_\theta \\ \partial_t \end{pmatrix} V \right\|_{H^k_\theta(\mathbb{T}^d)} \right\|_{L^2_t(]a,\infty[)} \\
\geq C \left\| \left\| \begin{pmatrix} \nabla_\theta \\ \partial_t \end{pmatrix}^{k'-1} \begin{pmatrix} \mathrm{N}^T\nabla_\theta \\ \partial_t \end{pmatrix} V \right\|_{L^\infty_\theta(\mathbb{T}^d)} \right\|_{L^2_t(]a,\infty[)} &\geq C \left\| \left\| \nabla^{k'} v \right\|_{L^\infty_{z'}(\mathbb{R}^{d-1})} \right\|_{L^2_{z_d}(]a,\infty[)}. \qquad (4.26)
\end{aligned}$$

We resort to the latter in section 4.5.2 with $k' = 1$ or 2.

4.3 Poisson's integral representation of the variational solution

One can always assume that $a = 0$. We do so for the rest of the paper (except in section 4.6.2), even if it means to work with $v^a_{bl} := v_{bl}(\cdot + an)$ instead of v_{bl} solving (4.1). The main advantage of this assumption is that the domain $\Omega_n = \{y \cdot n > 0\}$ is invariant under the scaling $y \mapsto \varepsilon y$ for $\varepsilon > 0$. The purpose of this section (see in particular section 4.3.2) is to prove uniqueness for the solution to (4.1) in a larger class than that of [GVM11]. This result (theorem 4.10 below) allows to represent the variational solution by the mean of Poisson's kernel.

4.3.1 Estimates on Green's and Poisson's kernels

Let $G = G(y, \tilde{y}) \in M_N(\mathbb{R})$ solving, for all $\tilde{y} \in \Omega_n$, the elliptic system

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y G(y, \tilde{y}) &= \delta(y - \tilde{y})\,\mathrm{I}_N, \quad y \cdot n > 0 \\ \qquad\qquad G(y, \tilde{y}) &= 0, \qquad\qquad\quad\; y \cdot n = 0 \end{cases}, \qquad (4.27)$$

with source term $\delta(\cdot - \tilde{y})$ ($\delta(\cdot)$ is the Dirac distribution). The matrix G is called the Green kernel associated to the operator $-\nabla \cdot A(y)\nabla \cdot$ and to the domain Ω_n. Its existence in Ω_n is proved for $d \geq 3$ in [HK07] (see theorem 4.1), and for $d = 2$ in [DK09] (see theorem 2.12). Similarly, $G^* = G^*(y, \tilde{y}) \in M_N(\mathbb{R})$ is the Green kernel associated to the transposed operator $-\nabla \cdot A^*(y)\nabla \cdot$ and the domain Ω_n, A^* being the transpose of the tensor A defined for all $\alpha,\ \beta \in \{1, \ldots d\}$ and for all $i,\ j \in \{1, \ldots N\}$ by

$$(A^*)^{\alpha\beta}_{ij} = A^{\beta\alpha}_{ji} = \left(A^{\beta\alpha}\right)^T_{ij}.$$

From the smoothness of A and local regularity estimates, it follows that

$$G \in C^\infty\left(\Omega_n \times \Omega_n \setminus \{y = \tilde{y}\}\right).$$

Moreover, the following symmetry property holds: for all $y,\ \tilde{y} \in \Omega_n$,

$$G^T(y, \tilde{y}) = G^*(\tilde{y}, y).$$

Using Green's kernel, one defines another function, the Poisson kernel $P = P(x, \tilde{x}) \in M_N(\mathbb{R})$: for all $i,\ j \in \{1, \dots N\}$, for all $y \in \Omega_n$, $\tilde{y} \in \partial\Omega_n$,

$$P_{ij}(y,\tilde{y}) := -A^{\alpha\beta}_{kj}(\tilde{y})\partial_{\tilde{y}_\alpha} G_{ik}(y,\tilde{y}) n_\beta \tag{4.28a}$$

$$= -\left(A^*\right)^{\beta\alpha}_{jk}(\tilde{y})\partial_{\tilde{y}_\alpha} G^*_{ki}(\tilde{y},y) n_\beta \tag{4.28b}$$

$$= -\left[A^{*,\beta\alpha}(\tilde{y})\partial_{\tilde{y}_\alpha} G^*(\tilde{y},y) n_\beta\right]_{ji} \tag{4.28c}$$

$$= -\left[A^*(\tilde{y})\nabla_{\tilde{y}} G^*(\tilde{y},y)\cdot n\right]^T_{ij}. \tag{4.28d}$$

The kernel $P^* = P^*(y,\tilde{y}) \in M_N(\mathbb{R})$ is defined in the same way as G^*.

If one considers Green's kernel $G^0 = G^0(y,\tilde{y}) \in M_N(\mathbb{R})$ associated to the constant coefficients elliptic operator $-\nabla \cdot A^0\nabla\cdot$ and to the domain Ω_n, there exists $C > 0$ (see [ADN64] section 6 for a statement and [SW77] V.4.2 (Satz 3) for a proof) such that for all $\Lambda_1,\ \Lambda_2 \in \mathbb{N}^d$, for all $y,\ \tilde{y} \in \Omega_n$, $y \neq \tilde{y}$,

$$\left|G^0(y,\tilde{y})\right| \le C\left(|\ln|y-\tilde{y}|| + 1\right), \quad \text{if } d = 2, \tag{4.29a}$$

$$\left|G^0(y,\tilde{y})\right| \le \frac{C}{|y-\tilde{y}|^{d-2}}, \quad \text{if } d \ge 3, \tag{4.29b}$$

$$\left|\partial_y^{\Lambda_1}\partial_{\tilde{y}}^{\Lambda_2} G^0(y,\tilde{y})\right| \le \frac{C}{|y-\tilde{y}|^{d-2+|\Lambda_1|+|\Lambda_2|}}, \quad \text{if } |\Lambda_1| + |\Lambda_2| \ge 1. \tag{4.29c}$$

One has similar estimates on Poisson's kernel $P^0 = P^0(y,\tilde{y}) \in M_N(\mathbb{R})$ and its derivatives. Such bounds on Green's kernel and its derivatives are not known for operators with non constant coefficients. Let us temporarily assume that neither **(A2)** nor **(A3)** hold. We know then from [KK10] (see theorem 3.3) that under certain smoothness assumptions on the coefficients there exists $R_{max} \in]0,\infty]$ and $C > 0$, such that for all $y,\ \tilde{y} \in \Omega_n$, $|y-\tilde{y}| < R_{max}$ implies

$$|G(y,\tilde{y})| \le \frac{C}{|y-\tilde{y}|^{d-2}}. \tag{4.30}$$

If $R_{max} < \infty$, estimate (4.30) does not provide a control of $G(y,\tilde{y})$ for y and $\tilde{y}$ far from each other. However, under the periodicity assumption **(A2)**, we have the following improved global bounds:

Lemma 4.6 ([AL87a] theorem 13 and lemma 21, [GVM12] lemma 6). *There exists $C > 0$, such that*

1. *for all $y,\ \tilde{y} \in \Omega_n$, $y \neq \tilde{y}$,*

$$|G(y,\tilde{y})| \le C\left(|\ln|y-\tilde{y}|| + 1\right), \quad \textit{if } d = 2, \tag{4.31a}$$

$$|G(y,\tilde{y})| \le \frac{C}{|y-\tilde{y}|^{d-2}}, \quad \textit{if } d \ge 3, \tag{4.31b}$$

and for all $d \ge 2$,

$$|G(y,\tilde{y})| \le C\frac{(y\cdot n)(\tilde{y}\cdot n)}{|y-\tilde{y}|^d}, \tag{4.31c}$$

$$|\nabla_y G(y,\tilde{y})| \le \frac{C}{|y-\tilde{y}|^{d-1}}, \tag{4.31d}$$

$$|\nabla_y G(y,\tilde{y})| \le C\left(\frac{\tilde{y}\cdot n}{|y-\tilde{y}|^d} + \frac{(y\cdot n)(\tilde{y}\cdot n)}{|y-\tilde{y}|^{d+1}}\right), \tag{4.31e}$$

$$|\nabla_y\nabla_{\tilde{y}} G(y,\tilde{y})| \le \frac{C}{|y-\tilde{y}|^d}; \tag{4.31f}$$

2. *for all $y \in \Omega_n$, for all $\tilde{y} \in \partial\Omega_n$, for all $d \geq 2$,*

$$|P(y,\tilde{y})| \leq C\frac{y \cdot n}{|y-\tilde{y}|^d}, \tag{4.32a}$$

$$|\nabla_y P(y,\tilde{y})| \leq C\left(\frac{1}{|y-\tilde{y}|^d} + \frac{y \cdot n}{|y-\tilde{y}|^{d+1}}\right). \tag{4.32b}$$

By continuity of G and its derivatives, up to the boundary, the estimates on Green's kernel naturally extend to $y \neq \tilde{y} \in \overline{\Omega_n}$. These bounds are of constant use in our work: for a proof in the half-space see [GVM12] (appendix A). The key argument is due to Avellaneda and Lin. The large scale description of G boils down to an homogenization problem. In the paper [AL87a], the authors face this homogenization problem under the periodicity assumption **(A2)** and manage to get uniform local estimates on $u^\varepsilon = u^\varepsilon(x)$ satisfying

$$\left\{\begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon & = f, & x \in D(0,1) \\ u^\varepsilon & = g, & x \in \Gamma(0,1) \end{array}\right. . \tag{4.33}$$

We recall the two local estimates useful in the sequel:

Theorem 4.7 (local boundary estimate, [AL87a] lemma 12)**.** *Let μ, δ be positive real numbers, $\mu < 1$, $F \in L^{d+\delta}(D(0,1))$ and $g \in C^{0,1}(\Gamma(0,1))$. Assume that $f = \nabla \cdot F$. There exists $C > 0$ such that for all $\varepsilon > 0$, if u^ε belongs to $L^2(D(0,1))$ and satisfies* (4.33)*, then*

$$\|u^\varepsilon\|_{C^{0,\mu}\left(D\left(0,\frac{1}{2}\right)\right)} \leq C\left[\|u^\varepsilon\|_{L^2(D(0,1))} + \|F\|_{L^{d+\delta}(D(0,1))} + \|g\|_{C^{0,1}(\Gamma(0,1))}\right]. \tag{4.34}$$

Theorem 4.8 (local boundary gradient estimate, [AL87a] lemma 20)**.** *Let ν, δ be positive real numbers, $\nu < 1$, $f \in L^{d+\delta}(D(0,1))$ and $g \in C^{1,\nu}(\Gamma(0,1))$. There exists $C > 0$ such that for all $\varepsilon > 0$, if u^ε belongs to $L^\infty(D(0,1))$ and satisfies* (4.33)*, then*

$$\|\nabla u^\varepsilon\|_{L^\infty\left(D\left(0,\frac{1}{2}\right)\right)} \leq C\left[\|u^\varepsilon\|_{L^\infty(D(0,1))} + \|f\|_{L^{d+\delta}(D(0,1))} + \|g\|_{C^{1,\nu}(\Gamma(0,1))}\right]. \tag{4.35}$$

4.3.2 Integral representation formula

We aim at showing that the solution v_{bl} of (4.1) can be represented in terms of an integral formula involving Poisson's kernel. One of the main difficulties arises from the fact that the Dirichlet data v_0 of (4.1) is not compactly supported in the boundary. The function $v = v(z) \in \mathbb{R}^N$, such that for all $z \in \mathbb{R}^{d-1} \times \mathbb{R}^*_+$, $v(z) := v_{bl}(\mathrm{M}z)$, solves (4.15). Let us now recall what we consider as a solution of (4.15). In fact, we have two kinds of solutions. The first corresponds to the variational construction in [GVM11]: see section 4.2.2, in particular theorem 4.5. Poisson's kernel $P = P(y,\tilde{y}) \in M_N(\mathbb{R})$ associated to the domain Ω_n and the operator $-\nabla \cdot A(y)\nabla\cdot$ makes it possible to define a second function $w_{bl} = w_{bl}(y) \in \mathbb{R}^N$ solving (4.1). For all $y \in \Omega_n$, we define $w_{bl}(y)$ by

$$w_{bl}(y) = \int_{\tilde{y}\cdot n=0} P(y,\tilde{y})v_0(\tilde{y})d\tilde{y}$$

and w by for all $z \in \mathbb{R}^{d-1}\times]0,\infty[$,

$$w(z) := w_{bl}(\mathrm{M}z) = \int_{\partial\Omega_n} P(\mathrm{M}z,\tilde{y})v_0(\tilde{y})d\tilde{y} = \int_{\tilde{z}_d=0} P(\mathrm{M}z,\mathrm{M}\tilde{z})v_0(\mathrm{M}\tilde{z})d\tilde{z}.$$

Proposition 4.9. *The function w belongs to $C^\infty\left(\mathbb{R}^{d-1}\times[0,\infty[\right)$ and satisfies* (4.15). *Furthermore,*

$$\|\nabla_z w\|_{L^\infty\left(\mathbb{R}^{d-1}\times]t,\infty[\right)} \overset{t\to\infty}{\longrightarrow} 0, \tag{4.36a}$$

$$\partial_{z_d} w \in L^2_{z_d}\left(\mathbb{R}^*_+, L^\infty_{z'}\left(\mathbb{R}^{d-1}\right)\right). \tag{4.36b}$$

Let us give a sketch of how to deduce these properties from the bound on $\nabla_y P$ given in (4.32). For all $z\in\mathbb{R}^{d-1}\times]0,\infty[$,

$$\begin{aligned}
|\nabla_z w(z)| &\leq \int_{\tilde z_d=0} \left|\mathrm{M}^T\nabla_y P(\mathrm{M}z,\mathrm{M}\tilde z)v_0(\mathrm{M}\tilde z)\right| d\tilde z\\
&\leq C\int_{\tilde z_d=0}\left(\frac{1}{|z-\tilde z|^d}+\frac{z_d}{|z-\tilde z|^{d+1}}\right)d\tilde z\\
&\leq C\int_{\mathbb{R}^{d-1}}\left(\frac{1}{\left(z_d^2+|z'-\tilde z'|^2\right)^{\frac{d}{2}}}+\frac{z_d}{\left(z_d^2+|z'-\tilde z'|^2\right)^{\frac{d+1}{2}}}\right)d\tilde z'\\
&\leq C\frac{1}{z_d}\int_{\mathbb{R}^{d-1}}\left(\frac{1}{\left(1+|z'-\tilde z'|^2\right)^{\frac{d}{2}}}+\frac{1}{\left(1+|z'-\tilde z'|^2\right)^{\frac{d+1}{2}}}\right)d\tilde z',
\end{aligned}$$

from which one gets (4.36a) as well as (4.36b).

Our goal is now to show that the variational solution and the Poisson solution coincide.

Theorem 4.10. *We have $v = w$.*

We work on the difference $u := v - w$, which is a C^∞ solution of

$$\left\{\begin{array}{rll} -\nabla\cdot B(\mathrm{M}z)\nabla u &= 0, & z_d>0\\ u &= 0, & z_d = 0\end{array}\right. .$$

We intend to show that u is zero proceeding by duality. Let $f\in C^\infty_c\left(\mathbb{R}^{d-1}\times]0,\infty[\right)$. We take $U = U(z)\in\mathbb{R}^N$ the solution to the elliptic boundary value problem

$$\left\{\begin{array}{rll} -\nabla\cdot B^*(\mathrm{M}z)\nabla U &= f, & z_d>0\\ U &= 0, & z_d = 0\end{array}\right.$$

given by Green's representation formula: for all $z\in\mathbb{R}^{d-1}\times]0,\infty[$,

$$U(z) = \int_{\tilde z_d>0} G^*(\mathrm{M}z,\mathrm{M}\tilde z)f(\tilde z)d\tilde z.$$

The bound (4.31c) on G^* and the bound (4.31d) on its first-order derivative imply:

Lemma 4.11. *There is $C>0$ such that for sufficiently large $z\in\mathbb{R}^{d-1}\times]0,\infty[$, i.e. far enough from the support of f,*

$$|U(z)| \leq C\frac{z_d}{\left(z_d^2+|z'|^2\right)^{\frac{d}{2}}}, \tag{4.37a}$$

$$|\nabla U(z)| \leq C\frac{1}{\left(z_d^2+|z'|^2\right)^{\frac{d}{2}}}. \tag{4.37b}$$

Moreover, thanks to (4.25b) and (4.36b), one manages to estimate u in L^∞: there is $C>0$, such that for all $z\in\mathbb{R}^{d-1}\times]0,\infty[$,

$$|u(z)| \leq \int_0^{z_d} |\partial_{z_d} u(z',t)|\,dt \leq z_d^{\frac{1}{2}} \left(\int_0^{z_d} |\partial_{z_d} u(z',t)|^2\,dt\right)^{\frac{1}{2}} \leq C z_d^{\frac{1}{2}}. \tag{4.38}$$

We now carry out integrations by parts:

$$\begin{aligned}
\int_{z_d>0} u(z)f(z)dz &= -\int_{z_d>0} u(z)\nabla\cdot B^*(\mathrm{M}z)\nabla U(z)dz\\
&= \int_{z_d>0} B^{\alpha\beta}(\mathrm{M}z)\partial_{z_\beta}u(z)\partial_{z_\alpha}U(z)dz\\
&= -\int_{z_d>0} \partial_{z_\alpha}\left(B^{\alpha\beta}(\mathrm{M}z)\partial_{z_\beta}u(z)\right)U(z)dz = 0.
\end{aligned}$$

To fully justify the preceding equalities we have to integrate by parts on the bounded domain $[-R,R]^{d-1}\times[0,R]$ and prove that the boundary integrals vanish in the limit $R\to\infty$. We actually show that

$$\int_{\partial\left([-R,R]^{d-1}\times[0,R]\right)} u(z)\left(B^*(\mathrm{M}z)\nabla U(z)\right)\cdot n(z)dz \overset{R\to\infty}{\longrightarrow} 0, \tag{4.39a}$$

$$\int_{\partial\left([-R,R]^{d-1}\times[0,R]\right)} \left[B(\mathrm{M}z)\nabla u(z)\right]\cdot n(z)U(z)\,dz \overset{R\to\infty}{\longrightarrow} 0. \tag{4.39b}$$

On the one hand, (4.38) together with (4.37b) yields

$$\begin{aligned}
&\left|\int_{\partial\left([-R,R]^{d-1}\times[0,R]\right)} u(z)\left(B^*(\mathrm{M}z)\nabla U(z)\right)\cdot n(z)dz\right|\\
&\qquad\leq C\int_{\partial\left([-R,R]^{d-1}\times[0,R]\right)\setminus\left([-R,R]^{d-1}\times\{0\}\right)} z_d^{\frac{1}{2}}\frac{1}{|z|^d}dz \leq C\frac{1}{R^{d-\frac{1}{2}}}R^{d-1} = \frac{C}{R^{\frac{1}{2}}},
\end{aligned}$$

which gives (4.39a). On the other hand, it follows from (4.25a), (4.36a) and (4.37a) that

$$\begin{aligned}
&\left|\int_{[-R,R]^{d-1}\times\{R\}} \left[B(\mathrm{M}z)\nabla u(z)\right]\cdot n(z)U(z)\,dz\right|\\
&\leq C\int_{[-R,R]^{d-1}} |\nabla u(z',R)|\,|U(z',R)|\,dz'\\
&\leq C\int_{[-R,R]^{d-1}} \|\nabla u\|_{L^\infty\left(\mathbb{R}^{d-1}\times]R,\infty[\right)}\frac{R}{(R^2+|z'|^2)^{\frac{d}{2}}}dz'\\
&\leq C\|\nabla u\|_{L^\infty\left(\mathbb{R}^{d-1}\times]R,\infty[\right)} \overset{R\to\infty}{\longrightarrow} 0
\end{aligned}$$

and that

$$\begin{aligned}
&\left|\int_{\{R\}\times[-R,R]^{d-2}\times[0,R]} \left[B(\mathrm{M}z)\nabla u(z)\right]\cdot n(z)U(z)\,dz\right|\\
&\leq C\int_{[-R,R]^{d-2}\times[0,R]} \|\nabla u\|_{L^\infty\left(\mathbb{R}^{d-1}\times]z_d,\infty[\right)}\frac{z_d}{(R^2+|z_2|^2+\dots\ |z_d|^2)^{\frac{d}{2}}}dz_2\dots\ dz_d\\
&\leq C\int_{[-1,1]^{d-2}\times[0,1]} \|\nabla u\|_{L^\infty\left(\mathbb{R}^{d-1}\times]Rz_d,\infty[\right)}\frac{z_d}{(1+|z_2|^2+\dots\ |z_d|^2)^{\frac{d}{2}}}dz_2\dots\ dz_d
\end{aligned}$$

tend to 0 by dominated convergence, which yields (4.39b) and terminates the proof of theorem 4.10.

For practical convenience, we have argued that $v = w$. Yet, theorem 4.10 proves that the variational solution v_{bl} of (4.1) equals the Poisson solution w_{bl}. This allows to work, for the rest of the paper, with the solution v_{bl} of (4.1) satisfying (4.14a) and (4.14b), no matter whether this solution is constructed variationally or via Poisson's kernel. Thanks to the bound (4.32a), v_{bl} is seen to be bounded on Ω_n.

4.3.3 Homogenization problem

Studying the tail of v_{bl}, i.e. the limit when $y \cdot n \to \infty$ of $v_{bl}(y)$, boils down to describing the asymptotics of $P(y, \tilde{y})$ for y far away from the boundary $\partial\Omega_n$. One of the main focus of our paper is thus to expand $P(y, \tilde{y})$ for $|y - \tilde{y}| \gg 1$, where $y \in \Omega_n$ and $\tilde{y} \in \partial\Omega_n$, i.e. to describe the large scale asymptotics of P. Let $y \in \Omega_n$, $\tilde{y} \in \partial\Omega_n$ and

$$\varepsilon := \frac{1}{|y - \tilde{y}|}.$$

If $|y - \tilde{y}| \gg 1$, then $\varepsilon \ll 1$ is a small parameter. Introducing the rescaled variables

$$x := \varepsilon y \in \Omega_n \quad \text{and} \quad \tilde{x} := \varepsilon \tilde{y} \in \Omega_n$$

yields $|x - \tilde{x}| = 1$. Such a scaling transforms our initial question of the large scale asymptotic description of P into the study of $\frac{1}{\varepsilon^{d-1}} P\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right)$ for $\varepsilon \to 0$ and $|x - \tilde{x}|$ close to 1.

Lemma 4.12. *Let $\varepsilon > 0$ and call G^ε (resp. P^ε) the Green (resp. Poisson) kernel associated to the operator $L^\varepsilon = -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla \cdot$ and the domain Ω_n.*
Then,

1. *for all $x, \ \tilde{x} \in \Omega_n$,*
$$G^\varepsilon(x, \tilde{x}) = \frac{1}{\varepsilon^{d-2}} G\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right); \tag{4.40}$$
2. *for all $x \in \Omega_n$, $\tilde{x} \in \partial\Omega_n$,*
$$P^\varepsilon(x, \tilde{x}) = \frac{1}{\varepsilon^{d-1}} P\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right). \tag{4.41}$$

Proof. This lemma follows easily from Green's integral representation formula. Let $f \in C^\infty_c(\Omega_n)$, $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^N$ the solution of

$$\begin{cases} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon &= \frac{1}{\varepsilon^2} f\left(\frac{x}{\varepsilon}\right), \quad x \cdot n > 0 \\ u^\varepsilon &= 0, \qquad\qquad\quad x \cdot n = 0 \end{cases}$$

and $u = u(y) \in \mathbb{R}^N$ the solution of

$$\begin{cases} -\nabla \cdot A(y) \nabla u &= f, \quad y \cdot n > 0 \\ u &= 0, \quad y \cdot n = 0 \end{cases}.$$

For all $x \in \Omega_n$,

$$\begin{aligned} \int_{\Omega_n} G^\varepsilon(x, \tilde{x}) \frac{1}{\varepsilon^2} f\left(\frac{\tilde{x}}{\varepsilon}\right) d\tilde{x} &= u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right) \\ &= \int_{\Omega_n} G\left(\frac{x}{\varepsilon}, \tilde{y}\right) f(\tilde{y}) d\tilde{y} = \int_{\Omega_n} \frac{1}{\varepsilon^d} G\left(\frac{x}{\varepsilon}, \frac{\tilde{x}}{\varepsilon}\right) f\left(\frac{\tilde{x}}{\varepsilon}\right) d\tilde{x}, \end{aligned}$$

which yields (4.40); (4.41) easily follows from analogous ideas. □

It immediately follows from the definition of G^ε, that for all $\tilde{x} \in \Omega_n$, $G^\varepsilon(\cdot, \tilde{x})$ solves the system

$$\left\{ \begin{array}{rll} -\nabla_x \cdot A\left(\frac{x}{\varepsilon}\right) \nabla_x G^\varepsilon(x, \tilde{x}) & = \delta(x-\tilde{x})\, \mathrm{I}_N, & x \cdot n > 0 \\ G^\varepsilon(x, \tilde{x}) & = 0, & x \cdot n = 0 \end{array} \right. . \tag{4.42}$$

The Poisson kernel P^ε satisfies: for all $i,\ j \in \{1, \dots N\}$, for all $x \in \Omega_n$, $\tilde{x} \in \partial\Omega_n$,

$$P^\varepsilon_{ij}(x, \tilde{x}) = -A^{\alpha\beta}_{kj}\left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{\tilde{x}_\alpha} G^\varepsilon_{ik}(x, \tilde{x}) n_\beta \tag{4.43a}$$

$$= -\left(A^*\right)^{\beta\alpha}_{jk}\left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{\tilde{x}_\alpha} G^{*,\varepsilon}_{ki}(\tilde{x}, x) n_\beta \tag{4.43b}$$

$$= -\left[A^{*,\beta\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{\tilde{x}_\alpha} G^{*,\varepsilon}(\tilde{x}, x) n_\beta\right]_{ji} \tag{4.43c}$$

$$= -\left[A^*\left(\frac{\tilde{x}}{\varepsilon}\right) \nabla_{\tilde{x}} G^{*,\varepsilon}(\tilde{x}, x) \cdot n\right]^T_{ij}. \tag{4.43d}$$

where $G^{*,\varepsilon}$ is the Green kernel associated to the operator $L^{*,\varepsilon} = -\nabla \cdot A^*\left(\frac{x}{\varepsilon}\right) \nabla \cdot$ and the domain Ω_n.

The estimates (4.31) and (4.32) can be rescaled. In particular, there exists $C > 0$ such that for all $\varepsilon > 0$, for all $x,\ \tilde{x} \in \overline{\Omega_n}$, $\tilde{x} \neq x$, for all $d \geq 2$,

$$|G^\varepsilon(x, \tilde{x})| \leq C\left(|\ln|x-\tilde{x}|| + 1\right), \quad \text{if } d = 2 \quad \text{(see [AL87a] theorem 13 (ii))}, \tag{4.44a}$$

$$|G^\varepsilon(x, \tilde{x})| \leq \frac{C}{|x-\tilde{x}|^{d-2}}, \quad \text{if } d \geq 3, \tag{4.44b}$$

$$|G^\varepsilon(x, \tilde{x})| \leq C\frac{(x \cdot n)(\tilde{x} \cdot n)}{|x-\tilde{x}|^d}, \tag{4.44c}$$

$$|\nabla_{\tilde{x}} G^\varepsilon(x, \tilde{x})| \leq \frac{C}{|x-\tilde{x}|^{d-1}}, \tag{4.44d}$$

and for all $x \in \Omega_n$, $\tilde{x} \in \partial\Omega_n$,

$$|P^\varepsilon(x, \tilde{x})| \leq C\frac{x \cdot n}{|x-\tilde{x}|^d}, \tag{4.44e}$$

$$|\nabla_x P^\varepsilon(x, \tilde{x})| \leq C\left(\frac{1}{|x-\tilde{x}|^d} + \frac{x \cdot n}{|x-\tilde{x}|^{d+1}}\right). \tag{4.44f}$$

According to lemma 4.12, we now deal with highly oscillating kernels G^ε and P^ε, instead of looking at G and P. Hence the asymptotic description of G (resp. P) at large scales, is replaced by an homogenization problem on G^ε (resp. P^ε). This fact, which has been stressed by Avellaneda and Lin (see [AL91] corollary on p. 903), is the cornerstone of our method.

4.4 Homogenization in the half-space

This section is concerned with the asymptotics, for small ε, of $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^N$ solving

$$\left\{ \begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon & = f, & x \cdot n > 0 \\ u^\varepsilon & = 0, & x \cdot n = 0 \end{array} \right. . \tag{4.45}$$

The study of this dual homogenization problem is preparatory to the expansion of the Green and Poisson kernels in the next section. In the introduction, we defined the interior and boundary layer correctors to u^ε up to the order ε and reviewed some error estimates in the case of a bounded domain Ω. The purpose of this section is to show similar estimates, yet in the unbounded domain Ω_n.

Let $f \in C_c^\infty\left(\overline{\Omega_n}\right)$; note that the support of f may intersect the boundary. We define $u^0 = u^0(x) \in \mathbb{R}^N$ as the solution of

$$\left\{\begin{array}{rll} -\nabla \cdot A^0 \nabla u^0 & = f, & x \cdot n > 0 \\ u^0 & = 0, & x \cdot n = 0 \end{array}\right., \tag{4.46}$$

$u^1 = u^1(x,y) \in \mathbb{R}^N$ by $u^1(x,y) := \chi^\alpha(y)\partial_{x_\alpha} u^0(x)$, for all $x \in \Omega_n$ and $y \in \mathbb{T}^d$, and $u_{bl}^{1,\varepsilon} = u_{bl}^{1,\varepsilon}(x) \in \mathbb{R}^N$ as the Poisson solution to

$$\left\{\begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u_{bl}^{1,\varepsilon} & = 0, & x \cdot n > 0 \\ u_{bl}^{1,\varepsilon} & = -u^1\left(x, \frac{x}{\varepsilon}\right) = -\chi^\alpha\left(\frac{x}{\varepsilon}\right)\partial_{x_\alpha} u^0(x), & x \cdot n = 0 \end{array}\right..$$

The variational solution u^ε (resp. u^0) coincides with the solution given by Green's integral formula. Besides, u^ε, u^0, as well as $u_{bl}^{1,\varepsilon}$ belong to $C^\infty\left(\overline{\Omega_n}\right)$, thanks to the smoothness of the boundary $\partial\Omega_n$, using local regularity estimates from [ADN64]. The rest of this section is devoted to the proof of the following proposition:

Proposition 4.13. *Let $r_{bl}^{1,\varepsilon} := u^\varepsilon(x) - u^0(x) - \varepsilon\chi^\alpha\left(\frac{x}{\varepsilon}\right)\partial_{x_\alpha} u^0 - \varepsilon u_{bl}^{1,\varepsilon}(x)$, and $\delta > 0$. Then, there exists a constant $C > 0$, such that for all $\varepsilon > 0$,*

$$\left\| r_{bl}^{1,\varepsilon} \right\|_{L^\infty(\Omega_n)} \leq C\varepsilon \|f\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)}, \tag{4.47a}$$

$$\left\| u_{bl}^{1,\varepsilon} \right\|_{L^\infty(\Omega_n)} \leq C \|f\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)}, \tag{4.47b}$$

$$\left\| u^\varepsilon - u^0 \right\|_{L^\infty(\Omega_n)} \leq C\varepsilon \|f\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)}. \tag{4.47c}$$

The proof of (4.47a) relies on estimates in L^∞ of $r_{bl}^{1,\varepsilon}$ solution of the following elliptic system

$$\left\{\begin{array}{rll} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla r_{bl}^{1,\varepsilon} & = f^\varepsilon, & x \cdot n > 0 \\ r_{bl}^{1,\varepsilon} & = 0, & x \cdot n = 0 \end{array}\right.,$$

where $f^\varepsilon = f^\varepsilon(x) \in \mathbb{R}^N$ is given by

$$f^\varepsilon := f + \nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^0 + \varepsilon \nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla\left(\chi^\alpha\left(\frac{x}{\varepsilon}\right)\partial_{x_\alpha} u^0(x)\right).$$

The idea is to use the integral representation provided by Green's formula in order to bound $r_{bl}^{1,\varepsilon}$. However, as such, f^ε does not seem to be of order ε. Let us thus work on f^ε to make its structure more explicit. Expanding f^ε, we get for all $x \in \Omega_n$

$$\begin{aligned} f^\varepsilon(x) = & \frac{1}{\varepsilon}\left[\nabla_y \cdot A(y)\nabla_x u^0 + \nabla_y \cdot A(y)\nabla_y u^1\right]\left(x, \frac{x}{\varepsilon}\right) \\ & + \left[f + \nabla_x \cdot A(y)\nabla_x u^0 + \nabla_x \cdot A(y)\nabla_y u^1 + \nabla_y \cdot A(y)\nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right) \\ & + \varepsilon\left[\nabla_x \cdot A(y)\nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right). \end{aligned} \tag{4.48}$$

We aim to remove the terms of order ε^{-1} and ε^0 in (4.48). The term of order ε^{-1} easily cancels thanks to (4.4):

$$\nabla_y \cdot A(y)\nabla_x u^0 + \nabla_y \cdot A(y)\nabla_y u^1 = \left[\partial_{y_\alpha}\left(A^{\alpha\gamma}(y)\right) + \partial_{y_\alpha}\left(A^{\alpha\beta}(y)\partial_{y_\beta}\chi^\gamma(y)\right)\right]\partial_{x_\gamma} u^0 = 0.$$

For the term of order ε^0 in the r.h.s. of (4.48)

$$\begin{aligned}\left[f + \nabla_x \cdot A(y)\nabla_x u^0 + \nabla_x \cdot A(y)\nabla_y u^1 + \nabla_y \cdot A(y)\nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right) \\ = \left[f + \nabla_x \cdot v + \nabla_y \cdot A(y)\nabla_x u^1\right]\left(x, \frac{x}{\varepsilon}\right),\end{aligned} \tag{4.49}$$

where $v = v(x,y) := A(y)\nabla_x u^0 + A(y)\nabla_y u^1 = A(y)\left[\nabla_x u^0 + \nabla_y u^1\right]$, we notice that

$$\fint_{\mathbb{T}^d} \left(v - A^0\nabla u^0\right) = 0.$$

As $v - A^0\nabla u^0$ factors into $\Phi\nabla u^0$, with

$$\Phi = \Phi(y) := A(y)\left(\mathrm{I} + \nabla_y \chi(y)\right) - \int_{\mathbb{T}^d} A(\tilde{y})\left(\mathrm{I} + \nabla_y \chi(\tilde{y})\right) d\tilde{y} \in \mathbb{R}^{N^2\times d^2},$$

one can take advantage of the classical lemma:

Lemma 4.14. *There exists a smooth function $\Psi = \Psi(y) \in \mathbb{R}^{N^2\times d^3}$ such that for all $y \in \mathbb{T}^d$,*

$$\Phi(y) = \nabla_y \cdot \Psi(y).$$

Via lemma 4.14, (4.49) becomes

$$\begin{aligned} f + \nabla_x \cdot v + \nabla_y \cdot A(y)\nabla_x u^1 &= \nabla_x \cdot \left[v - A^0\nabla_x u^0\right] + \nabla_y \cdot A(y)\nabla_x u^1 \\ &= \nabla_x \cdot \left(\nabla_y \cdot \left(\Psi(y)\right)\nabla u^0\right) + \nabla_y \cdot A(y)\nabla_x u^1 \\ &= \nabla_y \cdot \left(\Psi(y)\nabla^2 u^0\right) + \Psi(y)\nabla^3 u^0. \end{aligned}$$

Subsequently, for all $x \in \Omega_n$,

$$\begin{aligned} f^\varepsilon(x) &= \left[\nabla_y \cdot \left(\Psi(y)\nabla^2 u^0\right)\right]\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\chi\left(\frac{x}{\varepsilon}\right)\nabla^2 u^0\right) \\ &= \left[\nabla_y \cdot \left(\Psi(y)\nabla^2 u^0\right)\right]\left(x, \frac{x}{\varepsilon}\right) \\ &\quad + \varepsilon\left[\nabla_x \cdot \left(\Psi(y)\nabla^2 u^0\right)\right]\left(x, \frac{x}{\varepsilon}\right) - \varepsilon\Psi\left(\frac{x}{\varepsilon}\right)\nabla^3 u^0 \\ &\quad + \varepsilon\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\chi\left(\frac{x}{\varepsilon}\right)\nabla^2 u^0\right) \\ &= \varepsilon\nabla \cdot \left(\Psi\left(\frac{x}{\varepsilon}\right)\nabla^2 u^0\right) + \varepsilon\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\chi\left(\frac{x}{\varepsilon}\right)\nabla^2 u^0\right) - \varepsilon\Psi\left(\frac{x}{\varepsilon}\right)\nabla^3 u^0. \end{aligned}$$

Hence $f^\varepsilon = \varepsilon\nabla \cdot h^\varepsilon + \varepsilon g^\varepsilon$, with

$$\begin{aligned} h^\varepsilon &= h^\varepsilon(x) := \Psi\left(\frac{x}{\varepsilon}\right)\nabla^2 u^0 + A\left(\frac{x}{\varepsilon}\right)\chi\left(\frac{x}{\varepsilon}\right)\nabla^2 u^0, \\ g^\varepsilon &= g^\varepsilon(x) := -\Psi\left(\frac{x}{\varepsilon}\right)\nabla^3 u^0. \end{aligned}$$

The next lemma contains estimates on h^ε and g^ε for large x.

Lemma 4.15. *There is a constant $C > 0$, such that for all x sufficiently large, for all $\varepsilon > 0$,*

$$|h^\varepsilon(x)| \leq C\frac{1}{|x|^d}, \tag{4.50a}$$

$$|g^\varepsilon(x)| \leq C\frac{1}{|x|^{d+1}}. \tag{4.50b}$$

Proof. Let $\Lambda \in \mathbb{N}^d$, $2 \leq |\Lambda| \leq 3$, and $R > 0$ such that the support of f is included in $B(0,R)$. It follows from (4.29) that for x large enough,

$$\begin{aligned}
\left|\partial_x^\Lambda u^0(x)\right| &\leq \int_{\Omega_n} \left|\partial_x^\Lambda G^0(x,\tilde{x})\right| |f(\tilde{x})| d\tilde{x} \\
&\leq C \int_{B(0,R)} \frac{1}{|x-\tilde{x}|^{d-2+|\Lambda|}} d\tilde{x} \\
&\leq C \int_{B(0,R)} \frac{1}{(|x|-R)^{d-2+|\Lambda|}} d\tilde{x} \\
&\leq \frac{C}{(|x|-R)^{d-2+|\Lambda|}} \overset{\infty}{\cong} O\left(\frac{1}{|x|^{d-2+|\Lambda|}}\right),
\end{aligned}$$

which ends the proof. □

These preliminaries being done, we now turn to the estimation of $r_{bl}^{1,\varepsilon}$. Let $x \in \Omega_n$ be fixed. Green's formula yields

$$r_{bl}^{1,\varepsilon}(x) = \varepsilon \int_{\Omega_n} G^\varepsilon(x,\tilde{x}) \left(\nabla \cdot h^\varepsilon + g^\varepsilon\right)(\tilde{x}) d\tilde{x}. \tag{4.51}$$

We concentrate on each term of the r.h.s. of (4.51) separately. The strategy in both cases is to split the integral in two parts:

1. for $\tilde{x}$ close to x, one relies on Young inequalities to bound this part in L^∞;
2. for $\tilde{x}$ far from x, one uses (4.50a) and (4.50b) to show that this part can be made arbitrarily small uniformly in x.

Let $R > 0$ and assume for the moment $d \geq 3$. An integration by parts

$$\int_{\Omega_n} G^\varepsilon(x,\tilde{x}) \nabla \cdot h^\varepsilon(\tilde{x}) d\tilde{x} = -\int_{\Omega_n} \left(\nabla_{\tilde{x}} G^\varepsilon\right)(x,\tilde{x}) h^\varepsilon(\tilde{x}) d\tilde{x},$$

together with (4.44b) and (4.44d) gives

$$\begin{aligned}
&\left|r_{bl}^{1,\varepsilon}(x)\right| \leq \varepsilon \int_{\Omega_n} \frac{C}{|x-\tilde{x}|^{d-1}} |h^\varepsilon(\tilde{x})| \, d\tilde{x} + \varepsilon \int_{\Omega_n} \frac{C}{|x-\tilde{x}|^{d-2}} |g^\varepsilon(\tilde{x})| \, d\tilde{x} \\
&\leq \varepsilon \int_{\mathbb{R}^d} \frac{C}{|x-\tilde{x}|^{d-1}} 1_{B(0,R)}(x-\tilde{x}) \left|\widetilde{h}^\varepsilon(\tilde{x})\right| d\tilde{x} + \varepsilon \int_{\mathbb{R}^d} \frac{C}{|x-\tilde{x}|^{d-1}} 1_{B(0,R)^c}(x-\tilde{x}) \left|\widetilde{h}^\varepsilon(\tilde{x})\right| d\tilde{x} \\
&\quad + \varepsilon \int_{\mathbb{R}^d} \frac{C}{|x-\tilde{x}|^{d-2}} 1_{B(0,R)}(x-\tilde{x}) \left|\widetilde{g}^\varepsilon(\tilde{x})\right| d\tilde{x} + \varepsilon \int_{\mathbb{R}^d} \frac{C}{|x-\tilde{x}|^{d-2}} 1_{B(0,R)^c}(x-\tilde{x}) \left|\widetilde{g}^\varepsilon(\tilde{x})\right| d\tilde{x},
\end{aligned}$$

where $\widetilde{h}^\varepsilon$ (resp. $\widetilde{g}^\varepsilon$) is the extension of h^ε (resp. g^ε) to $\mathbb{R}^d$ by 0 outside of Ω_n. Let us first concentrate on the terms involving $\widetilde{h}^\varepsilon$. First, it simply follows from lemma 4.15 that $\left|\widetilde{h}^\varepsilon(\tilde{x})\right|$ is $O\left(\frac{1}{|\tilde{x}|^d}\right)$ in a neighbourhood of ∞. One can find $p,\ p' \geq 1$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad p > \frac{d}{d-1}, \qquad \text{and} \quad p' > 1.$$

Therefore,

$$\left\| \frac{1}{|\tilde{x}|^{d-1}} 1_{B(0,R)^c}(\tilde{x}) \right\|_{L^p(\mathbb{R}^d)} \overset{R\to\infty}{\longrightarrow} 0$$

and $\widetilde{h}^\varepsilon$ is bounded uniformly in ε in $L^{p'}\left(\mathbb{R}^d\right)$. Thanks to Young's inequality,

$$\left|\int_{\mathbb{R}^d} \frac{C}{|x-\tilde{x}|^{d-1}} 1_{B(0,R)^c}(x-\tilde{x}) \left|\widetilde{h}^\varepsilon(\tilde{x})\right| d\tilde{x}\right| \\ \leq \left\| \frac{1}{|\tilde{x}|^{d-1}} 1_{B(0,R)^c}(\tilde{x}) \right\|_{L^p\left(\mathbb{R}^d\right)} \left\|\widetilde{h}^\varepsilon\right\|_{L^{p'}\left(\mathbb{R}^d\right)} \overset{R\to\infty}{\longrightarrow} 0 \quad (4.52)$$

uniformly in $x \in \Omega_n$. The r.h.s. of (4.52) can be made less than $\|f\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)}$ for an $R > 0$ large enough. We now carry out the analysis of the integral on $|x-\tilde{x}| < R$. An adequate choice of the exponents in Young's inequality leads to (4.47a). Indeed, for all $1 \leq q < \frac{d}{d-1}$, $\frac{1}{|\tilde{x}|^{d-1}} 1_{B(0,R)} \in L^q\left(\mathbb{R}^d\right)$. From the condition $1 = \frac{1}{q} + \frac{1}{q'}$ on Young exponents, one deduces $q' > d$. Yet, for all $\delta > 0$, for all $q' := d + \delta$, thanks to elliptic regularity and Sobolev's injection

$$\left\|\widetilde{h}^\varepsilon\right\|_{L^{q'}(\Omega_n)} \leq C \left\|\nabla^2 u^0\right\|_{L^{q'}(\Omega_n)} \leq C \left\|\nabla^2 u^0\right\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)} \\ \leq C \left\|u^0\right\|_{W^{3,\frac{d}{2}+\delta}(\Omega_n)} \leq C \|f\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)}.$$

Young's inequality finally gives

$$\int_{\mathbb{R}^d} \frac{1}{|x-\tilde{x}|^{d-1}} 1_{B(0,R)}(x-\tilde{x}) \left|\widetilde{h}^\varepsilon(\tilde{x})\right| d\tilde{x} \leq C \|f\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)}.$$

The reasoning for $\widetilde{g}^\varepsilon$ is almost the same, except for the exponents in Young's inequalities which have to be adapted.

Let us briefly indicate how to treat the case $d = 2$. Each term can be estimated as above, except the ones involving g^ε. As before, we split the integral:

$$\int_{\Omega_n} G^\varepsilon(x,\tilde{x}) g^\varepsilon(\tilde{x}) d\tilde{x} = \int_{\Omega_n} G^\varepsilon(x,\tilde{x}) 1_{D(0,R)}(x-\tilde{x}) g^\varepsilon(\tilde{x}) d\tilde{x} \\ + \int_{\Omega_n} G^\varepsilon(x,\tilde{x}) 1_{D(0,R)^c}(x-\tilde{x}) g^\varepsilon(\tilde{x}) d\tilde{x}.$$

For x close to $\tilde{x}$ we bound G^ε by (4.44a)

$$\int_{\Omega_n} G^\varepsilon(x,\tilde{x}) 1_{D(0,R)}(x-\tilde{x}) g^\varepsilon(\tilde{x}) d\tilde{x} \leq C \int_{\mathbb{R}^d} \left(|\ln|x-\tilde{x}|| + 1\right) 1_{B(0,R)}(x-\tilde{x}) \widetilde{g}^\varepsilon(\tilde{x}) d\tilde{x},$$

and for $|x-\tilde{x}| > R$ we have recourse to (4.44c) instead of (4.44b)

$$\left|\int_{\Omega_n} G^\varepsilon(x,\tilde{x}) 1_{D(0,R)^c}(x-\tilde{x}) g^\varepsilon(\tilde{x}) d\tilde{x}\right| \leq C \int_{\mathbb{R}^d} \frac{|x\cdot n||\tilde{x}\cdot n|}{|x-\tilde{x}|^2} 1_{D(0,R)^c}(x-\tilde{x}) \left|\widetilde{g}^\varepsilon(\tilde{x})\right| d\tilde{x} \\ \leq C \int_{\mathbb{R}^d} \left(\frac{|(x-\tilde{x})\cdot n||\tilde{x}\cdot n|}{|x-\tilde{x}|^2} + \frac{|\tilde{x}\cdot n|^2}{|x-\tilde{x}|^2} \right) 1_{B(0,R)^c}(x-\tilde{x}) \left|\widetilde{g}^\varepsilon(\tilde{x})\right| d\tilde{x} \\ \leq C \int_{\mathbb{R}^d} \frac{1}{|x-\tilde{x}|} 1_{B(0,R)^c}(x-\tilde{x}) |\tilde{x}| \left|\widetilde{g}^\varepsilon(\tilde{x})\right| d\tilde{x} \\ + \int_{\mathbb{R}^d} \frac{1}{|x-\tilde{x}|^2} 1_{B(0,R)^c}(x-\tilde{x}) |\tilde{x}|^2 \left|\widetilde{g}^\varepsilon(\tilde{x})\right| d\tilde{x}$$

and estimate these terms, uniformly in x and ε, via Young's inequality. The bound (4.47a) on $r_{bl}^{1,\varepsilon}$ is shown.

As elliptic regularity and Sobolev injections imply

$$\left\| \chi^\alpha \left(\frac{\cdot}{\varepsilon}\right) \partial_{x_\alpha} u^0 \right\|_{L^\infty(\Omega_n)} \leq C \left\| \nabla u^0 \right\|_{W^{2,\frac{d}{2}+\delta}(\Omega_n)} \leq C \left\| f \right\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)},$$

it remains to establish (4.47b) in order to prove (4.47c). Let $x \in \Omega_n$. Poisson's representation formula for $u_{bl}^{1,\varepsilon}$ yields

$$u_{bl}^{1,\varepsilon}(x) = -\int_{\partial\Omega_n} P^\varepsilon(x,\tilde{x}) \chi^\alpha \left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{x_\alpha} u^0(\tilde{x}) d\tilde{x}.$$

From the bound (4.44e) on P^ε, one gets

$$\begin{aligned}
\left| u_{bl}^{1,\varepsilon}(x) \right| &\leq \int_{\partial\Omega_n} \left| P^\varepsilon(x,\tilde{x}) \right| \left| \chi^\alpha \left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{x_\alpha} u^0(\tilde{x}) \right| d\tilde{x} \\
&\leq C \int_{\partial\Omega_n} \frac{x \cdot n}{|x-\tilde{x}|^d} \left| \partial_{x_\alpha} u^0(\tilde{x}) \right| d\tilde{x} \\
&\leq C \int_{\mathbb{R}^{d-1}} \frac{\hat{x}_d}{\left(\hat{x}_d^2 + |\hat{x}' - \tilde{x}'|^2\right)^{\frac{d}{2}}} \left| \partial_{x_\alpha} u^0 \left(\mathrm{M}\left(\tilde{x}', 0\right)\right) \right| d\tilde{x}' \\
&\leq C \int_{\mathbb{R}^{d-1}} \frac{1}{\left(1 + |\tilde{x}'|^2\right)^{\frac{d}{2}}} \left| \partial_{x_\alpha} u^0 \left(\mathrm{M}\left(\hat{x}' - \hat{x}_d \tilde{x}', 0\right)\right) \right| d\tilde{x}' \\
&\leq C \left\| \nabla u^0 \right\|_{L^\infty(\Omega_n)} \int_{\mathbb{R}^{d-1}} \frac{1}{\left(1 + |\tilde{x}'|^2\right)^{\frac{d}{2}}} d\tilde{x}' \\
&\leq C \left\| \nabla u^0 \right\|_{W^{2,\frac{d}{2}+\delta}(\Omega_n)} \leq C \left\| f \right\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)},
\end{aligned}$$

with $\hat{x} := \mathrm{M}^T x$. This establishes (4.47a) and proposition 4.13.

4.5 Asymptotic expansion of Poisson's kernel

We intend to get the asymptotics of $P^\varepsilon = P^\varepsilon(x,\tilde{x})$, defined by (4.43), for $\tilde{x}$ in a neighbourhood of the boundary $\partial\Omega_n$ and $|x - \tilde{x}|$ close to 1. Our method is in three steps:

1. homogenization of u^ε solution of (4.45) for $f \in C_c^\infty\left(\overline{\Omega_n}\right)$;
2. expansion of G^ε, thanks to a duality argument;
3. expansion of P^ε via (4.43) and the expansion for G^ε.

The first point has been the purpose of section 4.4. We now turn to the second point.

4.5.1 Back to Green's kernel

Proposition 4.16. *For all $0 < \kappa < \frac{1}{d}$, there exists $C_\kappa > 0$, such that for all $\varepsilon > 0$, for all $x,\ \tilde{x} \in \overline{\Omega_n}$, $\frac{1}{4} \leq |x - \tilde{x}| \leq 4$ implies*

$$\left| G^\varepsilon(x,\tilde{x}) - G^0(x,\tilde{x}) \right| \leq C_\kappa \varepsilon^\kappa.$$

The idea of the proof is taken from [AL91]. We start from (4.47c) and proceed by a duality method to prove the estimate on the kernels. The key is as usual Green's representation formula. Let $x \in \Omega_n$ and $\varepsilon > 0$ be fixed for the rest of the proof. We look at

$$\sigma_{\varepsilon,x} := \sup_{\substack{\tilde{x} \in \Omega_n \\ \frac{1}{4} \leq |x-\tilde{x}| \leq 4}} \left| G^\varepsilon(x,\tilde{x}) - G^0(x,\tilde{x}) \right|.$$

The least upper bound $\sigma_{\varepsilon,x}$ is reached for at least one $\tilde{x}_{\varepsilon,x}$, which may be on the boundary $\partial\Omega_n$. From (4.44d), one obtains the existence of $C_1 > 0$, independent of ε and x, such that for all $\tilde{x} \in \Omega_n$, $\frac{1}{5} \leq |x - \tilde{x}| \leq 5$,

$$|\nabla_{\tilde{x}} G^\varepsilon(x, \tilde{x})| + \left|\nabla_{\tilde{x}} G^0(x, \tilde{x})\right| \leq C_1.$$

Let $\rho_{\varepsilon,x} := \frac{\sigma_{\varepsilon,x}}{2N^2 C_1}$. One can always increase C_1, so that $\rho_{\varepsilon,x} < 1$ and

$$B\left(\tilde{x}_{\varepsilon,x}, \rho_{\varepsilon,x}\right) \subset B(x, 5) \setminus \overline{B}\left(x, \frac{1}{5}\right).$$

Then:

Lemma 4.17. *There exists $i,\ j \in \{1, \dots N\}$ such that for all $\tilde{x} \in D\left(\tilde{x}_{\varepsilon,x}, \rho_{\varepsilon,x}\right)$,*

$$\left|G^\varepsilon_{ij}(x, \tilde{x}) - G^0_{ij}(x, \tilde{x})\right| \geq \frac{\sigma_{\varepsilon,x}}{2N^2}.$$

Proof. By equivalence of norms in finite dimension, one can always assume that

$$\left|G^\varepsilon(x, \tilde{x}) - G^0(x, \tilde{x})\right| = \sum_{i,j} \left|G^\varepsilon_{ij}(x, \tilde{x}) - G^0_{ij}(x, \tilde{x})\right|.$$

From $\left|G^\varepsilon(x, \tilde{x}_{\varepsilon,x}) - G^0(x, \tilde{x}_{\varepsilon,x})\right| = \sigma_{\varepsilon,x}$, it comes the existence of $i,\ j \in \{1, \dots N\}$ such that

$$\left|G^\varepsilon_{ij}(x, \tilde{x}_{\varepsilon,x}) - G^0_{ij}(x, \tilde{x}_{\varepsilon,x})\right| \geq \frac{\sigma_{\varepsilon,x}}{N^2}. \tag{4.53}$$

The integers $i,\ j$ are now fixed such as (4.53) holds. Then, by the reverse triangle inequality

$$\begin{aligned}
0 &\leq \frac{\sigma_{\varepsilon,x}}{N^2} - \left|G^\varepsilon_{ij}(x, \tilde{x}) - G^0_{ij}(x, \tilde{x})\right| \\
&\leq \left|G^\varepsilon_{ij}(x, \tilde{x}_{\varepsilon,x}) - G^0_{ij}(x, \tilde{x}_{\varepsilon,x})\right| - \left|G^\varepsilon_{ij}(x, \tilde{x}) - G^0_{ij}(x, \tilde{x})\right| \\
&\leq \left|G^\varepsilon_{ij}(x, \tilde{x}_{\varepsilon,x}) - G^\varepsilon_{ij}(x, \tilde{x})\right| + \left|G^0_{ij}(x, \tilde{x}_{\varepsilon,x}) - G^0_{ij}(x, \tilde{x})\right| \\
&\leq \left[\|\nabla_{\tilde{x}} G^\varepsilon\|_{L^\infty\left(\frac{1}{5} \leq |x - \tilde{x}| \leq 5\right)} + \left\|\nabla_{\tilde{x}} G^0\right\|_{L^\infty\left(\frac{1}{5} \leq |x - \tilde{x}| \leq 5\right)}\right] |\tilde{x}_{\varepsilon,x} - \tilde{x}| \\
&\leq C_1 \rho_{\varepsilon,x} = \frac{\sigma_{\varepsilon,x}}{2N^2},
\end{aligned}$$

and finally

$$\left|G^\varepsilon_{ij}(x, \tilde{x}) - G^0_{ij}(x, \tilde{x})\right| \geq \frac{\sigma_{\varepsilon,x}}{2N^2}.$$

□

Take $\varphi \in C^\infty_c\left(B(\tilde{x}_{\varepsilon,x}, 1)\right)$, with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B\left(\tilde{x}_{\varepsilon,x}, \frac{1}{2}\right)$. Note that the support of φ may intersect the boundary $\partial\Omega_n$. For $\rho > 0$, we define φ_ρ by $\varphi_\rho := \varphi\left(\frac{\cdot}{\rho}\right) \in C^\infty_c\left(B\left(\tilde{x}_{\varepsilon,x}, \rho\right)\right)$; we have:

$$\|\varphi_\rho\|_{L^\infty(\Omega_n)} \leq C \quad \text{and} \qquad \|\nabla \varphi_\rho\|_{L^\infty(\Omega_n)} \leq \frac{C}{\rho}.$$

For $\rho = \rho_{\varepsilon,x}$, the constants above do not depend on ε or x.

Let $i,\ j$ the integers given by lemma 4.17. The intermediate value theorem implies that $G^\varepsilon_{ij}(x, \tilde{x}) - G^0_{ij}(x, \tilde{x})$ has a constant sign on $D\left(\tilde{x}_{\varepsilon,x}, \rho_{\varepsilon,x}\right)$. Up to the change of f in $-f$ in what follows, one can always assume that $G^\varepsilon_{ij}(x,\tilde{x}) - G^0_{ij}(x,\tilde{x}) \geq 0$, which automatically yields

$$G^\varepsilon_{ij}(x,\tilde{x}) - G^0_{ij}(x,\tilde{x}) \geq \frac{\sigma_{\varepsilon,x}}{2N^2}$$

for all $\tilde{x} \in D\left(\tilde{x}_{\varepsilon,x}, \rho_{\varepsilon,x}\right)$. We now carry out the duality argument. For this purpose, take $f = f(\tilde{x}) := \varphi_{\rho_{\varepsilon,x}}(\tilde{x}) e_j$, where e_j is the j^{th} vector of the canonical basis of $\mathbb{R}^N$ and denote by $u^\varepsilon = u^\varepsilon(\tilde{x}) \in \mathbb{R}^N$ and $u^0 = u^0(\tilde{x}) \in \mathbb{R}^N$ the solutions of (4.45) and (4.46) with r.h.s. f. We remind that f, u^ε as well as u^0 depend on x. Thanks to Green's representation formula,

$$\begin{aligned}
\left[u^\varepsilon(x) - u^0(x)\right]_i &= \int_{\Omega_n} \left[\left(G^\varepsilon(x,\tilde{x}) - G^0(x,\tilde{x})\right) f(\tilde{x})\right]_i d\tilde{x} \\
&= \int_{D(\tilde{x}_{\varepsilon,x},\rho_{\varepsilon,x})} \left[G^\varepsilon_{ij}(x,\tilde{x}) - G^0_{ij}(x,\tilde{x})\right] \varphi_{\rho_{\varepsilon,x}}(\tilde{x}) d\tilde{x} \\
&\geq \int_{D\left(\tilde{x}_{\varepsilon,x},\frac{\rho_{\varepsilon,x}}{2}\right)} G^\varepsilon_{ij}(x,\tilde{x}) - G^0_{ij}(x,\tilde{x}) d\tilde{x} \\
&\geq \int_{D\left(\tilde{x}_{\varepsilon,x},\frac{\rho_{\varepsilon,x}}{2}\right)} \frac{\sigma_{\varepsilon,x}}{2N^2} d\tilde{x} \geq C\rho_{\varepsilon,x}^{d+1}.
\end{aligned}$$

Yet, we know from (4.47c), an estimate of $u^\varepsilon - u^0$:

$$\begin{aligned}
\left\|u^\varepsilon - u^0\right\|_{L^\infty(\Omega_n)} &\leq C_\delta \varepsilon \left\|f\right\|_{W^{1,\frac{d}{2}+\delta}(\Omega_n)} \\
&= C_\delta \varepsilon \left[\left\|\varphi_{\rho_{\varepsilon,x}}\right\|_{L^{\frac{d}{2}+\delta}(\Omega_n)} + \left\|\nabla\varphi_{\rho_{\varepsilon,x}}\right\|_{L^{\frac{d}{2}+\delta}(\Omega_n)}\right] \leq C_\delta \varepsilon \rho_{\varepsilon,x}^{\frac{d-2\delta}{d+2\delta}}.
\end{aligned}$$

Putting together these bounds, we get

$$C\rho_{\varepsilon,x}^{d+1} \;\leq\; \left|\left[u^\varepsilon(x) - u^0(x)\right]_i\right| \;\leq\; \left|u^\varepsilon(x) - u^0(x)\right| \;\leq\; \left\|u^\varepsilon - u^0\right\|_{L^\infty(\Omega_n)} \;\leq\; C_\delta \varepsilon \rho_{\varepsilon,x}^{\frac{d-2\delta}{d+2\delta}},$$

which summarizes in

$$\varepsilon \geq C_\delta \rho_{\varepsilon,x}^{-\frac{d-2\delta}{d+2\delta}+d+1} = C_\delta \rho_{\varepsilon,x}^{d+\frac{4\delta}{d+2\delta}}$$

for every $\delta > 0$, with a constant C_δ independent of ε and x. The inequalities

$$\sigma_{\varepsilon,x} \leq C_\delta \rho_{\varepsilon,x} \leq C_\delta \varepsilon^{\frac{1}{d+\frac{4\delta}{d+2\delta}}}$$

contain the asymptotic expansion of G^ε at zeroth order of proposition 4.16. One can follow the same reasoning as above, changing A in A^*, to obtain for all $0 < \kappa < \frac{1}{d}$, for all $x,\ \tilde{x} \in \Omega_n$, $\frac{1}{4} \leq |x - \tilde{x}| \leq 4$,

$$\left|G^{*,\varepsilon}(x,\tilde{x}) - G^{*,0}(x,\tilde{x})\right| \leq C_\kappa \varepsilon^\kappa. \tag{4.54}$$

4.5.2 Homogenization of Poisson's kernel

Let $0 < \varepsilon < 1$ and $x \in \Omega_n$ be fixed. Assume that x is close to the boundary, say $x \cdot n < 4$ to fix the ideas. According to (4.42), $G^{*,\varepsilon}(\cdot, x)$ satisfies

$$\begin{cases} -\nabla_{\tilde{x}} \cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right) \nabla_{\tilde{x}} G^{*,\varepsilon}(\tilde{x},x) &= 0, \quad \tilde{x} \in D(x,4) \setminus \overline{D}\left(x,\frac{1}{4}\right) \\ G^{*,\varepsilon}(\tilde{x},x) &= 0, \quad \tilde{x} \in \Gamma(x,4) \setminus \overline{\Gamma}\left(x,\frac{1}{4}\right) \end{cases}$$

This leads to the idea that one can apply theorem 4.8 to an expansion of $G^{*,\varepsilon}$ for which a local estimate in L^∞ is known. Doing so, one has to handle carefully the trace on $\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)$. Take for example $Z^{*,\varepsilon,x}=Z^{*,\varepsilon,x}(\tilde{x})\in M_N(\mathbb{R})$ defined for all $\tilde{x}\in D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)$ by

$$\begin{aligned}Z^{*,\varepsilon,x}(\tilde{x}):=G^{*,\varepsilon}(\tilde{x},x)-G^{*,0}(\tilde{x},x)-\varepsilon\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}G^{*,0}(\tilde{x},x)&\\-\varepsilon^2\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x).&\end{aligned}$$

Then, for all $0<\nu<1$,

$$\begin{aligned}&\|Z^{*,\varepsilon,x}\|_{C^{1,\nu}\left(\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)\right)}\\&=\varepsilon\left\|\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}G^{*,0}(\tilde{x},x)+\varepsilon\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\right\|_{C^{1,\nu}\left(\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)\right)}=O\left(\varepsilon^{-\nu}\right)\end{aligned}$$

which worsens our estimates. One way of getting around this difficulty is again to introduce a boundary layer term in the expansion . This term has to cancel the trace on the boundary due to the first-order term.

For all $\gamma\in\{1,\dots d\}$, let $v_{bl}^{*,\gamma}=v_{bl}^{*,\gamma}(\tilde{y})\in M_N(\mathbb{R})$ be the solution of

$$\begin{cases}-\nabla_{\tilde{y}}\cdot A^*(\tilde{y})\nabla_{\tilde{y}}v_{bl}^{*,\gamma}&=0,\qquad\quad \tilde{y}\in\Omega_n\\ \qquad\qquad v_{bl}^{*,\gamma}&=-\chi^{*,\gamma}(\tilde{y}),\quad\tilde{y}\in\partial\Omega_n\end{cases}$$

in the sense of theorem 4.5 or proposition 4.9, both being identical according to theorem 4.10. Let us recall that $\chi^{*,\gamma}=\chi^{*,\gamma}(y)\in M_N\left(\mathbb{R}\right)$ solves the system

$$\begin{cases}-\nabla_y\cdot A^*(y)\nabla_y\chi^{*,\gamma}&=\partial_{y_\alpha}A^{*,\alpha\gamma}\quad,\ y\in\mathbb{T}^d\\ \quad\int_{\mathbb{T}^d}\chi^{*,\gamma}(y)dy&=0\end{cases}.\tag{4.55}$$

Instead of $Z^{*,\varepsilon,x}$ one focuses now on $Z_{bl}^{*,\varepsilon,x}=Z_{bl}^{*,\varepsilon,x}(\tilde{x})\in M_N(\mathbb{R})$ such that

$$\begin{aligned}Z_{bl}^{*,\varepsilon,x}(\tilde{x}):=G^{*,\varepsilon}(\tilde{x},x)-G^{*,0}(\tilde{x},x)-\varepsilon\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}G^{*,0}(\tilde{x},x)-\varepsilon v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x)&\\-\varepsilon^2\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)-\varepsilon^2\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right)v_{bl}^{*,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)&\end{aligned}$$

for all $\tilde{x}\in D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)$. The method is now similar to the one, which led to the estimate on $r_{bl}^{1,\varepsilon}$: $Z_{bl}^{*,\varepsilon,x}$ satisfies

$$\begin{cases}-\nabla\cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right)\nabla Z_{bl}^{*,\varepsilon,x}&=F^\varepsilon+F_{bl}^\varepsilon,\qquad\qquad\qquad\qquad\qquad\qquad \tilde{x}\in D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\\ \qquad\qquad Z_{bl}^{*,\varepsilon,x}&=-\varepsilon^2\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\\ &\quad-\varepsilon^2\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right)v_{bl}^{*,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x),\tilde{x}\in\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)\end{cases}$$

where

$$\begin{aligned}F^\varepsilon:=\nabla_{\tilde{x}}\cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right)\nabla_{\tilde{x}}G^{*,0}(\tilde{x},x)+\varepsilon\nabla_{\tilde{x}}\cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right)\nabla_{\tilde{x}}\left(\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}G^{*,0}(\tilde{x},x)\right)&\\+\varepsilon^2\nabla_{\tilde{x}}\cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right)\nabla_{\tilde{x}}\left(\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\right)&\end{aligned}$$

and

$$F_{bl}^{\varepsilon} := \varepsilon \nabla_{\tilde{x}} \cdot A^* \left(\frac{\tilde{x}}{\varepsilon}\right) \nabla_{\tilde{x}} v_{bl}^{*,\varepsilon}(\tilde{x}) + \varepsilon^2 \nabla_{\tilde{x}} \cdot A^* \left(\frac{\tilde{x}}{\varepsilon}\right) \nabla_{\tilde{x}} \left(\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right) v_{bl}^{*,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta} G^{*,0}(\tilde{x},x)\right). \tag{4.56}$$

Applying (4.35), one gets

$$\begin{aligned}\|\nabla Z_{bl}^{*,\varepsilon,x}\|_{L^\infty\left(D(x,3)\setminus\overline{D}\left(x,\frac{1}{3}\right)\right)} \leq C\Bigg[&\|Z_{bl}^{*,\varepsilon,x}\|_{L^\infty\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)} \\ &+ \|F^{\varepsilon}\|_{L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)} + \|F_{bl}^{\varepsilon}\|_{L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)} \\ &+ \Bigg\|\varepsilon^2\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x) \\ &+\varepsilon^2\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right) v_{bl}^{*,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\Bigg\|_{C^{1,\nu}\left(\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)\right)}\Bigg].\end{aligned} \tag{4.57}$$

From (4.54) and the boundedness of $v_{bl}^{*,\gamma}$, one immediately obtains

$$\|Z_{bl}^{*,\varepsilon,x}\|_{L^\infty\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)} = O\left(\varepsilon^{\frac{1}{d}}\right). \tag{4.58}$$

Now, for $0<\nu<1$,

$$\begin{aligned}\|Z_{bl}^{*,\varepsilon,x}\|_{C^{1,\nu}\left(\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)\right)} = \varepsilon^2 \Bigg\|&\Gamma^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x) \\ &+\chi^{*,\alpha}\left(\frac{\tilde{x}}{\varepsilon}\right) v_{bl}^{*,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\Bigg\|_{C^{1,\nu}\left(\Gamma(x,4)\setminus\overline{\Gamma}\left(x,\frac{1}{4}\right)\right)} = O\left(\varepsilon^{1-\nu}\right).\end{aligned} \tag{4.59}$$

Expanding F^{ε} in powers of ε, one notices that the terms of order -1 and 0 in ε cancel and that for all $\tilde{x} \in D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)$

$$\begin{aligned}F^{\varepsilon}(\tilde{x}) = \varepsilon\Big[&\nabla_{\tilde{x}}\cdot A^*(\tilde{y})\nabla_{\tilde{x}}\left(\chi^{*,\alpha}(\tilde{y})\partial_{\tilde{x}_\alpha}G^{*,0}(\tilde{x},x)\right) \\ &+ \nabla_{\tilde{x}}\cdot A^*(\tilde{y})\nabla_{\tilde{y}}\left(\Gamma^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\right) \\ &+ \nabla_{\tilde{y}}\cdot A^*(\tilde{y})\nabla_{\tilde{x}}\left(\Gamma^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\right)\Big]\left(\tilde{x},\frac{\tilde{x}}{\varepsilon}\right) \\ &+ \varepsilon^2\left[\nabla_{\tilde{x}}\cdot A^*(\tilde{y})\nabla_{\tilde{x}}\left(\Gamma^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)\right)\right]\left(\tilde{x},\frac{\tilde{x}}{\varepsilon}\right).\end{aligned}$$

This demonstrates that F^{ε} is $O(\varepsilon)$ in $L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)$.

The source term F_{bl}^{ε} due to the boundary layer deserves more attention. The expansion of F_{bl}^{ε} in powers of ε is quite heavy. For the first term in the r.h.s. of (4.56) we get

$$\varepsilon\nabla_{\tilde{x}}\cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right)\nabla_{\tilde{x}}\left(v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x)\right)$$

$$= \partial_{\tilde{y}_\alpha}A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x) \tag{4.60a}$$

$$+ A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{y}_\beta}v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x) \tag{4.60b}$$

$$+ A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{y}_\alpha}v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x) \tag{4.60c}$$

$$+ \varepsilon A^{*,\alpha,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x). \tag{4.60d}$$

We write the second term in the r.h.s. of (4.56) as a sum of three terms

$$\varepsilon^2 \nabla_{\tilde{x}} \cdot A^*\left(\frac{\tilde{x}}{\varepsilon}\right) \nabla_{\tilde{x}} \left(\chi^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right) v_{bl}^{*,\eta}\left(\frac{\tilde{x}}{\varepsilon}\right) \partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta} G^{*,0}(\tilde{x},x)\right) \\ = T^0\left(\tilde{x},x,\frac{\tilde{x}}{\varepsilon}\right) + \varepsilon T^1\left(\tilde{x},x,\frac{\tilde{x}}{\varepsilon}\right) + \varepsilon^2 T^2\left(\tilde{x},x,\frac{\tilde{x}}{\varepsilon}\right),$$

with at order ε^0

$$T^0\left(\tilde{x},x,\frac{\tilde{x}}{\varepsilon}\right) := \left[\partial_{\tilde{y}_\alpha}\left(A^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{y}_\beta}\chi^{*,\gamma}(\tilde{y})\right) v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \tag{4.61a}$$

$$+ \left[A^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{y}_\beta}\chi^{*,\gamma}(\tilde{y})\partial_{\tilde{y}_\alpha} v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \tag{4.61b}$$

$$+ \left[\partial_{\tilde{y}_\alpha}\left(A^{*,\alpha\beta}(\tilde{y})\chi^{*,\gamma}(\tilde{y})\right)\partial_{\tilde{y}_\beta} v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \tag{4.61c}$$

$$+ \left[A^{*,\alpha\beta}(\tilde{y})\chi^{*,\gamma}(\tilde{y})\partial_{\tilde{y}_\alpha}\partial_{\tilde{y}_\beta} v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x), \tag{4.61d}$$

at order ε^1

$$T^1\left(\tilde{x},x,\frac{\tilde{x}}{\varepsilon}\right) := \left[A^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{y}_\beta}\chi^{*,\gamma}(\tilde{y}) v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \tag{4.61e}$$

$$+ \left[A^{*,\alpha\beta}(\tilde{y})\chi^{*,\gamma}(\tilde{y})\partial_{\tilde{y}_\beta} v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \tag{4.61f}$$

$$+ \left[\partial_{\tilde{y}_\alpha}\left(A^{*,\alpha\beta}(\tilde{y})\chi^{*,\gamma}(\tilde{y})\right) v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \tag{4.61g}$$

$$+ \left[A^{*,\alpha\beta}(\tilde{y})\chi^{*,\gamma}(\tilde{y})\partial_{\tilde{y}_\alpha} v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x), \tag{4.61h}$$

and at order ε^2

$$T^2\left(\tilde{x},x,\frac{\tilde{x}}{\varepsilon}\right) := \left[A^{*,\alpha\beta}(\tilde{y})\chi^{*,\gamma}(\tilde{y}) v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x). \tag{4.61i}$$

We intend to show that all these terms are $O\left(\varepsilon^\kappa\right)$, with $\kappa > 0$, in $L^{d+\delta}\left(D\left(x,4\right) \setminus \overline{D}\left(x,\frac{1}{4}\right)\right)$. This seems tricky for some terms in the expansion above. Indeed, (4.60a) and (4.61a) are of order $O(1)$, as we do not know more than $v_{bl}^{*,\eta} \in L^\infty(\Omega_n)$. However, the sum of (4.60a) and (4.61a) cancels thanks to the definition (4.55) of χ^*:

$$\left[\partial_{\tilde{y}_\alpha}\left(A^{*,\alpha\beta}(\tilde{y})\partial_{\tilde{y}_\beta}\chi^{*,\gamma}(\tilde{y})\right) v_{bl}^{*,\eta}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x) \\ = \left[\partial_{\tilde{y}_\alpha}\left(A^{*,\alpha\eta}(\tilde{y})\partial_{\tilde{y}_\eta}\chi^{*,\beta}(\tilde{y})\right) v_{bl}^{*,\gamma}(\tilde{y})\right]\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x) \\ = -\partial_{\tilde{y}_\alpha}A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right) v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x).$$

For the remaining terms, we use either (4.26), if at least one derivative of $v_{bl}^{*,\eta}$ is involved, or (4.38) if not. Therefore we proceed in the same manner for (4.60b), (4.60c), (4.61b), (4.61c), (4.61d), (4.61f), (4.61h) on the one hand, and (4.60d), (4.61e), (4.61g), (4.61i) on the other hand. Let us estimate (4.60b) in $L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)$: by (4.26) with $k'=1$,

$$\begin{aligned}
&\left\|A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{y}_\beta}v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x)\right\|_{L^1\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)}\\
&\leq\left(\int_{D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)}\left|\nabla_{\tilde{y}}v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\right|d\tilde{x}\right)\\
&\qquad\leq C\left(\int_0^8\sup_{(\tilde{z}_1,\ldots\tilde{z}_{d-1})\in\mathbb{R}^{d-1}}\left|\nabla_{\tilde{y}}v_{bl}^{*,\gamma}\left(\mathrm{M}\left(\tilde{z}_1,\ldots\tilde{z}_{d-1},\frac{t}{\varepsilon}\right)\right)\right|dt\right)\\
&\qquad\quad\leq C\left(\int_0^8\sup_{(\tilde{z}_1,\ldots\tilde{z}_{d-1})\in\mathbb{R}^{d-1}}\left|\nabla_{\tilde{y}}v_{bl}^{*,\gamma}\left(\mathrm{M}\left(\tilde{z}_1,\ldots\tilde{z}_{d-1},\frac{t}{\varepsilon}\right)\right)\right|^2dt\right)^{\frac{1}{2}}\\
&\qquad\qquad\leq C\varepsilon^{\frac{1}{2}}\left(\int_0^\infty\sup_{(\tilde{z}_1,\ldots\tilde{z}_{d-1})\in\mathbb{R}^{d-1}}\left|\nabla_{\tilde{y}}v_{bl}^{*,\gamma}\left(\mathrm{M}\left(\tilde{z}_1,\ldots\tilde{z}_{d-1},t\right)\right)\right|^2dt\right)^{\frac{1}{2}}\leq C\varepsilon^{\frac{1}{2}}
\end{aligned}$$

and

$$\left\|A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{y}_\beta}v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x)\right\|_{L^\infty\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)}=O(1)$$

from which we get, by interpolation,

$$\left\|A^{*,\alpha\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{y}_\beta}v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x)\right\|_{L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)}=O\left(\varepsilon^{\frac{1}{2(d+\delta)}}\right).$$

We have gained a positive power of ε for all corresponding terms listed above. In the other cases, the use of (4.38) deteriorates the exponent of ε in the estimate, though it remains positive. Let us bound (4.60d) in $L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)$:

$$\begin{aligned}
&\left\|\varepsilon A^{*,\alpha,\beta}\left(\frac{\tilde{x}}{\varepsilon}\right)v_{bl}^{*,\eta}\left(\frac{\tilde{x}}{\varepsilon}\right)\partial_{\tilde{x}_\alpha}\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x)\right\|_{L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)}\\
&\leq C\varepsilon\left(\int_{D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)}\left|v_{bl}^{*,\gamma}\left(\frac{\tilde{x}}{\varepsilon}\right)\right|^{d+\delta}d\tilde{x}\right)^{\frac{1}{d+\delta}}\leq C\varepsilon\left(\int_0^8\sqrt{\frac{t}{\varepsilon}}^{d+\delta}dt\right)^{\frac{1}{d+\delta}}\leq C\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Hence we have shown

$$\|F_{bl}^{\varepsilon}\|_{L^{d+\delta}\left(D(x,4)\setminus\overline{D}\left(x,\frac{1}{4}\right)\right)}=O\left(\varepsilon^{\frac{1}{2(d+\delta)}}\right).$$

This bound, with (4.58), (4.59), the estimate on F^ε and (4.57), boils down to

$$\|\nabla Z_{bl}^{*,\varepsilon,x}\|_{L^\infty\left(D(x,3)\setminus\overline{D}\left(x,\frac{1}{3}\right)\right)}\leq C\varepsilon^\kappa \tag{4.62}$$

with a positive, albeit small, $\kappa:=\min\left(1-\nu,\frac{1}{2(d+\delta)}\right)$. One can always take ν sufficiently small so that $1-\nu>\frac{1}{2(d+\delta)}$. As $\nabla Z_{bl}^{*,\varepsilon,x}$ is C^∞ up to the boundary $\partial\Omega_n$, it follows from (4.43) and (4.62) that one can expand $P^\varepsilon(x,\tilde{x})$. For all $\tilde{x}\in D(x,3)\setminus\overline{D}\left(x,\frac{1}{3}\right)$,

$$\left|P^\varepsilon(x,\tilde{x})-P^0\left(x,\tilde{x},\frac{\tilde{x}}{\varepsilon}\right)-\varepsilon P^1\left(x,\tilde{x},\frac{\tilde{x}}{\varepsilon}\right)-\varepsilon^2P^2\left(x,\tilde{x},\frac{\tilde{x}}{\varepsilon}\right)\right|\leq C\varepsilon^\kappa, \tag{4.63}$$

with at order ε^0

$$\left[P^0(x,\tilde{x},\tilde{y})\right]^T:=\left[A^{*,\alpha\beta}(\tilde{y})+A^{*,\alpha\gamma}(\tilde{y})\left(\partial_{\tilde{y}_\gamma}\chi^{*,\beta}(\tilde{y})+\partial_{\tilde{y}_\gamma}v_{bl}^{*,\beta}(\tilde{y})\right)\right]\partial_{\tilde{x}_\beta}G^{*,0}(\tilde{x},x)n_\alpha,$$

at order ε^1

$$\left[P^1(x,\tilde{x},\tilde{y})\right]^T := \Big[A^{*,\alpha\beta}(\tilde{y})\left(\chi^{*,\gamma}(\tilde{y}) + v_{bl}^{*,\gamma}(\tilde{y})\right) + A^{*,\alpha\eta}(\tilde{y})\partial_{\tilde{y}_\eta}\Gamma^{*,\gamma\beta}(\tilde{y}) + A^{*,\alpha\eta}(\tilde{y})\partial_{\tilde{y}_\eta}\left(\chi^{*,\gamma}(\tilde{y})v_{bl}^{*,\beta}(\tilde{y})\right)(\tilde{y})\Big]\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}G^{*,0}(\tilde{x},x)n_\alpha,$$

and at order ε^2

$$\left[P^2(x,\tilde{x},\tilde{y})\right]^T := A^{*,\alpha\beta}(\tilde{y})\left[\Gamma^{*,\gamma\eta}(\tilde{y}) + \chi^{*,\gamma}(\tilde{y})v_{bl}^{*,\eta}(\tilde{y})\right]\partial_{\tilde{x}_\beta}\partial_{\tilde{x}_\gamma}\partial_{\tilde{x}_\eta}G^{*,0}(\tilde{x},x)n_\alpha.$$

For $y\in\Omega_n$, $\tilde{y}\in\partial\Omega_n$ and $\varepsilon := \frac{1}{|y-\tilde{y}|}$, $x := \varepsilon y$ and $\tilde{x} := \varepsilon\tilde{y}$ are such that $|x-\tilde{x}| = 1$. Applying now (4.63) with x and $\tilde{x}$ defined like this, and rescaling the estimate in the variables y, $\tilde{y}$ using the scaling properties of P^ε and $G^{*,0}$ (see lemma 4.12), we finally have:

Theorem 4.18. *For all $0<\kappa<\frac{1}{2d}$, there exists $C_\kappa>0$, such that for all $y\in\Omega_{n,0}$ and $\tilde{y}\in\partial\Omega_{n,0}$,*

$$\Big|P^T(y,\tilde{y}) - A^*(\tilde{y})\nabla_{\tilde{y}}G^{*,0}(\tilde{y},y)\cdot n - A^*(\tilde{y})\nabla_{\tilde{y}}\left(\chi^*(\tilde{y})\cdot\nabla_{\tilde{y}}G^{*,0}(\tilde{y},y)\right)\cdot n - A^*(\tilde{y})\nabla_{\tilde{y}}\left(v_{bl}^*(\tilde{y})\cdot\nabla_{\tilde{y}}G^{*,0}(\tilde{y},y)\right)\cdot n - A^*(\tilde{y})\nabla_{\tilde{y}}\left(\Gamma^*(\tilde{y})\cdot\nabla^2_{\tilde{y}}G^{*,0}(\tilde{y},y)\right)\cdot n - A^*(\tilde{y})\nabla_{\tilde{y}}\left(\chi^*(\tilde{y})v_{bl}^*(\tilde{y})\cdot\nabla^2_{\tilde{y}}G^{*,0}(\tilde{y},y)\right)\cdot n\Big| \leq \frac{C_\kappa}{|y-\tilde{y}|^{d-1+\kappa}}. \tag{4.64}$$

4.6 Convergence towards a boundary layer tail

4.6.1 The convergence proof

It follows from theorem 4.10 that the variational solution v_{bl} of (4.1) in $\Omega_{n,0}$ (see section 4.2.2) can be expressed by the mean of Poisson's kernel associated to $-\nabla\cdot A(y)\nabla\cdot$ and Ω_n. Hence, by theorem 4.18, for all $y\in\Omega_n$, for all $i\in\{1,\dots N\}$,

$$\begin{aligned} v_{bl,i}(y) &= \int_{\partial\Omega_n} P_{ij}(y,\tilde{y})v_{0,j}(\tilde{y})d\tilde{y} = \int_{\partial\Omega_n} P^T_{ji}(y,\tilde{y})v_{0,j}(\tilde{y})d\tilde{y} \\ &= \int_{\partial\Omega_n} A^*_{jk}(\tilde{y})\nabla_{\tilde{y}}\Big[G^{*,0}(\tilde{y},y) + \chi^*(\tilde{y})\cdot\nabla_{\tilde{y}}G^{*,0}(\tilde{y},y) \\ &\quad + v_{bl}^*(\tilde{y})\cdot\nabla_{\tilde{y}}G^{*,0}(\tilde{y},y) + \Gamma^*(\tilde{y})\cdot\nabla^2_{\tilde{y}}G^{*,0}(\tilde{y},y) \\ &\quad + \chi^*(\tilde{y})v_{bl}^*(\tilde{y})\cdot\nabla^2_{\tilde{y}}G^{*,0}(\tilde{y},y)\Big]_{ki}\cdot n\ v_{0,j}(\tilde{y})d\tilde{y} + \int_{\partial\Omega_n} R_i(y,\tilde{y})d\tilde{y} \end{aligned} \tag{4.65}$$

where for all $\tilde{y}\in\partial\Omega_n$, $|R_i(y,\tilde{y})|\leq\frac{C}{|y-\tilde{y}|^{d-1+\kappa}}$. The bound on the remainder R_i yields

$$\begin{aligned} \left|\int_{\partial\Omega_n} R_i(y,\tilde{y})d\tilde{y}\right| &\leq C\int_{\partial\Omega_n}\frac{1}{|y-\tilde{y}|^{d-1+\kappa}}d\tilde{y} \\ &\leq C\int_{\mathbb{R}^{d-1}\times\{0\}}\frac{1}{|z-\tilde{z}|^{d-1+\kappa}}d\tilde{z} \leq \frac{C}{(y\cdot n)^\kappa}\int_{\mathbb{R}^{d-1}}\frac{1}{(1+|\tilde{z}'|^2)^{\frac{d-1+\kappa}{2}}}d\tilde{z}' \leq \frac{C}{(y\cdot n)^\kappa}\overset{y\cdot n\to\infty}{\longrightarrow}0. \end{aligned}$$

where $z := \mathrm{M}^T y$. It remains to handle the other terms. The boundary function v_0 is quasiperiodic along the boundary $\partial\Omega_n$ albeit not periodic. This suggests to use the following lemma to take advantage of the ergodic properties related to the quasiperiodic setting.

Lemma 4.19 ([Šub74] theorem S.3)**.** *Let $f = f(y) \in \mathbb{R}$ be a quasiperiodic function on $\mathbb{R}^d$. Then, there exists $\mathcal{M}\{f\} \in \mathbb{R}$ such that for all $\varphi \in L^1\left(\mathbb{R}^d\right)$,*

$$\int_{\mathbb{R}^d} \varphi(y) f\left(\frac{y}{\varepsilon}\right) dy \stackrel{\varepsilon\to 0}{\longrightarrow} \mathcal{M}\{f\} \int_{\mathbb{R}^d} \varphi(y) dy.$$

Let $i,\ j \in \{1,\dots\ N\}$ be fixed. We focus on the convergence when $y \cdot n \to \infty$ of

$$\int_{\partial\Omega_n} A^*_{jk}(\tilde{y})\nabla_{\tilde{y}} G^{*,0}_{ki}(\tilde{y}, y) \cdot n\ v_{0,j}(\tilde{y})d\tilde{y} = \int_{\partial\Omega_n} A^{*,\alpha\beta}_{jk}(\tilde{y})\partial_{\tilde{y}_\beta} G^{*,0}_{ki}(\tilde{y}, y) n_\alpha v_{0,j}(\tilde{y})d\tilde{y}.$$

For all $R > 0$, for all $y = y' + (y \cdot n)n \in \Omega_n$ with $y' \in \partial\Omega_n \cap B(0, R)$, taking $z := \mathrm{M}^T y$ and $\varepsilon := \frac{1}{y\cdot n} > 0$, we get

$$\begin{aligned}
&\int_{\partial\Omega_n} A^{*,\alpha\beta}_{jk}(\tilde{y})\partial_{\tilde{y}_\beta} G^{*,0}_{ki}(\tilde{y}, y) n_\alpha v_{0,j}(\tilde{y})d\tilde{y} \\
&= \int_{\mathbb{R}^{d-1}\times\{0\}} A^{*,\alpha\beta}_{jk}(\mathrm{M}\tilde{z})\partial_{1,\beta} G^{*,0}_{ki}(\mathrm{M}\tilde{z}, \mathrm{M}(z', z_d)) n_\alpha v_{0,j}(\mathrm{M}\tilde{z})d\tilde{z} \\
&= \int_{\mathbb{R}^{d-1}} A^{*,\alpha\beta}_{jk}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) \frac{1}{\varepsilon^{d-1}} \partial_{1,\beta} G^{*,0}_{ki}\left(\frac{\mathrm{M}(\tilde{z}', 0)}{\varepsilon}, \frac{\mathrm{M}(\varepsilon z', 1)}{\varepsilon}\right) n_\alpha v_{0,j}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) d\tilde{z}' \\
&= \int_{\mathbb{R}^{d-1}} \partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(0, 1)\right) n_\alpha A^{*,\alpha\beta}_{jk}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) v_{0,j}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) d\tilde{z}' \\
&\quad + \int_{\mathbb{R}^{d-1}} \Big[\partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(\varepsilon z', 1)\right) \\
&\qquad - \partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(0, 1)\right)\Big] n_\alpha A^{*,\alpha\beta}_{jk}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) v_{0,j}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) d\tilde{z}'.
\end{aligned} \tag{4.66}$$

Let us show that the second term in (4.66) tends to 0 when $\varepsilon \to 0$, uniformly in $z' \in B(0, R) \subset \mathbb{R}^{d-1}$. For ε sufficiently small such that $\varepsilon R \leq 1$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{d-1}} \left[\partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(\varepsilon z', 1)\right) - \partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(0, 1)\right)\right] \right. \\
&\qquad\qquad \left. n_\alpha A^{*,\alpha\beta}_{jk}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) v_{0,j}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon}, 0\right)\right) d\tilde{z}' \right| \\
&\leq C \int_{\mathbb{R}^{d-1}} \sup_{u' \in B(0,\varepsilon R)} \left|\nabla_1\nabla_2 G^{*,0}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(u', 1)\right)\right| \varepsilon |z'| d\tilde{z}' \\
&\leq C\varepsilon R \int_{\mathbb{R}^{d-1}} \sup_{u' \in B(0,1)} \frac{C}{(1 + |\tilde{z}' - u'|^2)^{\frac{d}{2}}} d\tilde{z}' \\
&\leq C\varepsilon.
\end{aligned}$$

From the bound (4.31c), we get for all $\tilde{y} \in \partial\Omega_n$,

$$\left|\nabla_1 G^{*,0}(\tilde{y}, y)\right| \leq C\frac{y \cdot n}{|y - \tilde{y}|^d},$$

which shows that

$$\left|\partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}', 0), \mathrm{M}(0, 1)\right)\right| \leq \frac{C}{(1 + |\tilde{z}'|^2)^{\frac{d}{2}}}.$$

Hence $\partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}',0),\mathrm{M}(0,1)\right) \in L^1_{\tilde{z}'}\left(\mathbb{R}^{d-1}\right)$ and one can apply lemma 4.19 to get the convergence of the first term in (4.66):

$$\int_{\mathbb{R}^{d-1}} \partial_{1,\beta} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}',0),\mathrm{M}(0,1)\right) n_\alpha A^{*,\alpha\beta}_{jk}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon},0\right)\right) v_{0,j}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon},0\right)\right) d\tilde{z}'$$
$$\stackrel{\varepsilon\to 0}{\longrightarrow} \mathcal{M}\left\{\begin{array}{ccc} \partial\Omega_n & \longrightarrow & \mathbb{R} \\ \tilde{y} & \longmapsto & A^{*,\alpha\beta}_{jk}(\tilde{y}) v_{0,j}(\tilde{y}) n_\alpha \end{array}\right\} \int_{\Omega_n} \partial_{\tilde{y}_\beta} G^{*,0}_{ki}(\tilde{y},n) d\tilde{y}.$$

The reasoning is identical for

$$\int_{\partial\Omega_n} A^*_{jk}(\tilde{y}) \nabla_{\tilde{y}}\left[\chi^*_{kl}(\tilde{y}) \nabla_{\tilde{y}} G^{*,0}_{li}(\tilde{y},y)\right]\cdot n v_{0,j}(\tilde{y}) d\tilde{y}, \tag{4.67a}$$
$$\int_{\partial\Omega_n} A^*_{jk}(\tilde{y}) \nabla_{\tilde{y}}\left[v^*_{bl,kl}(\tilde{y}) \nabla_{\tilde{y}} G^{*,0}_{li}(\tilde{y},y)\right]\cdot n v_{0,j}(\tilde{y}) d\tilde{y}, \tag{4.67b}$$

as both terms involve just one derivative of $G^{*,0}$. For (4.67b), we notice that $\tilde{y} \mapsto \partial_{\tilde{y}_\beta} v^{*,\gamma}_{bl}(\tilde{y})$ is quasiperiodic on $\partial\Omega_n$: we know from section 4.2.2 that there is a unique smooth $V^\gamma = V^\gamma(\theta,t) \in \mathbb{R}^N$, defined for $\theta \in \mathbb{T}^d$ and $t \geq 0$, such that for all $\tilde{y} \in \partial\Omega_n$,

$$v^{*,\gamma}_{bl}(\tilde{y}) = v^{*,\gamma}_{bl}(\mathrm{M}\tilde{z}) = V^\gamma(\mathrm{N}\tilde{z}',0).$$

The other terms in (4.65) involving strictly more than one derivative of $G^{*,0}$ tend to 0 when $\varepsilon \to 0$. Indeed, for all $R > 0$, for all $y = y' + (y\cdot n)n \in \Omega_n$ with $y' \in \partial\Omega_n \cap B(0,R)$, taking again $z := \mathrm{M}^T y$ and $\varepsilon := \frac{1}{y\cdot n} > 0$,

$$\begin{aligned}
&\int_{\partial\Omega_n} A^{*,\alpha\beta}_{jk}(\tilde{y}) \chi^{*,\gamma}_{kl}(\tilde{y}) \partial^2_{\tilde{y}_{\beta\gamma}} G^{*,0}_{li}(\tilde{y},y) n_\alpha v_{0,j}(\tilde{y}) d\tilde{y} \\
&= \int_{\mathbb{R}^{d-1}\times\{0\}} A^{*,\alpha\beta}_{jk}(\mathrm{M}\tilde{z}) \chi^{*,\gamma}_{kl}(\mathrm{M}\tilde{z}) \partial^2_{1,\beta\gamma} G^{*,0}_{li}(\mathrm{M}\tilde{z},\mathrm{M}(z',z_d)) n_\alpha v_{0,j}(\mathrm{M}\tilde{z}) d\tilde{z} \\
&= \varepsilon \int_{\mathbb{R}^{d-1}} A^{*,\alpha\beta}_{jk}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon},0\right)\right) \chi^{*,\gamma}_{kl}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon},0\right)\right) \partial^2_{1,\beta\gamma} G^{*,0}_{ki}\left(\mathrm{M}(\tilde{z}',0),\mathrm{M}(\varepsilon z',1)\right) \\
&\qquad\qquad n_\alpha v_{0,j}\left(\mathrm{M}\left(\frac{\tilde{z}'}{\varepsilon},0\right)\right) d\tilde{z}',
\end{aligned}$$

which is easily shown to be of order $O(\varepsilon)$. We proceed following the same method for the remaing terms. This demonstrates the convergence of the boundary layer towards a constant vector field v^∞_{bl}, the boundary layer tail: for all $y = y' + (y\cdot n)n \in \Omega_n$ with $y' \in \partial\Omega_n \cap B(0,R)$,

$$\begin{aligned}
v_{bl}(\tilde{y}) \stackrel{y\cdot n\to\infty}{\longrightarrow} v^\infty_{bl} := \int_{\partial\Omega_n} \partial_{\tilde{y}_\alpha} G^0(n,\tilde{y}) d\tilde{y} \Big[&\mathcal{M}\left\{A^{\beta\alpha}(\tilde{y}) v_0(\tilde{y}) n_\beta\right\} \\
&+ \mathcal{M}\left\{\partial_{\tilde{y}_\beta}\left(\chi^{*,\alpha}\right)^T(\tilde{y}) A^{\beta\gamma}(\tilde{y}) v_0(\tilde{y}) n_\gamma\right\} + \mathcal{M}\left\{\partial_{\tilde{y}_\beta}\left(v^{*,\alpha}_{bl}\right)^T(\tilde{y}) A^{\beta\gamma}(\tilde{y}) v_0(\tilde{y}) n_\gamma\right\}\Big]
\end{aligned} \tag{4.68}$$

locally uniformly in y'. Moreover, (4.68) yields an explicit expression for v^∞_{bl} in terms of the means $\mathcal{M}\{\cdot\}$ on $\partial\Omega_n$.

4.6.2 The boundary layer tail does not depend on a

In the preceding section we have shown the convergence of v_{bl} defined in $\Omega_{n,a}$ towards $v_{bl}^{a,\infty}$ when $y\cdot n \to \infty$. It remains to show that $v_{bl}^{a,\infty}$ is independent of a in order to complete the proof of theorem 4.2. The fact that $n \notin \mathbb{R}\mathbb{Q}^d$ is crucial. To do so, we generalize lemma 6 in [GVM11]:

Proposition 4.20. *Assume that $n \notin \mathbb{R}\mathbb{Q}^d$. Then,*

$$a \in \mathbb{R} \longmapsto v_{bl}^{a,\infty} \in \mathbb{R}^N$$

is Lipschitz continuous.

The proof in [GVM11] relies on the small divisors assumption (4.12) and energy estimates. We, instead, have recourse to Poisson's integral formula and estimates on Poisson's kernel. Let $a,\ a' \in \mathbb{R}$ and $\nu := |a' - a|$. We call G^a (resp. P^a) the Green (resp. Poisson) kernel associated to $-\nabla \cdot A(\cdot + an)\nabla\cdot$ and $\Omega_n = \Omega_{n,0}$. We define analogously $G^{a'}$ and $P^{a'}$. We also have the $*$-versions $G^{*,a}$ and $G^{*,a'}$ corresponding to the transposed operator (see section 4.3.1). The following lemma is an adaptation of the results due to Avellaneda and Lin (see [AL87a]), Kenig and Shen (see [KS11b] section 2), and Gérard-Varet and Masmoudi (see [GVM12] appendix A).

Lemma 4.21. *Let $0 < \mu < 1$. There exists $C > 0$ such that for all $y,\ \tilde{y} \in \Omega_n$, $y \neq \tilde{y}$,*

$$\left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| \leq C\nu \frac{1}{|y-\tilde{y}|^{d-2}}, \qquad \textit{if } d \geq 3, \tag{4.69a}$$

$$\left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| \leq C\nu \frac{(y\cdot n)^{\mu}\,(\tilde{y}\cdot n)^{\mu}}{|y-\tilde{y}|^{d-2+2\mu}}, \qquad \textit{for all } d \geq 2. \tag{4.69b}$$

Proof. The ideas for the proofs are in large part taken from the reference above. For (4.69a), we rely on a representation formula of $G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})$:

$$\begin{aligned} G^a(y,\tilde{y}) &- G^{a'}(y,\tilde{y}) \\ &= \int_{\Omega_n} \partial_{2,\alpha} G^a(y,\hat{y})\left(A^{\alpha\beta}(\hat{y}+na') - A^{\alpha\beta}(\hat{y}+na)\right)\partial_{1,\beta}G^{a'}(\hat{y},\tilde{y})d\hat{y}. \end{aligned} \tag{4.70}$$

This formula, which is proven in [HK07] (see corollary 3.5) for the domain $\mathbb{R}^d$, $d \geq 3$, is a consequence of Green's representation formula. The proof of Hofmann and Kim extends to the domain Ω_n. From (4.70) and (4.31d), one deduces that for all $d \geq 3$, $y,\ \tilde{y} \in \Omega_n$, $y \neq \tilde{y}$,

$$\begin{aligned} \left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| &\leq C\nu \int_{\Omega_n} \frac{1}{|y-\hat{y}|^{d-1}} \frac{1}{|\tilde{y}-\hat{y}|^{d-1}} d\hat{y} \\ &\leq C\nu \int_{\mathbb{R}^d} \frac{1}{|y-\tilde{y}-\hat{y}|^{d-1}} \frac{1}{|\hat{y}|^{d-1}} d\hat{y} \\ &\leq C\nu \frac{1}{|y-\tilde{y}|^{d-2}} \int_{\mathbb{R}^d} \frac{1}{\left|\frac{y-\tilde{y}}{|y-\tilde{y}|} - \hat{y}\right|^{d-1}} \frac{1}{|\hat{y}|^{d-1}} d\hat{y} \\ &\leq C\nu \frac{1}{|y-\tilde{y}|^{d-2}}. \end{aligned}$$

For (4.69b) we have recourse to the local boundary estimate (4.34). Assume that $d \geq 3$. For given $y,\ \tilde{y} \in \Omega_n$, $y \neq \tilde{y}$, we first establish the bound

$$\left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| \leq C\nu \frac{(y\cdot n)^{\mu}}{|y-\tilde{y}|^{d-2+\mu}}. \tag{4.71}$$

Let $r := |y-\tilde{y}|$. We distinguish between two cases. If $y \cdot n \geq \frac{r}{3}$, then (4.71) follows directly from (4.69a). Assume that $y \cdot n < \frac{r}{3}$ and let $\bar{y} \in \partial\Omega_n$ such that $y \cdot n = |y - \bar{y}|$. Then, from (4.27) it comes that $G^a(\cdot,\tilde{y}) - G^{a'}(\cdot,\tilde{y})$ satisfies

$$\begin{cases} -\nabla_y \cdot A(\cdot + na)\nabla_y \left(G^a(\cdot,\tilde{y}) - G^{a'}(\cdot,\tilde{y})\right) & = \nabla \cdot \left[A(\cdot+na) - A(\cdot+na')\right] \nabla G^{a'}(\cdot,\tilde{y}) \\ & \qquad \text{in} \quad D\left(\bar{y},\frac{r}{3}\right) \\ G^a(\cdot,\tilde{y}) - G^{a'}(\cdot,\tilde{y}) & = 0 \qquad \text{in} \quad \Gamma\left(\bar{y},\frac{r}{3}\right) \end{cases}. \tag{4.72}$$

Applying a rescaled version of (4.34), one gets using (4.69a) and (4.31d),

$$\begin{aligned}
&\left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| = \left|G^a(y,\tilde{y}) - G^{a'}(\bar{y},\tilde{y}) - \left(G^{a'}(y,\tilde{y}) - G^a(\bar{y},\tilde{y})\right)\right| \\
&\leq \left\|G^a(\cdot,\tilde{y}) - G^{a'}(\cdot,\tilde{y})\right\|_{C^{0,\mu}\left(D\left(\bar{y},\frac{r}{6}\right)\right)} \frac{|y\cdot n|^\mu}{r^\mu} r^\mu \\
&\leq C\left[\frac{1}{r^{\frac{d}{2}}}\left\|G^a(\cdot,\tilde{y}) - G^{a'}(\cdot,\tilde{y})\right\|_{L^2\left(D\left(\bar{y},\frac{r}{3}\right)\right)} \right.\\
&\qquad \left. + \frac{r}{r^{d+\delta}}\left\|\left[A(\cdot+na) - A(\cdot+na')\right]\nabla G^{a'}(\cdot,\tilde{y})\right\|_{L^{\frac{d}{d+\delta}}\left(D\left(\bar{y},\frac{r}{3}\right)\right)}\right] \frac{|y\cdot n|^\mu}{r^\mu} \\
&\leq C\left[\left\|G^a(\cdot,\tilde{y}) - G^{a'}(\cdot,\tilde{y})\right\|_{L^\infty\left(D\left(\bar{y},\frac{r}{3}\right)\right)} + r\nu\left\|\nabla G^{a'}(\cdot,\tilde{y})\right\|_{L^\infty\left(D\left(\bar{y},\frac{r}{3}\right)\right)}\right] \frac{|y\cdot n|^\mu}{r^\mu} \\
&\leq C\nu\left[\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{1}{|\hat{y}-\tilde{y}|^{d-2}} + r\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{1}{|\hat{y}-\tilde{y}|^{d-1}}\right] \frac{|y\cdot n|^\mu}{r^\mu} \\
&\leq C\nu\frac{|y\cdot n|^\mu}{r^{d-2+\mu}},
\end{aligned}$$

as for all $\hat{y} \in D\left(\bar{y},\frac{r}{3}\right)$, $|\hat{y}-\tilde{y}| > \frac{r}{6}$. This shows (4.71). We now turn to the proof of (4.69b) itself. If $\tilde{y}\cdot n \geq \frac{r}{3}$, then (4.69b) follows directly from (4.71). Assume that $\tilde{y}\cdot n < \frac{r}{3}$ and let $\bar{y} \in \partial\Omega_n$ such that $\tilde{y}\cdot n = |\tilde{y}-\bar{y}|$. Applying a rescaled version of (4.34) to $G^{*,a}(\cdot,y) - G^{*,a'}(\cdot,y)$ satisfying (4.72) with A (resp. $G^{a'}$, y, $\tilde{y}$) replaced by A^* (resp. $G^{*,a'}$, $\tilde{y}$, y), one gets using (4.71), (4.31e), and for all $\hat{y} \in D\left(\bar{y},\frac{r}{3}\right)$, $|\hat{y}-y| > \frac{r}{6}$,

$$\begin{aligned}
&\left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| = \left|G^{*,a}(\tilde{y},y) - G^{*,a'}(\tilde{y},y)\right| \\
&\leq C\left[\left\|G^{*,a}(\cdot,y) - G^{*,a'}(\cdot,y)\right\|_{L^\infty\left(D\left(\bar{y},\frac{r}{3}\right)\right)} + r\nu\left\|\nabla G^{*,a'}(\cdot,y)\right\|_{L^\infty\left(D\left(\bar{y},\frac{r}{3}\right)\right)}\right] \frac{|\tilde{y}\cdot n|^\mu}{r^\mu} \\
&\leq C\nu\left[\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \left|G^a(y,\hat{y}) - G^{a'}(y,\hat{y})\right| + r\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{y\cdot n}{|\hat{y}-y|^d}\right] \frac{|\tilde{y}\cdot n|^\mu}{r^\mu} \\
&\leq C\nu\left[\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{(y\cdot n)^\mu}{|\hat{y}-y|^{d-2+\mu}} + r\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{(y\cdot n)^\mu}{|\hat{y}-y|^{d-1+\mu}}\frac{(y\cdot n)^{1-\mu}}{|\hat{y}-y|^{1-\mu}}\right] \frac{|\tilde{y}\cdot n|^\mu}{r^\mu} \\
&\leq C\nu\left[\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{(y\cdot n)^\mu}{|\hat{y}-y|^{d-2+\mu}} + r\sup_{\hat{y}\in D\left(\bar{y},\frac{r}{3}\right)} \frac{(y\cdot n)^\mu}{|\hat{y}-y|^{d-1+\mu}}\right] \frac{|\tilde{y}\cdot n|^\mu}{r^\mu} \\
&\leq C\nu\frac{(y\cdot n)^\mu(\tilde{y}\cdot n)^\mu}{|y-\tilde{y}|^{d-2+2\mu}},
\end{aligned}$$

on condition that $y\cdot n \leq |\hat{y}-y|$, for all $\hat{y} \in D\left(\bar{y},\frac{r}{3}\right)$. If the latter is not true, i.e. if there is $\hat{y} \in D\left(\bar{y},\frac{r}{3}\right)$ such that $y\cdot n > |\hat{y}-y| > \frac{r}{6}$, then we apply the same reasoning as above with A (resp. $G^{a'}$, y, $\tilde{y}$) replaced by A^* (resp. $G^{*,a'}$, $\tilde{y}$, y) to get

$$\left|G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right| = \left|G^{*,a}(\tilde{y},y) - G^{*,a'}(\tilde{y},y)\right| \leq C\nu \frac{(\tilde{y}\cdot n)^\mu}{|y-\tilde{y}|^{d-2+\mu}}, \tag{4.73}$$

and deduce (4.69b) from (4.73) and $y \cdot n > \frac{r}{6}$. The two-dimensional bound follows from the three-dimensional one as explained in [GVM12] and [AL87a]. □

Let us notice that proposition 4.20 implies that for all $a,\ a' \in \mathbb{R}$, $v_{bl}^{a,\infty} = v_{bl}^{a',\infty}$. Indeed, for $\xi \in \mathbb{Z}^d$, v_{bl}^{ξ} solving

$$\begin{cases} -\nabla \cdot A(y)\nabla v_{bl}^{\xi} &= 0, \qquad\quad y\cdot n - a + \xi \cdot n > 0 \\ \qquad\qquad v_{bl}^{\xi} &= v_0(y), \quad y \cdot n - a + \xi\cdot n = 0 \end{cases}$$

satisfies, by periodicity of the coefficients and of the Dirichlet data, $v_{bl}^{\xi}(\cdot) := v_{bl}(\cdot + \xi)$. Hence, $v_{bl}^{a-\xi\cdot n,\infty} = v_{bl}^{a,\infty}$. As $n \notin \mathbb{R}\mathbb{Q}^d$, the set $\{\xi\cdot n,\ \xi \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}$. The independence of $v_{bl}^{a,\infty}$ of a follows now from the continuity of $a \mapsto v_{bl}^{a,\infty}$.

The rest of this section is devoted to the proof of proposition 4.20. Let as before $a,\ a' \in \mathbb{R}$ and $\nu := |a'-a|$. Let v_{bl} be the solution of (4.1) and v'_{bl} the solution of (4.1) in the domain $\Omega_{n,a'}$ instead of $\Omega_{n,a}$. Then $v_{bl} = v_{bl}^a(\cdot - an)$ (resp. $v_{bl} = v_{bl}^{a'}(\cdot - a'n)$) where v_{bl}^a (resp. $v_{bl}^{a'}$) solves

$$\begin{cases} -\nabla \cdot A(y+an)\nabla v_{bl}^{a} &= 0, \qquad\qquad\quad y\cdot n > 0 \\ \qquad\qquad\quad v_{bl}^{a} &= v_0(y+an), \quad y \cdot n = 0 \end{cases} \tag{4.74}$$

(resp. (4.74) with a replaced by a'). Take $\varphi \in C_c^\infty(\mathbb{R})$, compactly supported in $[-1,1]$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Then $\tilde{v}_{bl}^a := v_{bl}^a - \varphi(y\cdot n)v_0(y+na)$ (resp. $\tilde{v}_{bl}^{a'} := v_{bl}^{a'} - \varphi(y\cdot n)v_0(y+na')$) solves

$$\begin{cases} -\nabla \cdot A(y+an)\nabla \tilde{v}_{bl}^{a} &= \nabla\cdot A(y+na)\nabla\left(\varphi(y\cdot n)v_0(y+na)\right), \quad y\cdot n > 0 \\ \qquad\qquad\quad \tilde{v}_{bl}^{a} &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y \cdot n = 0 \end{cases} \tag{4.75}$$

(resp. (4.75) with a replaced by a'). One important point is that the source term in (4.75) is compactly supported in the direction normal to the boundary. We now estimate

$$v_{bl}^a(y) - v_{bl}^{a'}(y) = \tilde{v}_{bl}^a(y) - \tilde{v}_{bl}^{a'}(y) + \varphi(y\cdot n)\left[v_0(y+na) - v_0(y+na')\right]$$

for all $y \in \Omega_n$. We have,

$$\begin{aligned} &\tilde{v}_{bl}^a(y) - \tilde{v}_{bl}^{a'}(y) \\ &\quad= \int_{\Omega_n} G^a(y,\tilde{y})\nabla\cdot A(\tilde{y}+na)\nabla\left(\varphi(\tilde{y}\cdot n)v_0(\tilde{y}+na)\right)d\tilde{y} \\ &\qquad - \int_{\Omega_n} G^{a'}(y,\tilde{y})\nabla\cdot A(\tilde{y}+na')\nabla\left(\varphi(\tilde{y}\cdot n)v_0(\tilde{y}+na')\right)d\tilde{y} \\ &\quad= \int_{\Omega_n} \left[G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y})\right]\nabla\cdot A(\tilde{y}+na)\nabla\left(\varphi(\tilde{y}\cdot n)v_0(\tilde{y}+na)\right)d\tilde{y} &&(4.76a) \\ &\qquad + \int_{\Omega_n} G^{a'}(y,\tilde{y})\nabla\cdot\left[A(\tilde{y}+na) - A(\tilde{y}+na')\right]\nabla\left(\varphi(\tilde{y}\cdot n)v_0(\tilde{y}+na)\right)d\tilde{y} &&(4.76b) \\ &\qquad + \int_{\Omega_n} G^{a'}(y,\tilde{y})\nabla\cdot A(\tilde{y}+na')\nabla\left(\varphi(\tilde{y}\cdot n)\left[v_0(\tilde{y}+na) - v_0(\tilde{y}+na')\right]\right)d\tilde{y}. &&(4.76c) \end{aligned}$$

We analyse the terms in r.h.s. separately. The first, (4.76a) deserves more attention. By estimate (4.69b) and the usual change of variables $\tilde{z} = \mathrm{M}^T \tilde{y}$, for $y \cdot n \geq 1$,

$$\begin{aligned}
&\left| \int_{\Omega_n} \left[G^a(y,\tilde{y}) - G^{a'}(y,\tilde{y}) \right] \nabla \cdot A(\tilde{y}+na) \nabla \left(\varphi(\tilde{y}\cdot n) v_0(\tilde{y}+na) \right) d\tilde{y} \right| \\
&\qquad \leq C\nu \int_{\Omega_n} \frac{(y\cdot n)^{\mu} (\tilde{y}\cdot n)^{\mu}}{|y-\tilde{y}|^{d-2+2\mu}} \left| \nabla \cdot A(\tilde{y}+na) \nabla \left(\varphi(\tilde{y}\cdot n) v_0(\tilde{y}+na) \right) \right| d\tilde{y} \\
&\qquad \leq C\nu \, (y\cdot n)^{\mu} \int_{\mathbb{R}^{d-1}} \frac{1}{\left[(y\cdot n - 1)^2 + |\tilde{z}'|^2 \right]^{\frac{d-2+2\mu}{2}}} d\tilde{z}' \\
&\qquad \leq C\nu \frac{1}{(y\cdot n)^{-1+\mu}} \int_{\mathbb{R}^{d-1}} \frac{1}{\left[1 + |u'|^2 \right]^{\frac{d-2+2\mu}{2}}} du'.
\end{aligned}$$

We need, $2\mu > 1$ for the integral to be convergent, and $\mu < 1$ for the r.h.s. to be bounded when $y\cdot n \to \infty$. We now work with such a μ. For (4.76b), we rely on (4.31c): for $y \cdot n \geq 1$,

$$\begin{aligned}
&\left| \int_{\Omega_n} G^{a'}(y,\tilde{y}) \nabla \cdot \left[A(\tilde{y}+na) - A(\tilde{y}+na') \right] \nabla \left(\varphi(\tilde{y}\cdot n) v_0(\tilde{y}+na) \right) d\tilde{y} \right| \\
&\qquad\qquad\qquad\qquad \leq C\nu \int_{\mathbb{R}^{d-1}} \frac{1}{\left[1 + |u'|^2 \right]^{\frac{d}{2}}} du'
\end{aligned}$$

which is a convergent integral. For (4.76c), we argue analogously. We end with

$$\left| v_{bl}^a(y) - v_{bl}^{a'}(y) \right| \leq \left| \tilde{v}_{bl}^a(y) - \tilde{v}_{bl}^{a'}(y) \right| + \varphi(y\cdot n) \left| v_0(y+na) - v_0(y+na') \right| \leq C\nu,$$

for $y \cdot n \geq 1$, which proves proposition 4.20 letting $y \cdot n \to \infty$. This concludes the proof of theorem 4.2.

4.7 Almost arbitrarily slow convergence

We exhibit examples in dimension $d = 2$ showing that in general the convergence of v_{bl} towards its boundary layer tail v_{bl}^{∞} can be nearly arbitrarily slow. Let us, for the rest of this section, focus on the case when $n \notin \mathbb{R}\mathbb{Q}^2$. We take $d = 2$, $N = 1$, $A = I_2$ and study the unique variational solution v of

$$\left\{ \begin{array}{rll} -\Delta_z v = & 0, & z_2 > 0 \\ v(z) = & v_0(\mathrm{N} z_1), & z_2 = 0 \end{array} \right. ,$$

where as usual $\mathrm{N} \in \mathbb{R}^2$ is the first column vector of an orthogonal matrix M sending e_2 on n. From theorem 4.5 we know that $v = v(z_1, z_2) = V(\mathrm{N} z_1, z_2)$, with $V = V(\theta, t) \in \mathbb{R}$, $(\theta, t) \in \mathbb{T}^2 \times \mathbb{R}_+$, solving

$$\left\{ \begin{array}{rll} -\left| \begin{matrix} \mathrm{N}\cdot\nabla_\theta \\ \partial_t \end{matrix} \right|^2 V = & 0, & t > 0 \\ V(\theta, t) = & v_0(\theta), & t = 0 \end{array} \right. . \tag{4.77}$$

Expanding v_0 in Fourier series yields for all $\theta \in \mathbb{T}^2$,

$$v_0(\theta) = \sum_{\xi \in \mathbb{Z}^2} \widehat{v_0}(\xi) e^{2i\pi \xi \cdot \theta},$$

where $\left(\widehat{v_0}(\xi)\right)_\xi \in l^2(\mathbb{Z};\mathbb{R})$. From (4.20), in particular $\partial_t V \in L^2(\mathbb{T}^2 \times \mathbb{R}_+)$, it comes for all $(\theta, t) \in \mathbb{T}^2 \times \mathbb{R}_+$,

$$V(\theta,t) = \sum_{\xi\in\mathbb{Z}^2} \widehat{v_0}(\xi) e^{-2\pi|\mathrm{N}\cdot\xi|t} e^{2i\pi\xi\cdot\theta}.$$

Parceval's equality

$$\left\|V(\theta,t) - \widehat{v_0}(0)\right\|^2_{L^2(\mathbb{T}^2)} = \sum_{\xi\in\mathbb{Z}^2\setminus\{0\}} \left|\widehat{v_0}(\xi)\right|^2 e^{-4\pi|\mathrm{N}\cdot\xi|t} \tag{4.78}$$

together with Lebesgue's dominated convergence theorem prove that

$$\left\|V(\theta,t) - \widehat{v_0}(0)\right\|^2_{L^2(\mathbb{T}^2)} \overset{t\to\infty}{\longrightarrow} 0.$$

As v_0 is a C^∞ function, its Fourier coefficients $\left(\widehat{v_0}(\xi)\right)_\xi$ go to zero when $|\xi| \to \infty$, faster than any negative power of $|\xi|$. It follows from this, for all $\alpha \in \mathbb{N}^2$,

$$\left\|\partial^\alpha_\theta \left(V(\theta,t) - \widehat{v_0}(0)\right)\right\|^2_{L^2(\mathbb{T}^2)} \overset{t\to\infty}{\longrightarrow} 0. \tag{4.79}$$

Using Sobolev's injections, one notices that (4.79) proves again the convergence of v towards the boundary layer tail $v^\infty_{bl} := \widehat{v_0}(0)$.

Assume for a moment that n satisfies the small divisors condition (4.12). Let us come back to (4.78). For all $m \in \mathbb{N}$,

$$\begin{aligned}
\left\|V(\theta,t) - \widehat{v_0}(0)\right\|^2_{L^2(\mathbb{T}^2)} &= t^{-m} \sum_{\xi\in\mathbb{Z}^2\setminus\{0\}} \left|\widehat{v_0}(\xi)\right|^2 t^m e^{-4\pi|\mathrm{N}\cdot\xi|t} \\
&= t^{-m} \sum_{\xi\in\mathbb{Z}^2\setminus\{0\}} \frac{\left|\widehat{v_0}(\xi)\right|^2}{|\mathrm{N}\cdot\xi|^m} \left(|\mathrm{N}\cdot\xi|t\right)^m e^{-4\pi|\mathrm{N}\cdot\xi|t} \\
&\leq Ct^{-m} \sum_{\xi\in\mathbb{Z}^2\setminus\{0\}} \left|\widehat{v_0}(\xi)\right|^2 |\xi|^{(2+\tau)m} \left(|\mathrm{N}\cdot\xi|t\right)^m e^{-4\pi|\mathrm{N}\cdot\xi|t} \\
&\leq Ct^{-m} \sum_{\xi\in\mathbb{Z}^2\setminus\{0\}} \left|\widehat{v_0}(\xi)\right|^2 |\xi|^{(2+\tau)m} \leq C_m t^{-m},
\end{aligned} \tag{4.80}$$

the function $t \mapsto \left(|\mathrm{N}\cdot\xi|t\right)^m e^{-4\pi|\mathrm{N}\cdot\xi|t}$ being bounded on $\mathbb{R}_+$. We have shown that $V(\cdot,t)$ converges to $\widehat{v_0}(0)$ in $L^2\left(\mathbb{T}^2\right)$, faster than every negative power of t.

Assume now that $n \notin \mathbb{R}\mathbb{Q}^2$ does not verify (4.12) and let $l > 0$. Hence there are points of the lattice $\mathbb{Z}^2$, except 0, which are as close to the line $\mathrm{N}\cdot y = 0$ as we wish. The sum in the r.h.s. of (4.78) does not necessarily keep the trace of the exponential behaviour of its terms. We show that the convergence is at least as slow as the convergence of $t \mapsto t^{-l}$ towards 0 at ∞. In fact we aim at proving:

Theorem 4.22. *Assume that $n \notin \mathbb{R}\mathbb{Q}^2$ does not satisfy* (4.12).
Then, for every $l > 0$, there exists a smooth v_0 and a strictly increasing sequence $(t_M)_{M\geq 1}$ of positive real numbers, tending to ∞, such that for all $\alpha \in \mathbb{N}^2$, for all $M \in \mathbb{N}\setminus\{0\}$,

$$\left\|\partial^\alpha_\theta \left(V(\theta,t_M) - \widehat{v_0}(0)\right)\right\|_{L^2(\mathbb{T}^2)} \geq t_M^{-l},$$

where V is the solution of (4.77) *associated to v_0.*

Let us insist on the fact that theorem 4.22 holds for any $n \notin \mathbb{R}\mathbb{Q}^2$, which does not satisfy the small divisors assumption. The idea of the proof is to choose a family $\left(\widehat{v_0}(\xi)\right)_\xi$, whose support, that is the set of subscripts $\xi \in \mathbb{Z}^2$ such that $\widehat{v_0}(\xi) \neq 0$, is sufficiently close to the line $\mathrm{N} \cdot y = 0$. We now construct a suitable sequence $(\xi_M)_{M\geq 1}$ of $\mathbb{Z}^2 \setminus \{0\}$. The small divisors assumptions being not verified, for all $M \in \mathbb{N} \setminus \{0\}$, there exists $\xi \in \mathbb{Z}^2 \setminus \{0\}$ such that

$$|\mathrm{N} \cdot \xi| < \frac{1}{M}|\xi|^{-M}. \tag{4.81}$$

One can construct (ξ_M) recursively:

$$\xi_1 := \underset{\substack{\xi\in\mathbb{Z}^2\setminus\{0\}\\ |\mathrm{N}\cdot\xi|<|\xi|^{-1}}}{\operatorname{argmin}} |\xi|, \quad \xi_2 := \underset{\substack{\xi\in\mathbb{Z}^2\setminus\{0\}\\ |\xi_2|>|\xi_1|+1\\ |\mathrm{N}\cdot\xi|<\frac{1}{2}|\xi|^{-2}}}{\operatorname{argmin}} |\xi|, \quad \ldots, \quad \xi_M := \underset{\substack{\xi\in\mathbb{Z}^2\setminus\{0\}\\ |\xi_M|>|\xi_{M-1}|+1\\ |\mathrm{N}\cdot\xi|<\frac{1}{M}|\xi|^{-M}}}{\operatorname{argmin}} |\xi|,$$

where argmin stands for a minimizor. The existence of a minimizor is ensured by (4.81). For ξ_1 the reasoning is straightforward. Let us sketch the proof of the existence of ξ_2, which immediately applies, with minor modifications, for all ξ_M. We assume that for all $|\xi| > |\xi_1| + 1$, $|\mathrm{N} \cdot \xi| \geq \frac{1}{2}|\xi|^{-2}$. Then, for all $M \geq 2$, $|\mathrm{N} \cdot \xi| \geq \frac{1}{2}|\xi|^{-2} \geq \frac{1}{M}|\xi|^{-M}$. According to (4.81), this shows that there exists $\xi \in \mathbb{Z}^2 \setminus \{0\}$, $|\xi| \leq |\xi_1| + 1$, such that, for all $M \geq 1$, $|\mathrm{N} \cdot \xi| < \frac{1}{M}|\xi|^{-M}$. For this $\xi \neq 0$, $\mathrm{N} \cdot \xi = 0$, which is incompatible with $n \notin \mathbb{R}\mathbb{Q}^2$. We have thus built a sequence $(\xi_M)_{M\geq 1}$ of vectors of $\mathbb{Z}^2 \setminus \{0\}$ satisfying:

1. $(|\xi_M|)_M$ is strictly increasing;
2. for all $M \geq 1$, $|\xi_M| \geq M$;
3. for all $M \geq 1$, $|\mathrm{N} \cdot \xi_M| < \frac{1}{M}|\xi_M|^{-M}$.

We now come to the construction of v_0 keeping in mind that v_0 has to be smooth. For all $\xi \in \mathbb{Z}^2$, we define

$$\widehat{v_0}(\xi) := \begin{cases} 0, & \text{if } \xi \neq \xi_M,\ -\xi_M \text{ for all } M \geq 1 \\ M^{-l}|\xi_M|^{-Ml}, & \text{if } \xi = \xi_M \text{ or } -\xi_M \end{cases}.$$

Thanks to the construction of the sequence $(\xi_M)_M$, for all $m \in \mathbb{N}$, $\widehat{v_0}(\xi) = O\left(|\xi|^{-m}\right)$. Thus v_0 defined like this is a $C^\infty\left(\mathbb{T}^2\right)$ function and $V = V(\theta, t)$ defined by for all $\theta \in \mathbb{T}^2$, for all $t \geq 0$,

$$V(\theta, t) := 2\sum_{M\geq 1} M^{-l}|\xi_M|^{-Ml} e^{-2\pi|\mathrm{N}\cdot\xi_M|t} \cos(2\pi\xi_M \cdot \theta)$$

is a smooth solution to (4.77).

For all $M \geq 1$, let $t_M := l\frac{M|\xi_M|^M}{2\pi}$. The final step in the proof of theorem 4.22 is to estimate $\left\|\partial_\theta^\alpha \left(V(\theta, t_M) - \widehat{v_0}(0)\right)\right\|_{L^2(\mathbb{T}^2)}$ for $\alpha \in \mathbb{N}^2$ and $M \geq 1$. Of course, in our example $\widehat{v_0}(0) = 0$. Yet one is free to modify this coefficient without changing anything to the nature of the problem. Let $\alpha \in \mathbb{N}^2$. By Lebegue's dominated convergence theorem, for all $(\theta, t) \in \mathbb{T}^2 \times \mathbb{R}_+$,

$$\partial_\theta^\alpha V(\theta, t) = 2\sum_{M\geq 1} M^{-l}|\xi_M|^{-Ml}(2\pi)^{|\alpha|}\xi_M^\alpha e^{-2\pi|\mathrm{N}\cdot\xi_M|t} \cos^{(|\alpha|)}(2\pi\xi_M \cdot \theta),$$

and Parceval's inequality yields

$$\left\|\partial_\theta^\alpha V(\theta, t)\right\|_{L^2(\mathbb{T}^2)}^2 = 2\sum_{M\geq 1} M^{-2l}|\xi_M|^{-2Ml}(2\pi)^{2|\alpha|}|\xi_M^\alpha|^2 e^{-4\pi|\mathrm{N}\cdot\xi_M|t}. \tag{4.82}$$

Due to our choice of sequence $(\xi_M)_M$, in particular because of the second property of (ξ_M) listed above, there exists $M^{(0)} \geq 1$, such that for all $M \geq M^{(0)}$, $\xi_{M,1} \neq 0$ and $\xi_{M,2} \neq 0$, where $\xi_M = (\xi_{M,1}, \xi_{M,2})$; for all $M \geq M^{(0)}$, $|\xi_M^\alpha| = |\xi_{M,1}^{\alpha_1}||\xi_{M,2}^{\alpha_2}| \geq 1$. Therefore, (4.82) yields for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\left\|\partial_\theta^\alpha V(\theta,t)\right\|^2_{L^2(\mathbb{T}^2)} &\geq 2\sum_{M\geq M^{(0)}} M^{-2l}|\xi_M|^{-2Ml}(2\pi)^{2|\alpha|}|\xi_M^\alpha|^2 e^{-4\pi|\mathrm{N}\cdot\xi_M|t}\\ &\geq 2\sum_{M\geq M^{(0)}} M^{-2l}|\xi_M|^{-2Ml} e^{-4\pi|\mathrm{N}\cdot\xi_M|t}\\ &\geq \frac{2}{(4\pi)^{2l}}t^{-2l}\sum_{M\geq M^{(0)}}\left(\frac{4\pi t}{M|\xi_M|^M}\right)^{2l} e^{-\frac{4\pi}{M|\xi_M|^M}t},\end{aligned}$$

and for all $M \geq M^{(0)}$, $\left\|\partial_\theta^\alpha V(\theta,t_M)\right\|_{L^2(\mathbb{T}^2)} \geq \sqrt{2}\left(\frac{e^{-1}l}{2\pi}\right)^l t_M^{-l}$, which proves theorem 4.22.

Theorem 4.22 prevents $V(\theta,t)$ from decaying fast towards v_{bl}^∞ when $t \to \infty$: in particular, (4.80) is impossible, in general, when n does not meet the small divisors assumption. However, theorem 4.3 cannot be deduced from theorem 4.22. Let us turn to estimates in L^∞ norm for v. Uniformity in the tangential variable θ is replaced by local uniformity in z_1. Let $R > 0$. We slightly modify $\left(\widehat{v_0}(\xi)\right)_\xi$. As for all $|z_1| \leq R$,

$$2\pi|\xi_M\cdot\mathrm{N}||z_1| < \frac{2\pi R}{M}|\xi_M|^{-M} \leq \frac{2\pi R}{M} < \frac{\pi}{4},$$

for all $M \geq M^{(1)}$ sufficiently large, depending on R, we consider $\tilde{v}_0$ defined by

$$\widehat{\tilde{v}_0}(\xi) := \begin{cases} 0, & \text{if } \xi \neq \xi_M,\ -\xi_M \text{ for all } M \geq 1\\ 0, & \text{if } \xi = \xi_M \text{ or } -\xi_M \text{ for } 1 \leq M < M^{(1)}\\ M^{-l}|\xi_M|^{-Ml}, & \text{if } \xi = \xi_M \text{ or } -\xi_M \text{ for } M \geq M^{(1)}\end{cases}.$$

Then, for all $|z_1| \leq R$, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}v(z_1,t) &= 2\sum_{M\geq M^{(1)}} M^{-l}|\xi_M|^{-Ml}e^{-2\pi|\mathrm{N}\cdot\xi_M|t}\cos(2\pi\xi_M\cdot\mathrm{N}z_1)\\ &\geq \frac{2}{(2\pi)^l}t^{-l}\sum_{M\geq M^{(1)}}\left(\frac{2\pi t}{M|\xi_M|^M}\right)^l e^{-\frac{2\pi}{M|\xi_M|^M}t}\end{aligned}$$

which demonstrates theorem 4.3 when evaluated in $t = t_M$ for $M \geq M^{(1)}$.

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Chapter 5

Well-posedness of the Stokes-Coriolis system in the half-space over a rough surface

This chapter corresponds to the paper [DP13], submitted.
Joint work with Anne-Laure Dalibard

Contents

Abstract This chapter is devoted to the well-posedness of the stationary 3d Stokes-Coriolis system set in a half-space with rough bottom and Dirichlet data which does not decrease at space infinity. Our system is a linearized version of the Ekman boundary layer system. We look for a solution of infinite energy in a space of Sobolev regularity. Following an idea of Gérard-Varet and Masmoudi, the general strategy is to reduce the problem to a bumpy channel bounded in the vertical direction thanks a transparent boundary condition involving a Dirichlet to Neumann operator. Our analysis emphasizes some strong

singularities of the Stokes-Coriolis operator at low tangential frequencies. One of the main features of our work lies in the definition of a Dirichlet to Neumann operator for the Stokes-Coriolis system with data in the Kato space $H^{1/2}_{uloc}$.

5.1 Introduction

The goal of the present paper is to prove the existence and uniqueness of solutions to the Stokes-Coriolis system

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u|_\Gamma &= u_0 \end{cases} \tag{5.1}$$

where

$$\Omega := \{x \in \mathbb{R}^3,\ x_3 > \omega(x_h)\},$$
$$\Gamma = \partial\Omega = \{x \in \mathbb{R}^3,\ x_3 = \omega(x_h)\}$$

and $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth bounded function.

When ω has some structural properties, such as periodicity, existence and uniqueness of solutions are easy to prove: our aim here is to prove well-posedness when the function ω is arbitrary, say $\omega \in W^{1,\infty}(\mathbb{R}^2)$, and when the boundary data u_0 is not square integrable. More precisely, we wish to work with u_0 in a space of infinite energy of Sobolev regularity, such as Kato spaces. We refer to the end of this introduction for a definition of these uniformly locally Sobolev spaces L^2_{uloc}, H^s_{uloc}.

The interest for such function spaces to study fluid systems goes back to the papers by Lieumarié-Rieusset [LR99, LR02], in which existence is proved for weak solutions of the Navier-Stokes equations in $\mathbb{R}^3$ with initial data in L^2_{uloc}. These works fall into the analysis of fluid flows with infinite energy, which is an field of intense research. Without being exhaustive, let us quote the works of:

- Cannon and Knightly [CK70], Giga, Inui and Matsui [GIM99], Solonnikov [Sol03], Bae and Jin [BJ12] (local solutions), Giga, Matsui and Sawada [GMS01] (global solutions) on the nonstationary Navier-Stokes system in the whole space or in the half-space with initial data in L^∞ or in BUC (bounded uniformly continuous);
- Basson [Bas06], Maekawa and Terasawa [MT06] on local solutions of the nonstationary Navier-Stokes system in the whole space with initial data in L^p_{uloc} spaces;
- Giga and Miyakawa [GM89], Taylor [Tay92] (global solutions), Kato [Kat92] on local solutions to the nonstationary Navier-Stokes system, and Gala [Gal05] on global solutions to a quasi-geostrophic equation, with initial data in Morrey spaces;
- Gallagher and Planchon [GP02] on the nonstationary Navier-Stokes system in $\mathbb{R}^2$ with initial data in the homogeneous Besov space $\dot{B}^{2/r-1}_{r,q}$;
- Giga and co-authors [GIM$^+$07] on the nonstationary Ekman system in $\mathbb{R}^3_+$ with initial data in the Besov space $\dot{B}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$, for $2 < p < \infty$; see also [GIMM06] (local solutions), [GIMS08] (global solutions) on the Navier-Stokes-Coriolis system in $\mathbb{R}^3$ and the survey of Yoneda [Yon09] for initial data spaces containing almost-periodic functions;
- Konieczny and Yoneda [KY11] on the stationary Navier-Stokes system in Fourier-Besov spaces.

Despite this huge literature on initial value problems in fluid mechanics in spaces of infinite energy, we are not aware of such work concerning stationary systems and non homogeneous boundary value problems in $\mathbb{R}^3_+$. Let us emphasize that the derivation of L^2 bounds in stationary and time dependent settings are rather different: indeed, in a time dependent setting, boundedness of the solution at time t follows from boundedness of the initial data and of the associated semi-group. In a stationary setting, to the best of our knowledge, the only way to derive estimates without assuming any structure on the function ω is based on the arguments of Ladyzhenskaya and Solonnikov [LS80] (see also [GVM10] for the Stokes system in a bumped half plane).

In the present case, our motivation comes from the asymptotic analysis of highly rotating fluids near a rough boundary. Indeed, consider the system

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + \dfrac{1}{\varepsilon} e_3 \times u^\varepsilon + \nabla p^\varepsilon &= 0 \qquad \text{in } \Omega^\varepsilon, \\ \operatorname{div} u^\varepsilon &= 0 \qquad \text{in } \Omega^\varepsilon, \\ u^\varepsilon|_{\Gamma^\varepsilon} &= 0, \\ u^\varepsilon|_{x_3=1} &= (V_h, 0), \end{cases} \tag{5.2}$$

where $\Omega^\varepsilon := \{x \in \mathbb{R}^3,\ \varepsilon\omega(x_h/\varepsilon) < x_3 < 1\}$ and $\Gamma^\varepsilon := \partial\Omega^\varepsilon \setminus \{x_3 = 1\}$. Then it is expected that u^ε is the sum of a two-dimensional interior flow $(u^{int}(x_h), 0)$ balancing the rotation with the pressure term and a boundary layer flow $u^{BL}(x/\varepsilon; x_h)$, located in the vicinity of the lower boundary. In this case, the equation satisfied by u^{BL} is precisely (5.1), with $u_0(y_h; x_h) = -(u^{int}(x_h), 0)$. Notice that x_h is the macroscopic variable and is a parameter in the equation on u^{BL}.

The system (5.2) models large-scale geophysical fluid flows in the linear régime. In order to get a physical insight into the physics of rotating fluids, we refer to the book by Greenspan [Gre69] (rotating fluids in general, including an extensive study of the linear régime) and to the one by Pedlosky [Ped87] (focus on geophysical fluids). In [Ekm05], Ekman analyses the effect of the interplay between viscous forces and the Coriolis acceleration on geophysical fluid flows.

For further remarks on the system (5.2), we refer to the book [CDGG06, section 7] by Chemin, Desjardins, Gallagher and Grenier, and to [CDGG02], where a model with anisotropic viscosity is studied and an asymptotic expansion for u^ε is obtained.

Studying (5.1) with an arbitrary function ω is more realistic from a physical point of view, and also allows us to bring to light some bad behaviours of the system at low horizontal frequencies, which are masked in a periodic setting.

Our main result is the following.

Theorem 5.1. *Let $\omega \in W^{1,\infty}(\mathbb{R}^2)$, and let $u_{0,h} \in H^2_{uloc}(\mathbb{R}^2)^2$, $u_{0,3} \in H^1_{uloc}(\mathbb{R}^2)$. Assume that there exists $U_h \in H^{1/2}_{uloc}(\mathbb{R}^2)^2$ such that*

$$u_{0,3} - \nabla_h\omega \cdot u_{0,h} = \nabla_h \cdot U_h. \tag{5.3}$$

Then there exists a unique solution u of (5.1) such that

$$\forall a > 0, \quad \sup_{l\in\mathbb{Z}^2} \|u\|_{H^1(((l+[0,1]^2)\times(-1,a))\cap\Omega)} < \infty,$$
$$\sup_{l\in\mathbb{Z}^2} \sum_{\alpha\in\mathbb{N}^3, |\alpha|=q} \int_1^\infty \int_{l+[0,1]^2} |\nabla^\alpha u|^2 < \infty$$

for some integer q sufficiently large, which does not depend on ω nor u_0 (say $q \geq 4$).

Let us comment on this theorem:

- Assumption (5.3) is a compatibility condition, which stems from singularities at low horizontal frequencies in the system. When the bottom is flat, it merely becomes $u_{0,3} = \nabla_h \cdot U_h$. Notice that this condition only bears on the normal component of the velocity at the boundary: in particular, if $u_0 \cdot n|_\Gamma = 0$, then (5.3) is satisfied. We also stress that (5.3) is satisfied in the framework of highly rotating fluids near a rough boundary, since in this case $u_{0,3} = 0$ and $u_{0,h}$ is constant with respect to the microscopic variable.
- The singularities at low horizontal frequencies also account for the possible lack of integrability of the gradient far from the rough boundary: we were not able to prove that
$$\sup_{l\in\mathbb{Z}^2} \int_1^\infty \int_{l+[0,1]^2} |\nabla u|^2 < \infty$$
although this estimate is true for the Stokes system. In fact, looking closely at our proof, it seems that non-trivial cancellations should occur for such a result to hold in the Stokes-Coriolis case.
- Concerning the regularity assumptions on ω and u_0, it is classical to assume Lipschitz regularity on the boundary. The regularity required on u_0, however, may not be optimal, and stems in the present context from an explicit lifting of the boundary condition. It is possible that the regularity could be lowered if a different type of lifting were used.
- The same tools can be used to prove a similar result for the Stokes system in three dimensions (we recall that the paper [GVM10] is concerned with the Stokes system in two dimensions). In fact, the treatment of the Stokes system is easier, because the associated kernel is homogeneous and has no singularity at low frequencies. The results proved in Section 5.2 can be obtained thanks to the Green function associated with the Stokes system in three dimensions (see [Gal94]). On the other hand, the arguments of sections 5.3 and 5.4 of the present paper can be transposed as such to the Stokes system in 3d. The main novelties of these sections, which rely on careful energy estimates, are concerned with the higher dimensional space rather than with the presence of the rotation term (except for Lemma 5.32).

The statement of Theorem 5.1 is very close to one of the main results of the paper [GVM10] by Gérard-Varet and Masmoudi, namely the well-posedness of the Stokes system in a bumped half-plane with boundary data in $H^{1/2}_{uloc}(\mathbb{R})$. Of course, it shares the main difficulties of [GVM10]: spaces of functions of infinite energy, lack of a Poincaré inequality, irrelevancy of scalar tools (Harnack inequality, maximum principle) which do not apply to systems. But two additional problems are encountered when studying (5.1):

1. First, (5.1) is set in three dimensions, whereas the study of [GVM10] took place in 2d. This complicates the derivation of energy estimates. Indeed, the latter are based on the truncation method by Ladyzhenskaya and Solonnikov [LS80], which consists more or less in multiplying (5.1) by $\chi_k u$, where $\chi_k \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1})$ is a cut-off function in the horizontal variables such that $\operatorname{Supp} \chi_k \subset B_{k+1}$ and $\chi_k \equiv 1$ on B_k, for $k \in \mathbb{N}$. If $d = 2$, the size of the support of $\nabla \chi_k$ is bounded, while it is unbounded when $d = 3$. This has a direct impact on the treatment of some commutator terms.

2. Somewhat more importantly, the kernel associated with the Stokes-Coriolis operator has a more complicated expression than the one associated with the Stokes operator (see [Gal94, Chapter IV] for the computation of the Green function associated to the Stokes system in the half-space). In the case of the Stokes-Coriolis operator, the kernel is not homogeneous, which prompts us to distinguish between high and low

horizontal frequencies throughout the paper. Moreover, it exhibits strong singularities at low horizontal frequencies, which have repercussions on the whole proof and account for assumption (5.3).

The proof of Theorem 5.1 follows the same general scheme as in [GVM10] (this scheme has also been successfully applied in [DGV11] in the case of a Navier slip boundary condition on the rough bottom): we first perform a thorough analysis of the Stokes-Coriolis system in $\mathbb{R}^3_+$, and we define the associated Dirichlet to Neumann operator for boundary data in $H^{1/2}_{uloc}$. In particular, we derive a representation formula for solutions of the Stokes-Coriolis system in $\mathbb{R}^3_+$, based on a decomposition of the kernel which distinguishes high and low frequencies, and singular/regular terms. We also prove a similar representation formula for the Dirichlet to Neumann operator. Then, we derive an equivalent system to (5.1), set in a domain which is bounded in x_3 and in which a transparent boundary condition is prescribed on the upper boundary. These two preliminary steps are performed in Section 5.2. We then work with the equivalent system, for which we derive energy estimates in H^1_{uloc}; this allows us to prove existence in Section 5.3. Eventually, we prove uniqueness in Section 5.4. An Appendix gathers several technical lemmas used throughout the paper.

Notations

We will be working with spaces of uniformly locally integrable functions, called Kato spaces, whose definition we now recall. For $d \in \mathbb{N}$, $p \in [1,\infty)$, $s > 0$, we set

$$\begin{aligned} L^p_{uloc}(\mathbb{R}^d) &:= \{u \in L^p_{loc}(\mathbb{R}^d),\ \sup_{k\in\mathbb{Z}^d} \|u\|_{L^p(k+(0,1)^d)} < \infty\},\\ H^s_{uloc}(\mathbb{R}^d) &:= \{u \in H^s_{loc}(\mathbb{R}^d),\ \sup_{k\in\mathbb{Z}^d} \|u\|_{H^s(k+(0,1)^d)} < \infty\}. \end{aligned}$$

We will also work in the domain $\Omega^b := \{x \in \mathbb{R}^3,\ \omega(x_h) < x_3 < 0\}$, assuming that ω takes values in $(-1,0)$. With a slight abuse of notation, we will write

$$\begin{aligned} \|u\|_{L^p_{uloc}(\Omega^b)} &:= \sup_{k\in\mathbb{Z}^2} \|u\|_{L^p(((k+[0,1]^2)\times(\inf\omega,0))\cap\Omega^b)},\\ \|u\|_{H^s_{uloc}(\Omega^b)} &:= \sup_{k\in\mathbb{Z}^2} \|u\|_{H^s(((k+[0,1]^2)\times(\inf\omega,0))\cap\Omega^b)}, \end{aligned}$$

and $H^s_{uloc}(\Omega^b) = \{u \in H^s_{loc}(\Omega^b),\ \|u\|_{H^s_{uloc}(\Omega^b)} < \infty\}$, $L^p_{uloc}(\Omega^b) = \{u \in L^p_{loc}(\Omega^b),\ \|u\|_{L^p_{uloc}(\Omega^b)} < \infty\}$.

Throughout the proof, we will often use the notation $|\nabla^q u|$, where $q \in \mathbb{N}$, for the quantity

$$\sum_{\alpha\in\mathbb{N}^d, |\alpha|=q} |\nabla^\alpha u|,$$

where $d = 2$ or 3, depending on the context.

5.2 Presentation of a reduced system and main tools

Following an idea of David Gérard-Varet and Nader Masmoudi [GVM10], the first step is to transform (5.1) so as to work in a domain bounded in the vertical direction (rather than a half-space). This allows us eventually to use Poincaré inequalities, which are paramount in the proof. To that end, we introduce an artificial flat boundary above the rough surface Γ, and we replace the Stokes-Coriolis system in the half-space above the artificial boundary by a transparent boundary condition, expressed in terms of a Dirichlet to Neumann operator.

In the rest of the article, without loss of generality, we assume that $\sup\omega =: \alpha < 0$ and $\inf\omega \geq -1$, and we place the artificial boundary at $x_3 = 0$. We set

$$\begin{aligned}\Omega^b &:= \{x \in \mathbb{R}^3,\ \omega(x_h) < x_3 < 0\},\\ \Sigma &:= \{x_3 = 0\}.\end{aligned}$$

The Stokes-Coriolis system differs in several aspects from the Stokes system; in the present paper, the most crucial differences are the lack of an explicit Green function, and the bad behaviour of the system at low horizontal frequencies. The main steps of the proof are as follows:

1. Prove existence and uniqueness of a solution of the Stokes-Coriolis system in a half-space with a boundary data in $H^{1/2}(\mathbb{R}^2)$;
2. Extend this well-posedness result to boundary data in $H^{1/2}_{uloc}(\mathbb{R}^2)$;
3. Define the Dirichlet to Neumann operator for functions in $H^{1/2}(\mathbb{R}^2)$, and extend it to functions in $H^{1/2}_{uloc}(\mathbb{R}^2)$;
4. Define an equivalent problem in Ω^b, with a transparent boundary condition at Σ, and prove the equivalence between the problem in Ω^b and the one in Ω;
5. Prove existence and uniqueness of solutions of the equivalent problem.

Items 1-4 will be proved in the current section, and item 5 in sections 5.3 and 5.4.

5.2.1 The Stokes-Coriolis system in a half-space

The first step is to study the properties of the Stokes-Coriolis system in $\mathbb{R}^3_+$, namely

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p &= 0 \quad \text{in } \mathbb{R}^3_+,\\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3_+,\\ u|_{x_3=0} &= v_0. \end{cases} \tag{5.4}$$

In order to prove the result of Theorem 5.1, we have to prove the existence and uniqueness of a solution u of the Stokes-Coriolis system in $H^1_{loc}(\mathbb{R}^3_+)$ such that for some $q \in \mathbb{N}$ sufficiently large,

$$\sup_{l\in\mathbb{Z}^2} \int_{l+(0,1)^2} \int_1^\infty |\nabla^q u|^2 < \infty$$

However, the Green function for the Stokes-Coriolis is far from being explicit, and its Fourier transform, for instance, is much less well-behaved than the one of the Stokes system (which is merely the Poisson kernel). Therefore such a result is not so easy to prove. In particular, because of the singularities of the Fourier transform of the Green function at low frequencies, we are not able to prove that

$$\sup_{l\in\mathbb{Z}^2} \int_{l+(0,1)^2} \int_1^\infty |\nabla u|^2 < \infty.$$

- We start by solving the system when $v_0 \in H^{1/2}(\mathbb{R}^2)$. We have the following result:

Proposition 5.2. *Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ such that*

$$\int_{\mathbb{R}^2} \frac{1}{|\xi|} |\hat{v}_{0,3}(\xi)|^2 \, d\xi < \infty. \tag{5.5}$$

Then the system (5.4) *admits a unique solution* $u \in H^1_{loc}(\mathbb{R}^3_+)$ *such that*

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 < \infty.$$

Remark 5.3. The condition (5.5) stems from a singularity at low frequencies of the Stokes-Coriolis system, which we will encounter several times in the proof. Notice that (5.5) is satisfied in particular when $v_{0,3} = \nabla_h \cdot V_h$ for some $V_h \in H^{1/2}(\mathbb{R}^2)^2$, which is sufficient for further purposes.

Proof. • *Uniqueness.* Consider a solution whose gradient is in $L^2(\mathbb{R}^3_+)$ and with zero boundary data on $x_3 = 0$. Then, using the Poincaré inequality, we infer that

$$\int_0^a \int_{\mathbb{R}^2} |u|^2 \le C_a \int_0^a \int_{\mathbb{R}^2} |\nabla u|^2 < \infty,$$

and therefore we can take the Fourier transform of u in the horizontal variables. Denoting by $\xi \in \mathbb{R}^2$ the Fourier variable associated with x_h, we get

$$\begin{cases} (|\xi|^2 - \partial_3^2)\hat{u}_h + \hat{u}_h^\perp + i\xi\hat{p} &= 0, \\ (|\xi|^2 - \partial_3^2)\hat{u}_3 + \partial_3\hat{p} &= 0, \\ i\xi \cdot \hat{u}_h + \partial_3\hat{u}_3 &= 0, \end{cases} \tag{5.6}$$

and

$$\hat{u}|_{x_3=0} = 0.$$

Eliminating the pressure, we obtain

$$(|\xi|^2 - \partial_3^2)^2\hat{u}_3 - i\partial_3\xi^\perp \cdot \hat{u}_h = 0.$$

Taking the scalar product of the first equation in (5.6) with $(\xi^\perp, 0)$, and using the divergence-free condition, we are led to

$$(|\xi|^2 - \partial_3^2)^3\hat{u}_3 - \partial_3^2\hat{u}_3 = 0. \tag{5.7}$$

Notice that the solutions of this equation have a slightly different nature when $\xi \neq 0$ or when $\xi = 0$ (if $\xi = 0$, the associated characteristic polynomial has a multiple root at zero). Therefore, as in [GVM10] we introduce a function $\varphi = \varphi(\xi) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that the support of φ does not contain zero. Then $\varphi\hat{u}_3$ satisfies the same equation as $\hat{u}_3$, and vanishes in a neighbourhood of $\xi = 0$.

For $\xi \neq 0$, the solutions of (5.7) are linear combinations of $\exp(-\lambda_k x_3)$ (with coefficients depending on ξ), where $(\lambda_k)_{1\le k\le 6}$ are the complex valued solutions of the equation

$$(\lambda^2 - |\xi|^2)^3 + \lambda^2 = 0. \tag{5.8}$$

Notice that none of the roots of this equation is purely imaginary, and that if λ is a solution of (5.8), so are $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$. Additionally (5.8) has exactly one real valued positive solution. Therefore, without loss of generality we assume that λ_1, λ_2, λ_3 have strictly positive real part, while $\lambda_4, \lambda_5, \lambda_6$ have strictly negative real part, and $\lambda_1 \in \mathbb{R}$, $\bar{\lambda}_2 = \lambda_3$, with $\Im(\lambda_2) > 0$, $\Im(\lambda_3) < 0$.

On the other hand, the integrability condition on the gradient becomes

$$\int_{\mathbb{R}^3_+} (|\xi|^2|\hat{u}(\xi, x_3)|^2 + |\partial_3\hat{u}(\xi, x_3)|^2)d\xi\, dx_3 < \infty.$$

We infer immediately that $\varphi \hat{u}_3$ is a linear combination of $\exp(-\lambda_k x_3)$ for $1 \leq k \leq 3$: there exist $A_k : \mathbb{R}^2 \to \mathbb{C}^3$ for $k = 1, 2, 3$ such that

$$\varphi(\xi)\hat{u}_3(\xi, x_3) = \sum_{k=1}^{3} A_k(\xi) \exp(-\lambda_k(\xi) x_3).$$

Going back to (5.6), we also infer that

$$\begin{aligned} \varphi(\xi)\xi \cdot \hat{u}_h(\xi, x_3) &= -i \sum_{k=1}^{3} \lambda_k(\xi) A_k(\xi) \exp(-\lambda_k(\xi) x_3), \\ \varphi(\xi)\xi^\perp \cdot \hat{u}_h(\xi, x_3) &= i \sum_{k=1}^{3} \frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} A_k(\xi) \exp(-\lambda_k(\xi) x_3). \end{aligned} \tag{5.9}$$

Notice that by (5.8),

$$\frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} = \frac{\lambda_k}{|\xi|^2 - \lambda_k^2} \quad \text{for } k = 1, 2, 3.$$

Thus the boundary condition $\hat{u}|_{x_3=0} = 0$ becomes

$$M(\xi) \begin{pmatrix} A_1(\xi) \\ A_2(\xi) \\ A_3(\xi) \end{pmatrix} = 0,$$

where

$$M := \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \dfrac{(|\xi|^2 - \lambda_1^2)^2}{\lambda_1} & \dfrac{(|\xi|^2 - \lambda_2^2)^2}{\lambda_2} & \dfrac{(|\xi|^2 - \lambda_3^2)^2}{\lambda_3} \end{pmatrix}.$$

We have the following lemma:

Lemma 5.4.

$$\det M = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(|\xi| + \lambda_1 + \lambda_2 + \lambda_3).$$

Since the proof of the result is a mere calculation, we have postponed it to Appendix 5.A. It is then clear that M is invertible for all $\xi \neq 0$: indeed it is easily checked that all the roots of (5.8) are simple, and we recall that $\lambda_1, \lambda_2, \lambda_3$ have positive real part.

We conclude that $A_1 = A_2 = A_3 = 0$, and thus $\varphi(\xi)\hat{u}(\xi, x_3) = 0$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ supported far from $\xi = 0$. Since $\hat{u} \in L^2(\mathbb{R}^2 \times (0, a))^3$ for all $a > 0$, we infer that $\hat{u} = 0$.

• *Existence.* Now, given $v_0 \in H^{1/2}(\mathbb{R}^2)$, we define u through its Fourier transform in the horizontal variable. It is enough to define the Fourier transform for $\xi \neq 0$, since it is square integrable in ξ. Following the calculations above, we define coefficients A_1, A_2, A_3 by the equation

$$M(\xi) \begin{pmatrix} A_1(\xi) \\ A_2(\xi) \\ A_3(\xi) \end{pmatrix} = \begin{pmatrix} \hat{v}_{0,3} \\ i\xi \cdot \hat{v}_{0,h} \\ -i\xi^\perp \cdot \hat{v}_{0,h} \end{pmatrix} \quad \forall \xi \neq 0. \tag{5.10}$$

As stated in Lemma 5.4, the matrix M is invertible, so that A_1, A_2, A_3 are well defined. We then set

$$\begin{aligned} \hat{u}_3(\xi, x_3) &:= \sum_{k=1}^{3} A_k(\xi) \exp(-\lambda_k(\xi) x_3), \\ \hat{u}_h(\xi, x_3) &:= \frac{i}{|\xi|^2} \sum_{k=1}^{3} A_k(\xi) \left(-\lambda_k(\xi)\xi + \frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} \xi^\perp \right) \exp(-\lambda_k(\xi) x_3). \end{aligned} \tag{5.11}$$

We have to check that the corresponding solution is sufficiently integrable, namely

$$\begin{aligned}&\int_{\mathbb{R}^3_+} (|\xi|^2|\hat{u}_h(\xi,x_3)|^2 + |\partial_3\hat{u}_h(\xi,x_3)|^2) d\xi\, dx_3 < \infty,\\&\int_{\mathbb{R}^3_+} (|\xi|^2|\hat{u}_3(\xi,x_3)|^2 + |\partial_3\hat{u}_3(\xi,x_3)|^2) d\xi\, dx_3 < \infty.\end{aligned} \tag{5.12}$$

Notice that by construction, $\partial_3\hat{u}_3 = -i\xi\cdot\hat{u}_h$ (divergence-free condition), so that we only have to check three conditions.

To that end, we need to investigate the behaviour of λ_k, A_k for ξ close to zero and for $\xi\to\infty$. We gather the results in the following lemma, whose proof is once again postponed to Appendix 5.A:

Lemma 5.5.

– *As $\xi\to\infty$, we have*

$$\begin{aligned}\lambda_1 &= |\xi| - \frac{1}{2}|\xi|^{-\frac{1}{3}} + O\left(|\xi|^{-\frac{5}{3}}\right),\\\lambda_2 &= |\xi| - \frac{j^2}{2}|\xi|^{-\frac{1}{3}} + O\left(|\xi|^{-\frac{5}{3}}\right),\\\lambda_3 &= |\xi| - \frac{j}{2}|\xi|^{-\frac{1}{3}} + O\left(|\xi|^{-\frac{5}{3}}\right),\end{aligned}$$

where $j=\exp(2i\pi/3)$, so that

$$\begin{pmatrix}A_1(\xi)\\A_2(\xi)\\A_3(\xi)\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1&1&1\\1&j&j^2\\1&j^2&j\end{pmatrix}\begin{pmatrix}\hat{v}_{0,3}\\-2|\xi|^{1/3}(i\xi\cdot\hat{v}_{0,h} - |\xi|\hat{v}_{0,3}) + O(|\hat{v}_0|)\\-|\xi|^{-1/3}i\xi^\perp\cdot\hat{v}_{0,h} + O(|\hat{v}_0|)\end{pmatrix}. \tag{5.13}$$

– *As $\xi\to 0$, we have*

$$\begin{aligned}\lambda_1 &= |\xi|^3 + O\left(|\xi|^7\right),\\\lambda_2 &= e^{i\frac{\pi}{4}} + O\left(|\xi|^2\right),\\\lambda_3 &= e^{-i\frac{\pi}{4}} + O\left(|\xi|^2\right).\end{aligned}$$

As a consequence, for ξ close to zero,

$$\begin{aligned}A_1(\xi) &= \hat{v}_{0,3}(\xi) - \frac{\sqrt{2}}{2}\left(i\xi\cdot\hat{v}_{0,h} + i\xi^\perp\hat{v}_{0,h} + |\xi|\hat{v}_{0,3}\right) + O(|\xi|^2|\hat{v}_0(\xi)|),\\A_2(\xi) &= \frac{1}{2}\left(e^{-i\pi/4}i\xi\cdot\hat{v}_{0,h} + e^{i\pi/4}(i\xi^\perp\hat{v}_{0,h} + |\xi|\hat{v}_{0,3})\right) + O(|\xi|^2|\hat{v}_0(\xi)|),\\A_3(\xi) &= \frac{1}{2}\left(e^{i\pi/4}i\xi\cdot\hat{v}_{0,h} + e^{-i\pi/4}(i\xi^\perp\hat{v}_{0,h} + |\xi|\hat{v}_{0,3})\right) + O(|\xi|^2|\hat{v}_0(\xi)|).\end{aligned} \tag{5.14}$$

– *For all $a\geq 1$, there exists a constant $C_a>0$ such that*

$$a^{-1}\leq|\xi|\leq a \Longrightarrow \begin{cases}|\lambda_k(\xi)| + |\Re(\lambda_k(\xi))|^{-1}\leq C_a,\\|A(\xi)|\leq C_a|\hat{v}_0(\xi)|.\end{cases}$$

We then decompose each integral in (5.12) into three pieces, one on $\{|\xi|>a\}$, one on $\{|\xi|<a^{-1}\}$ and the last one on $\{|\xi|\in(a^{-1},a)\}$. All the integrals on $\{a^{-1}\leq|\xi|\leq a\}$ are bounded by

$$C_a\int_{a^{-1}<|\xi|<a}|\hat{v}_0(\xi)|^2\,d\xi \leq C_a\|v_0\|^2_{H^{1/2}(\mathbb{R}^2)}.$$

We thus focus on the two other pieces. We only treat the term

$$\int_{\mathbb{R}^3_+} |\xi|^2 |\hat{u}_3(\xi, x_3)|^2 \, d\xi \, dx_3,$$

since the two other terms can be evaluated using similar arguments.

$\triangleright$ On the set $\{|\xi| > a\}$, the difficulty comes from the fact that the contributions of the three exponentials compensate one another; hence a rough estimate is not possible. In order to simplify the calculations, we introduce the following notation: we set

$$\begin{aligned} B_1 &= A_1 + A_2 + A_3, \\ B_2 &= A_1 + j^2 A_2 + j A_3, \\ B_3 &= A_1 + j A_2 + j^2 A_3, \end{aligned} \tag{5.15}$$

so that

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

Hence we have $A_k = (B_1 + \alpha_k B_2 + \alpha_k^2 B_3)/3$, where $\alpha_1 = 1, \alpha_2 = j, \alpha_3 = j^2$. Notice that $\alpha_k^3 = 1$ and $\sum_k \alpha_k = 0$. According to Lemma 5.5,

$$\begin{aligned} B_1 &= \hat{v}_{0,3}, \\ B_2 &= -2|\xi|^{1/3}(i\xi \cdot \hat{v}_{0,h} - |\xi|\hat{v}_{0,3}) + O(|\hat{v}_0|), \\ B_3 &= -|\xi|^{-1/3} i\xi^\perp \cdot \hat{v}_{0,h} + O(|\hat{v}_0|). \end{aligned}$$

For all $\xi \in \mathbb{R}^2$, $|\xi| > a$, we have

$$|\xi|^2 \int_0^\infty |\hat{u}_3(\xi, x_3)|^2 dx_3 = |\xi|^2 \sum_{1 \le k,l \le 3} A_k \bar{A}_l \frac{1}{\lambda_k + \bar{\lambda}_l}.$$

Using the asymptotic expansions in Lemma 5.5, we infer that

$$\frac{1}{\lambda_k + \bar{\lambda}_l} = \frac{1}{2|\xi|} \left(1 + \frac{\alpha_k^2 + \bar{\alpha}_l^2}{2} |\xi|^{-4/3} + O(|\xi|^{-8/3}) \right).$$

Therefore, we obtain for $|\xi| \gg 1$

$$\begin{aligned} |\xi|^2 \sum_{1 \le k,l \le 3} A_k \bar{A}_l \frac{1}{\lambda_k + \bar{\lambda}_l} &= \frac{|\xi|}{2} \sum_{1 \le k,l \le 3} A_k \bar{A}_l \left(1 + \frac{\alpha_k^2 + \bar{\alpha}_l^2}{2} |\xi|^{-4/3} + O(|\xi|^{-8/3}) \right) \\ &= \frac{|\xi|}{2} \left(|B_1|^2 + \frac{1}{2}(B_2 \bar{B}_1 + \bar{B}_2 B_1)|\xi|^{-4/3} + O(|\hat{v}_0|^2) \right) \\ &= O(|\xi| \, |\hat{v}_0|^2). \end{aligned}$$

Hence, since $v_0 \in H^{1/2}(\mathbb{R}^2)$, we deduce that

$$\int_{|\xi| > a} \int_0^\infty |\xi|^2 |\hat{u}_3|^2 \, dx_3 \, d\xi < +\infty.$$

$\triangleright$ On the set $|\xi| \le a$, we can use a crude estimate: we have

$$\int_{|\xi| \le a} \int_0^\infty |\xi|^2 |\hat{u}_3(\xi, x_3)|^2 dx_3 \, d\xi \le C \sum_{k=1}^3 \int_{|\xi| \le a} |\xi|^2 \frac{|A_k(\xi)|^2}{2\Re(\lambda_k(\xi))} \, d\xi.$$

Using the estimates of Lemma 5.5, we infer that

$$\begin{aligned}
&\int_{|\xi|\leq a}\int_0^\infty |\xi|^2|\hat{u}_3(\xi,x_3)|^2 dx_3\, d\xi \\
\leq\ & C\int_{|\xi|\leq a}|\xi|^2\left((|\hat{v}_{0,3}(\xi)|^2+|\xi|^2|\hat{v}_{0,h}(\xi)|^2)\frac{1}{|\xi|^3}+|\xi|^2|\hat{v}_0(\xi)|^2\right)d\xi \\
\leq\ & C\int_{|\xi|\leq a}\left(\frac{|\hat{v}_{0,3}(\xi)|^2}{|\xi|}+|\xi|\,|\hat{v}_{0,h}(\xi)|^2\right)d\xi<\infty
\end{aligned}$$

thanks to the assumption (5.5) on $\hat{v}_{0,3}$. In a similar way, we have

$$\begin{aligned}
&\int_{|\xi|\leq a}\int_0^\infty |\xi|^2|\hat{u}_h(\xi,x_3)|^2 dx_3\, d\xi \leq C\int_{|\xi|\leq a}\left(\frac{|\hat{v}_{0,3}(\xi)|^2}{|\xi|}+|\xi|\,|\hat{v}_{0,h}(\xi)|^2\right)d\xi, \\
&\int_{|\xi|\leq a}\int_0^\infty |\partial_3\hat{u}_h(\xi,x_3)|^2 dx_3\, d\xi \leq C\int_{|\xi|\leq a}|\hat{v}_0|^2\, d\xi.
\end{aligned}$$

Gathering all the terms, we deduce that

$$\int_{\mathbb{R}^3_+}(|\xi|^2|\hat{u}(\xi,x_3)|^2+|\partial_3\hat{u}(\xi,x_3)|^2)d\xi\, dx_3<\infty,$$

so that $\nabla u\in L^2(\mathbb{R}^3_+)$. □

Remark 5.6. Notice that thanks to the exponential decay in Fourier space, for all $p\in\mathbb{N}$ with $p\geq 2$, there exists a constant $C_p>0$ such that

$$\int_1^\infty\int_{\mathbb{R}^2}|\nabla^p u|^2\leq C_p\|v_0\|^2_{H^{1/2}}.$$

• We now extend the definition of a solution to boundary data in $H^{1/2}_{uloc}(\mathbb{R}^2)$. We introduce the sets

$$\begin{aligned}
\mathcal{K} &:= \left\{u\in H^{1/2}_{uloc}(\mathbb{R}^2),\ \exists U_h\in H^{1/2}_{uloc}(\mathbb{R}^2)^2,\ u=\nabla_h\cdot U_h\right\}, \\
\mathbb{K} &:= \left\{u\in H^{1/2}_{uloc}(\mathbb{R}^2)^3,\ u_3\in\mathcal{K}\right\}.
\end{aligned}$$

In order to extend the definition of solutions to data which are only locally square integrable, we will first derive a representation formula for $v_0\in H^{1/2}(\mathbb{R}^2)$. We will prove that the formula still makes sense when $v_0\in\mathbb{K}$, and this will allow us to define a solution with boundary data in $\mathbb{K}$.

To that end, let us introduce some notation. According to the proof of Proposition 5.2, there exists $L_1,L_2,L_3:\mathbb{R}^2\to\mathcal{M}_3(\mathbb{C})$ and $q_1,q_2,q_3:\mathbb{R}^2\to\mathbb{C}^3$ such that

$$\begin{aligned}
\hat{u}(\xi,x_3) &= \sum_{k=1}^3 L_k(\xi)\hat{v}_0(\xi)\exp(-\lambda_k(\xi)x_3), \\
\hat{p}(\xi,x_3) &= \sum_{k=1}^3 q_k(\xi)\cdot\hat{v}_0(\xi)\exp(-\lambda_k(\xi)x_3).
\end{aligned}\tag{5.16}$$

For further reference, we state the following lemma:

Lemma 5.7. *For all $k \in \{1,2,3\}$, for all $\xi \in \mathbb{R}^2$, the following identities hold*

$$(|\xi|^2 - \lambda_k^2)L_k + \begin{pmatrix} -L_{k,21} & -L_{k,22} & -L_{k,23} \\ L_{k,11} & L_{k,12} & L_{k,13} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} i\xi_1 q_{k,1} & i\xi_1 q_{k,2} & i\xi_1 q_{k,3} \\ i\xi_2 q_{k,1} & i\xi_2 q_{k,2} & i\xi_2 q_{k,3} \\ -\lambda_k q_{k,1} & -\lambda_k q_{k,2} & -\lambda_k q_{k,3} \end{pmatrix} = 0$$

and for $j = 1,2,3,\ k = 1,2,3$,

$$i\xi_1 L_{k,1j} + i\xi_2 L_{k,2j} - \lambda_k L_{k,3j} = 0.$$

Proof. Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ such that $v_{0,3} = \nabla_h \cdot V_h$ for some $V_h \in H^{1/2}(\mathbb{R}^2)$. Then, according to Proposition 5.2, the couple (u,p) defined by (5.16) is a solution of (5.4). Therefore it satisfies (5.6). Plugging the definition (5.16) into (5.6), we infer that for all $x_3 > 0$,

$$\int_{\mathbb{R}^2} \sum_{k=1}^{3} \exp(-\lambda_k x_3) \mathcal{A}_k(\xi) \hat{v}_0(\xi)\, d\xi = 0, \tag{5.17}$$

where

$$\mathcal{A}_k := (|\xi|^2 - \lambda_k^2)L_k + \begin{pmatrix} -L_{k,21} & -L_{k,22} & -L_{k,23} \\ L_{k,11} & L_{k,12} & L_{k,13} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} i\xi_1 q_{k,1} & i\xi_1 q_{k,2} & i\xi_1 q_{k,3} \\ i\xi_2 q_{k,1} & i\xi_2 q_{k,2} & i\xi_2 q_{k,3} \\ -\lambda_k q_{k,1} & -\lambda_k q_{k,2} & -\lambda_k q_{k,3} \end{pmatrix}.$$

Since (5.17) holds for all v_0, we obtain

$$\sum_{k=1}^{3} \exp(-\lambda_k x_3) \mathcal{A}_k(\xi) = 0 \quad \forall \xi\ \forall x_3,$$

and since $\lambda_1, \lambda_2, \lambda_3$ are distinct for all $\xi \neq 0$, we deduce eventually that $\mathcal{A}_k(\xi) = 0$ for all ξ and for all k.

The second identity follows in a similar fashion from the divergence-free condition. □

Our goal is now to derive a representation formula for u, based on the formula satisfied by its Fourier transform, in such a way that the formula still makes sense when $v_0 \in \mathbb{K}$. The crucial part is to understand the action of the pseudo-differential operators $\mathrm{Op}(L_k(\xi)\phi(\xi))$ on L^2_{uloc} functions, where $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. To that end, we will need to decompose $L_k(\xi)$ for ξ close to zero into several terms.

Lemma 5.5 provides asymptotic developments of L_1, L_2, L_3 and $\alpha_1, \alpha_2, \alpha_3$ as $|\xi| \ll 1$ or $|\xi| \gg 1$. In particular, we have, for $|\xi| \ll 1$,

$$\begin{aligned} L_1(\xi) &= \frac{\sqrt{2}}{2|\xi|} \begin{pmatrix} \xi_2(\xi_2 - \xi_1) & -\xi_2(\xi_2 + \xi_1) & -i\sqrt{2}\xi_2 \\ \xi_1(\xi_1 - \xi_2) & \xi_1(\xi_2 + \xi_1) & i\sqrt{2}\xi_1 \\ i|\xi|(\xi_2 - \xi_1) & -i|\xi|(\xi_2 + \xi_1) & \sqrt{2}|\xi| \end{pmatrix} \\ &\quad + \begin{pmatrix} O(|\xi|^2) & O(|\xi|^2) & O(|\xi|) \end{pmatrix}, \end{aligned} \tag{5.18}$$

$$\begin{aligned} L_2(\xi) &= \frac{1}{2} \begin{pmatrix} 1 & i & \dfrac{2i(-\xi_1 + \xi_2)}{|\xi|} \\ -i & 1 & \dfrac{-2i(\xi_1 + \xi_2)}{|\xi|} \\ i(\xi_1 e^{-i\pi/4} - \xi_2 e^{i\pi/4}) & i(\xi_2 e^{-i\pi/4} + i\xi_1 e^{i\pi/4}) & e^{i\pi/4} \end{pmatrix} \\ &\quad + \begin{pmatrix} O(|\xi|^2) & O(|\xi|^2) & O(|\xi|) \end{pmatrix}, \end{aligned}$$

$$L_3(\xi) = \frac{1}{2}\begin{pmatrix} 1 & -i & \dfrac{2i(\xi_1+\xi_2)}{|\xi|} \\ i & 1 & \dfrac{-2i(\xi_1-\xi_2)}{|\xi|} \\ i(\xi_1 e^{i\pi/4} - \xi_2 e^{-i\pi/4}) & i(\xi_2 e^{i\pi/4} + i\xi_1 e^{-i\pi/4}) & e^{-i\pi/4} \end{pmatrix} + \begin{pmatrix} O(|\xi|^2) & O(|\xi|^2) & O(|\xi|) \end{pmatrix}.$$

The remainder terms are to be understood column-wise. Notice that the third column of L_k, i.e. $L_k e_3$, always acts on $\hat{v}_{0,3} = i\xi\cdot\hat{V}_h$. We thus introduce the following notation: for $k=1,2,3$, $M_k := (L_k e_1\ L_k e_2) \in \mathcal{M}_{3,2}(\mathbb{C})$, and $N_k := iL_k e_3\,{}^t\xi \in \mathcal{M}_{3,2}(\mathbb{C})$. M_k^1 (resp. N_k^1) denotes the 3×2 matrix whose coefficients are the nonpolynomial and homogeneous terms of order one in M_k (resp. N_k) for ξ close to zero. For instance,

$$M_1^1 := \frac{\sqrt{2}}{2|\xi|}\begin{pmatrix} \xi_2(\xi_2-\xi_1) & -\xi_2(\xi_2+\xi_1) \\ -\xi_1(\xi_2-\xi_1) & \xi_1(\xi_2+\xi_1) \\ 0 & 0 \end{pmatrix}, \quad N_1^1 := \frac{i}{|\xi|}\begin{pmatrix} -\xi_2\xi_1 & \xi_2^2 \\ \xi_1^2 & \xi_1\xi_2 \\ 0 & 0 \end{pmatrix}.$$

We also set $M_k^{rem} = M_k - M_k^1$, $N_k^{rem} := N_k - N_k^1$, so that for ξ close to zero,

$$M_1^{rem} = O(|\xi|), \quad \text{and for } k=2,\ 3, \quad M_k^{rem} = O(1),$$
$$\forall k \in \{1,2,3\}, \quad N_k^{rem} = O(|\xi|).$$

There are polynomial terms of order one in M_1^{rem} and N_k^{rem} (resp. of order 0 and 1 in M_k^{rem} for $k=2,\ 3$) which account for the fact that the remainder terms are not $O(|\xi|^2)$. However, these polynomial terms do not introduce any singularity when there are differentiated and thus, using the results of Appendix 5.B, we get, for any integer $q\geq 1$,

$$\left|\nabla_\xi^q M_k^{rem}\right|,\ \left|\nabla_\xi^q N_k^{rem}\right| = O(|\xi|^{2-q}+1) \quad \text{for } |\xi|\ll 1. \tag{5.19}$$

▷ Concerning the Fourier multipliers of order one M_k^1 and N_k^1, we will rely on the following lemma, which is proved in Appendix 5.C:

Lemma 5.8. *There exists a constant C_I such that for all $i,j\in\{1,2\}$, for any function $g\in\mathcal{S}(\mathbb{R}^2)$, for all $\zeta\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ and for all $K>0$,*

$$\begin{aligned} &\mathrm{Op}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\right)g(x) \\ &= C_I\int_{\mathbb{R}^2} dy\left[\frac{\delta_{i,j}}{|x-y|^3} - 3\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}\right]\times \\ &\qquad\times\left\{\rho * g(x) - \rho * g(y) - \nabla\rho * g(x)\cdot(x-y)\mathbf{1}_{|\mathbf{x}-\mathbf{y}|\leq\mathbf{K}}\right\}, \end{aligned} \tag{5.20}$$

where $\rho := \mathcal{F}^{-1}\zeta \in \mathcal{S}(\mathbb{R}^2)$.

Definition 5.9. If L is a homogeneous, nonpolynomial function of order one in $\mathbb{R}^2$, say

$$L(\xi) = \sum_{1\leq i,j\leq 2} a_{ij}\frac{\xi_i\xi_j}{|\xi|},$$

then we define, for $\varphi\in W^{2,\infty}(\mathbb{R}^2)$,

$$\mathcal{I}[L]\varphi(x) := \sum_{1\leq i,j\leq 2} a_{ij}\int_{\mathbb{R}^2} dy\gamma_{ij}(x-y)\left\{\varphi(x)-\varphi(y)-\nabla\varphi(x)\cdot(x-y)\mathbf{1}_{|\mathbf{x}-\mathbf{y}|\leq\mathbf{K}}\right\},$$

where

$$\gamma_{i,j}(x) = C_I\left(\frac{\delta_{i,j}}{|x|^3} - 3\frac{x_ix_j}{|x|^5}\right).$$

Remark 5.10. The value of the number K in the formula (5.20) and in Definition 5.9 is irrelevant, since for all $\varphi \in W^{2,\infty}(\mathbb{R}^2)$, for all $0 < K < K'$,

$$\int_{\mathbb{R}^2} dy \gamma_{ij}(x-y) \nabla\varphi(x) \cdot (x-y) \mathbf{1}_{\mathbf{K} < |\mathbf{x}-\mathbf{y}| \leq \mathbf{K}'} = 0$$

by symmetry arguments.

We then have the following bound:

Lemma 5.11. *Let $\varphi \in W^{2,\infty}(\mathbb{R}^2)$. Then for all $1 \leq i,j \leq 2$,*

$$\left\| \mathcal{I}\left[\frac{\xi_i \xi_j}{|\xi|}\right] \varphi \right\|_{L^\infty(\mathbb{R}^2)} \leq C \|\varphi\|_\infty^{1/2} \|\nabla^2 \varphi\|_\infty^{1/2}.$$

Remark 5.12. We will often apply the above Lemma with $\varphi = \rho * g$, where $\rho \in \mathcal{C}^2(\mathbb{R}^2)$ is such that ρ and $\nabla^2\rho$ have bounded second order moments in L^2, and $g \in L^2_{uloc}(\mathbb{R}^2)$. In this case, we have

$$\begin{aligned} \|\varphi\|_\infty &\leq C \|g\|_{L^2_{uloc}} \|(1+|\cdot|^2)\rho\|_{L^2(\mathbb{R}^2)}, \\ \|\nabla^2\varphi\|_\infty &\leq C \|g\|_{L^2_{uloc}} \|(1+|\cdot|^2)\nabla^2\rho\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Indeed,

$$\begin{aligned} \|\rho * g\|_{L^\infty} \leq & \sup_{x\in\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{1+|x-y|^4} |g(y)|^2 \, dy \right)^{1/2} \left(\int_{\mathbb{R}^2} (1+|x-y|^4) |\rho(x-y)|^2 \, dy \right)^{1/2} \\ \leq & C \|g\|_{L^2_{uloc}} \|(1+|\cdot|^2)\rho\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

The L^∞ norm of $\nabla^2\varphi$ is estimated exactly in the same manner, simply replacing ρ by $\nabla^2\rho$.

Proof of Lemma 5.11. We split the integral in (5.20) into three parts

$$\begin{aligned} \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\varphi(x) &= \int_{|x-y|\leq K} dy \gamma_{ij}(x-y) \left\{ \varphi(x) - \varphi(y) - \nabla\varphi(x)\cdot(x-y) \right\} \\ &+ \int_{|x-y|\geq K} dy \gamma_{ij}(x-y)\varphi(x) \\ &- \int_{|x-y|\geq K} dy \gamma_{ij}(x-y)\varphi(y) \\ &= \mathrm{A}(x) + \mathrm{B}(x) + \mathrm{C}(x). \end{aligned} \tag{5.21}$$

Concerning the first integral in (5.21), Taylor's formula implies

$$|\mathrm{A}(x)| \leq C \left\| \nabla^2\varphi \right\|_{L^\infty} \int_{|x-y|\leq K} \frac{dy}{|x-y|} \leq CK \left\| \nabla^2\varphi \right\|_{L^\infty}.$$

For the second and third integral in (5.21),

$$|\mathrm{B}(x)| + |\mathrm{C}(x)| \leq C \|\varphi\|_\infty \int_{|x-y|\geq K} \frac{dy}{|x-y|^3} \leq CK^{-1} \|\varphi\|_\infty.$$

We infer that for all $K > 0$,

$$\left\| \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\varphi \right\|_\infty \leq C \left(K \left\|\nabla^2\varphi\right\|_\infty + K^{-1}\|\varphi\|_\infty \right).$$

Optimizing in K (i.e. choosing $K = \|\varphi\|_\infty^{1/2} / \|\nabla^2\varphi\|_\infty^{1/2}$), we obtain the desired inequality. □

$\triangleright$ For the remainder terms M_k^{rem}, N_k^{rem} as well as the high-frequency terms, we will use the following estimates:

Lemma 5.13 (Kernel estimates). *Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\phi(\xi) = 1$ for $|\xi| \leq 1$. Define*

$$\begin{aligned}
\varphi_{HF}(x_h, x_3) &:= \mathcal{F}^{-1}\left(\sum_{k=1}^{3}(1-\phi)(\xi)L_k(\xi)\exp(-\lambda_k(\xi)x_3)\right),\\
\psi_1(x_h, x_3) &:= \mathcal{F}^{-1}\left(\sum_{k=1}^{3}\phi(\xi)M_k^{rem}(\xi)\exp(-\lambda_k(\xi)x_3)\right),\\
\psi_2(x_h, x_3) &:= \mathcal{F}^{-1}\left(\sum_{k=1}^{3}\phi(\xi)N_k^{rem}(\xi)\exp(-\lambda_k(\xi)x_3)\right).
\end{aligned}$$

Then the following estimates hold:

- *for all $q \in \mathbb{N}$, there exists $c_{0,q} > 0$, such that for all $\alpha, \beta > c_{0,q}$, there exists $C_{\alpha,\beta,q} > 0$ such that*
$$|\nabla^q \varphi_{HF}(x_h, x_3)| \leq \frac{C_{\alpha,\beta,q}}{|x_h|^\alpha + |x_3|^\beta};$$
- *for all $\alpha \in (0, 2/3)$, for all $q \in \mathbb{N}$, there $C_{\alpha,q} > 0$ such that*
$$|\nabla^q \psi_1(x_h, x_3)| \leq \frac{C_{\alpha,q}}{|x_h|^{3+q} + |x_3|^{\alpha+\frac{q}{3}}};$$
- *for all $\alpha \in (0, 2/3)$, for all $q \in \mathbb{N}$, there exists $C_{\alpha,q} > 0$ such that*
$$|\nabla^q \psi_2(x_h, x_3)| \leq \frac{C_{\alpha,q}}{|x_h|^{3+q} + |x_3|^{\alpha+\frac{q}{3}}}.$$

Proof. • Let us first derive the estimate on φ_{HF} for $q = 0$. We seek to prove that there exists $c_0 > 0$ such that

$$\forall(\alpha, \beta) \in (c_0, \infty)^2,\ \exists C_{\alpha,\beta},\ |\varphi_{HF}(x_h, x_3)| \leq \frac{C_{\alpha,\beta}}{|x_h|^\alpha + |x_3|^\beta}. \tag{5.22}$$

To that end, it is enough to show that for $\alpha \in \mathbb{N}^2$ and $\beta > 0$ with $|\alpha|, \beta \geq c_0$,

$$\sup_{x_3>0}\left(|x_3|^\beta \|\widehat{\varphi_{HF}}(\cdot, x_3)\|_{L^1(\mathbb{R}^2)} + \|\nabla_\xi^\alpha \widehat{\varphi_{HF}}(\cdot, x_3)\|_{L^1(\mathbb{R}^2)}\right) < \infty.$$

We recall that $\lambda_k(\xi) \sim |\xi|$ for $|\xi| \to \infty$. Moreover, using the estimates of Lemma 5.5, we infer that there exists $\gamma \in \mathbb{R}$ such that $L_k(\xi) = O(|\xi|^\gamma)$ for $|\xi| \gg 1$. Hence

$$\begin{aligned}
|x_3|^\beta|\hat{\varphi}_{HF}(\xi, x_3)| &\leq C|1-\phi(\xi)|\,|\xi|^\gamma \sum_{k=1}^{3}|x_3|^\beta \exp(-\Re(\lambda_k)x_3)\\
&\leq C|1-\phi(\xi)|\,|\xi|^{\gamma-\beta}\sum_{k=1}^{3}|\Re(\lambda_k)x_3|^\beta \exp(-\Re(\lambda_k)x_3)\\
&\leq C_\beta\,|\xi|^{\gamma-\beta}\mathbf{1}_{|\xi|\geq 1}.
\end{aligned}$$

Hence for β large enough, for all $x_3 > 0$,

$$|x_3|^\beta \|\hat{\varphi}_{HF}(\cdot, x_3)\|_{L^1(\mathbb{R}^2)} \leq C_\beta.$$

In a similar fashion, for $\alpha \in \mathbb{N}^2$, $|\alpha| \geq 1$, we have, as $|\xi| \to \infty$ (see Appendix 5.B)

$$\begin{aligned}\nabla^\alpha L_k(\xi) &= O\left(|\xi|^{\gamma-|\alpha|}\right),\\ \nabla^\alpha\left(\exp(-\lambda_k x_3)\right) &= O\left((|\xi|^{1-|\alpha|}x_3 + |x_3|^{|\alpha|})\exp(-\Re(\lambda_k)x_3)\right) = O\left(|\xi|^{-|\alpha|}\right).\end{aligned}$$

Moreover, we recall that $\nabla(1-\phi)$ is supported in a ring of the type $B_R \setminus B_1$ for some $R>1$. As a consequence, we obtain, for all $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 1$,

$$|\nabla^\alpha \widehat{\varphi_{HF}}(\xi, x_3)| \leq C_\alpha |\xi|^{\gamma-|\alpha|}\mathbf{1}_{|\xi|\geq 1},$$

so that

$$\|\nabla^\alpha \widehat{\varphi_{HF}}(\cdot, x_3)\|_{L^1(\mathbb{R}^2)} \leq C_\alpha.$$

Thus φ_{HF} satisfies (5.22) for $q=0$. For $q \geq 1$, the proof is the same, changing L_k into $|\xi|^{q_1}|\lambda_k|^{q_2}L_k$ with $q_1+q_2=q$.

• The estimates on ψ_1, ψ_2 are similar. The main difference lies in the degeneracy of λ_1 near zero. For instance, in order to derive an L^∞ bound on $|x_3|^{\alpha+q/3}\nabla^q\psi_1$, we look for an $L^\infty_{x_3}(L^1_\xi(\mathbb{R}^2))$ bound on $|x_3|^{\alpha+q/3}|\xi|^q\hat{\psi}_1(\xi, x_3)$. We have

$$\begin{aligned}&\left||x_3|^{\alpha+q/3}|\xi|^q\phi(\xi)\sum_{k=1}^3 M_k^{rem}\exp(-\lambda_k x_3)\right|\\ \leq\ & C|x_3|^{\alpha+q/3}|\xi|^q\sum_{k=1}^3 \exp(-\Re(\lambda_k)x_3)|M_k^{rem}|\mathbf{1}_{|\xi|\leq \mathbf{R}}\\ \leq\ & C|\xi|^q\sum_{k=1}^3 |\Re\lambda_k|^{-(\alpha+q/3)}|M_k^{rem}|\mathbf{1}_{|\xi|\leq \mathbf{R}}\\ \leq\ & C|\xi|^q(|\xi|^{1-3\alpha-q}+1)\mathbf{1}_{|\xi|\leq \mathbf{R}}.\end{aligned}$$

The right-hand side is in L^1 provided $\alpha < 2/3$. We infer that

$$\left|\ |x_3|^{\alpha+q/3}\nabla^q\psi_1(x)\right| \leq C_{\alpha,q} \quad \forall x\ \forall \alpha \in (0, 2/3).$$

The other bound on ψ_1 is derived in a similar way, using the fact that

$$\nabla_\xi^q M_1^{rem} = O(|\xi|^{2-q}+1)$$

for ξ in a neighbourhood of zero. □

▷ We are now ready to state our representation formula:

Proposition 5.14 (Representation formula). *Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ such that $v_{0,3} = \nabla_h \cdot V_h$ for some $V_h \in H^{1/2}(\mathbb{R}^2)$, and let u be the solution of (5.4). For all $x \in \mathbb{R}^3$, let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ on $B(x_h, 1)$. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ be a cut-off function as in Lemma 5.13, and let φ_{HF}, ψ_1, ψ_2 be the associated kernels. For $k=1,2,3$, set*

$$f_k(\cdot, x_3) := \mathcal{F}^{-1}\left(\phi(\xi)\exp(-\lambda_k x_3)\right).$$

Then

$$\begin{aligned}
u(x) &= \mathcal{F}^{-1}\left(\sum_{k=1}^{3} L_k(\xi)\begin{pmatrix}\widehat{\chi v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)(x)\\
&+ \sum_{k=1}^{3}\mathcal{I}[M_k^1]f_k(\cdot,x_3)*((1-\chi)v_{0,h})(x)\\
&+ \sum_{k=1}^{3}\mathcal{I}[N_k^1]f_k(\cdot,x_3)*((1-\chi)V_h)(x)\\
&+ \varphi_{HF}*\begin{pmatrix}(1-\chi)v_{0,h}\\ \nabla\cdot((1-\chi)V_h)\end{pmatrix}(x)\\
&+ \psi_1*((1-\chi)v_{0,h})(x)+\psi_2*((1-\chi)V_h)(x)
\end{aligned}$$

As a consequence, for all $a>0$, there exists a constant C_a such that

$$\sup_{k\in\mathbb{Z}^2}\int_{k+[0,1]^2}\int_0^a |u(x_h,x_3)|^2dx_3\,dx_h \le C_a\left(\|v_0\|^2_{H^{1/2}_{uloc}(\mathbb{R}^2)}+\|V_h\|^2_{H^{1/2}_{uloc}(\mathbb{R}^2)}\right).$$

Moreover, there exists $q\in\mathbb{N}$ such that

$$\sup_{k\in\mathbb{Z}^2}\int_{k+[0,1]^2}\int_1^\infty |\nabla^q u(x_h,x_3)|^2dx_3\,dx_h \le C\left(\|v_0\|^2_{H^{1/2}_{uloc}(\mathbb{R}^2)}+\|V_h\|^2_{H^{1/2}_{uloc}(\mathbb{R}^2)}\right).$$

Remark 5.15. The integer q in the above proposition is explicit and does not depend on v_0. One can take $q=4$ for instance.

Proof. The proposition follows quite easily from the preceding lemmas. We have, according to Proposition 5.2,

$$\begin{aligned}
u(x) &= \mathcal{F}^{-1}\left(\sum_{k=1}^{3} L_k(\xi)\begin{pmatrix}\widehat{\chi v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\right)(x)\\
&+ \mathcal{F}^{-1}\left(\sum_{k=1}^{3} L_k(\xi)\begin{pmatrix}\widehat{(1-\chi)v_{0,h}}(\xi)\\ \widehat{\nabla\cdot((1-\chi)V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\right)(x).
\end{aligned}$$

In the latter term, the cut-off function ϕ is introduced, writing simply $1=1-\phi+\phi$. We have, for the high-frequency term,

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\sum_{k=1}^{3}(1-\phi(\xi))L_k(\xi)\begin{pmatrix}\widehat{(1-\chi)v_{0,h}}(\xi)\\ \widehat{\nabla\cdot((1-\chi)V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\right)\\
&= \mathcal{F}^{-1}\left(\hat{\varphi}_{HF}(\xi,x_3)\begin{pmatrix}\widehat{(1-\chi)v_{0,h}}(\xi)\\ \widehat{\nabla\cdot((1-\chi)V_h)}(\xi)\end{pmatrix}\right)=\varphi_{HF}(\cdot,x_3)*\begin{pmatrix}(1-\chi)v_{0,h}(\xi)\\ \nabla\cdot((1-\chi)V_h)(\xi)\end{pmatrix}
\end{aligned}$$

Notice that $\nabla_h\cdot((1-\chi)V_h)=(1-\chi)v_{0,3}-\nabla_h\chi\cdot V_h\in H^{1/2}(\mathbb{R}^2)$.

In the low frequency terms, we distinguish between the horizontal and the vertical components of v_0. Let us deal with the vertical component, which is slightly more complicated: since $v_{0,3}=\nabla_h\cdot V_h$, we have

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\sum_{k=1}^{3}\phi(\xi)L_k(\xi)e_3\widehat{\nabla_h\cdot((1-\chi)V_h)}(\xi)\exp(-\lambda_k x_3)\right)\\
&= \mathcal{F}^{-1}\left(\sum_{k=1}^{3}\phi(\xi)L_k(\xi)e_3 i\xi\cdot\widehat{(1-\chi)V_h}(\xi)\exp(-\lambda_k x_3)\right).
\end{aligned}$$

We recall that $N_k = iL_k e_3{}^t\xi$, so that

$$L_k(\xi)e_3 i\xi \cdot \widehat{(1-\chi)V_h}(\xi) = N_k(\xi)\widehat{(1-\chi)V_h}(\xi).$$

Then, by definition of ψ_2 and f_k,

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\sum_{k=1}^3 \phi(\xi)N_k(\xi)\widehat{(1-\chi)V_h}(\xi)\exp(-\lambda_k x_3)\right)\\
=\;& \mathcal{F}^{-1}\left(\sum_{k=1}^3 \phi(\xi)N_k^1(\xi)\widehat{(1-\chi)V_h}(\xi)\exp(-\lambda_k x_3)\right)\\
&+\mathcal{F}^{-1}\left(\sum_{k=1}^3 \phi(\xi)N_k^{rem}(\xi)\widehat{(1-\chi)V_h}(\xi)\exp(-\lambda_k x_3)\right)\\
=\;& \sum_{k=1}^3 \mathcal{I}\left[N_k^1\right] f_k * ((1-\chi)\cdot V_h) + \mathcal{F}^{-1}\left(\hat{\psi}_2(\xi,x_3)\widehat{(1-\chi)\cdot V_h}(\xi)\right)\\
=\;& \sum_{k=1}^3 \mathcal{I}\left[N_k^1\right] f_k * ((1-\chi)\cdot V_h) + \psi_2 * ((1-\chi)\cdot V_h).
\end{aligned}$$

The representation formula follows.

There remains to bound every term occurring in the representation formula. In order to derive bounds on $(l+[0,1]^2)\times\mathbb{R}_+$ for some $l\in\mathbb{Z}^2$, we use the representation formula with a function $\chi_l \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi_l \equiv 1$ on $l+[-1,2]^2$, and we assume that the derivatives of χ_l are bounded uniformly in l (take for instance $\chi_l = \chi(\cdot + l)$ for some $\chi \in \mathcal{C}_0^\infty$).

– According to Proposition 5.2, we have [1]

$$\begin{aligned}
&\int_0^a \left\|\mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi_l v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi_l V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)\right\|_{L^2(\mathbb{R}^2)}^2 dx_3\\
\le\;& C_a\left(\|\chi_l v_{0,h}\|_{H^{1/2}}^2 + \|\nabla\chi_l\cdot V_h\|_{H^{1/2}}^2 + \|\chi_l v_{0,3}\|_{H^{1/2}(\mathbb{R}^2)}^2\right)\\
\le\;& C_a\left(\|v_0\|_{H^{1/2}_{uloc}}^2 + \|V_h\|_{H^{1/2}_{uloc}}^2\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int_0^\infty \left\|\nabla\mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi_l v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi_l V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)\right\|_{L^2(\mathbb{R}^2)}^2 dx_3\\
\le\;& C\left(\|v_0\|_{H^{1/2}_{uloc}}^2 + \|V_h\|_{H^{1/2}_{uloc}}^2\right).
\end{aligned}$$

Moreover, thanks to Remark 5.6, for any $q\ge 2$,

$$\begin{aligned}
&\int_1^\infty \left\|\nabla^q\mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi_l v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi_l V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)\right\|_{L^2(\mathbb{R}^2)}^2 dx_3\\
\le\;& C_q\left(\|v_0\|_{H^{1/2}_{uloc}}^2 + \|V_h\|_{H^{1/2}_{uloc}}^2\right).
\end{aligned}$$

1. We give in the next paragraph (see (5.31)) a proof of the inequality

$$\|\chi_l v_{0,h}\|_{H^{1/2}} \le C\|v_{0,h}\|_{H^{1/2}_{uloc}},$$

which is not entirely obvious since $H^{1/2}$ is a fractional Sobolev space.

- We now address the bounds of the terms involving the kernels $\varphi_{HF}, \psi_1, \psi_2$. According to Lemma 5.13, we have for instance, for all $x_3 > 0$, for all $x_h \in l + [0,1]^2$, for $\sigma \in \mathbb{N}^2$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \nabla^\sigma \varphi_{HF}(y_h, x_3) \begin{pmatrix} (1-\chi_l) v_{0,h} \\ \nabla \cdot ((1-\chi_l) V_h) \end{pmatrix} (x_h - y_h)\, dy_h \right| \\
\leq\ & C_{\alpha,\beta,|\sigma|} \int_{|y_h|\geq 1} |v_0(x_h - y_h)| \frac{1}{|y_h|^\alpha + x_3^\beta}\, dy_h \\
&+ C_{\alpha,\beta,|\sigma|} \int_{1\leq |y_h| \leq 2} |V_h(x_h - y_h)| \frac{1}{|y_h|^\alpha + x_3^\beta}\, dy_h \\
\leq\ & C \|V_h\|_{L^2_{uloc}} \frac{1}{1+x_3^\beta} + C \left(\int_{\mathbb{R}^2} \frac{|v_0(x_h - y_h)|^2}{1+|y_h|^\gamma} dy_h \right)^{1/2} \left(\int_{|y_h|\geq 1} \frac{1+|y_h|^\gamma}{\left(|y_h|^\alpha + x_3^\beta\right)^2}\, dy_h \right)^{1/2} \\
\leq\ & C \|V_h\|_{L^2_{uloc}} \frac{1}{1+x_3^\beta} + C \|v_0\|_{L^2_{uloc}} \inf\left(1, x_3^{\beta(\frac{2+\gamma}{2\alpha} - 1)}\right)
\end{aligned}$$

for all $\gamma > 2$ and for $\alpha,\ \beta > c_0$ and sufficiently large. In particular the $\dot{H}^q_{uloc}$ bound follows. The local bounds in L^2_{uloc} near $x_3 = 0$ are immediate since the right-hand side is uniformly bounded in x_3. The treatment of the terms with $\psi_1,\ \psi_2$ are analogous. Notice however that because of the slower decay of $\psi_1,\ \psi_2$ in x_3, we only have a uniform bound in $\dot{H}^q((l+[0,1]^2) \times (1,\infty))$ if q is large enough ($q \geq 2$ is sufficient).
- There remains to bound the terms involving $\mathcal{I}[M_k^1], \mathcal{I}[N_k^1]$, using Lemma 5.8 and Remark 5.12. We have for instance, for all $x_3 > 0$,

$$\begin{aligned}
&\left\| \mathcal{I}[N_k^1] f_k * ((1-\chi_l)V_h) \right\|_{L^2(l+[0,1]^2)} \\
\leq\ & C \|V_h\|_{L^2_{uloc}} \left(\|(1+|\cdot|^2) f_k(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} + \|(1+|\cdot|^2) \nabla_h^2 f_k(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} \right).
\end{aligned}$$

Using the Plancherel formula, we infer

$$\begin{aligned}
\|(1+|\cdot|^2) f_k(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} &\leq C \|\phi(\xi) \exp(-\lambda_k x_3)\|_{H^2(\mathbb{R}^2)} \\
&\leq C \|\exp(-\lambda_k x_3)\|_{H^2(B_R)} + C \exp(-\mu x_3),
\end{aligned}$$

where $R > 1$ is such that $\operatorname{Supp} \phi \subset B_R$ and μ is a positive constant depending only on ϕ. We have, for $k = 1, 2, 3$,

$$\left| \nabla^2 \exp(-\lambda_k x_3) \right| \leq C \left(x_3 |\nabla_\xi^2 \lambda_k| + x_3^2 |\nabla_\xi \lambda_k|^2 \right) \exp(-\lambda_k x_3).$$

The asymptotic expansions in Lemma 5.5 together with the results of Appendix 5.B imply that for ξ in any neighbourhood of zero,

$$\begin{aligned}
\nabla^2 \lambda_1 = O(|\xi|), &\qquad \nabla \lambda_1 = O(|\xi|^2), \\
\nabla^2 \lambda_k = O(1), &\quad \nabla \lambda_k = O(|\xi|) \text{ for } k = 2, 3.
\end{aligned}$$

In particular, if $k = 2, 3$, since λ_k is bounded away from zero in a neighbourhood of zero,

$$\int_0^\infty dx_3 \|\exp(-\lambda_k x_3)\|_{H^2(B_R)}^2 < \infty.$$

On the other hand, the degeneracy of λ_1 near $\xi = 0$ prevents us from obtaining the same result. Notice however that

$$\int_0^a \|\exp(-\lambda_1 x_3)\|_{H^2(B_R)}^2 \leq C_a$$

for all $a > 0$, and

$$\int_0^\infty \||\xi|^q \nabla^2 \exp(-\lambda_1 x_3)\|_{L^2(B_R)}^2 < \infty$$

for $q \in \mathbb{N}$ large enough ($q \geq 4$). Hence the $\dot{H}^q_{uloc}$ bound follows. □

▷ The representation formula, together with its associated estimates, now allows us to extend the notion of solution to locally integrable boundary data. Before stating the corresponding result, let us prove a technical lemma about some nice properties of operators of the type $\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]$, which we will use repeatedly:

Lemma 5.16. *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Then, for all $g \in L^2_{uloc}(\mathbb{R}^2)$, for all $\rho \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $\nabla^\alpha \rho$ has bounded second order moments in L^2 for $0 \leq \alpha \leq 2$,*

$$\int_{\mathbb{R}^2} \varphi \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \rho * g = \int_{\mathbb{R}^2} g \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \check{\rho} * \varphi,$$
$$\int_{\mathbb{R}^2} \nabla\varphi \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \rho * g = -\int_{\mathbb{R}^2} \varphi \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \nabla\rho * g.$$

Remark 5.17. Notice that the second formula merely states that

$$\nabla\left(\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \rho * g\right) = \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \nabla\rho * g$$

in the sense of distributions.

Proof. • The first formula is a consequence of Fubini's theorem: indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \rho * g \\
= \quad &\int_{\mathbb{R}^6} dx\, dy\, dt\, \gamma_{ij}(x-y) g(t) \varphi(x) \times \\
&\qquad\qquad \times \left\{\rho(x-t) - \rho(y-t) - \nabla\rho(x-t)\cdot(x-y)\mathbf{1}_{|\mathbf{x}-\mathbf{y}|\leq \mathbf{1}}\right\} \\
\underset{y'=x+t-y}{=} \quad &\int_{\mathbb{R}^6} dx\, dy'\, dt \gamma_{ij}(y'-t) g(t) \varphi(x) \times \\
&\qquad\qquad \times \left\{\rho(x-t) - \rho(x-y') - \nabla\rho(x-t)\cdot(y'-t)\mathbf{1}_{|\mathbf{y'}-\mathbf{t}|\leq \mathbf{1}}\right\}.
\end{aligned}$$

Integrating with respect to x, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \rho * g \\
= \quad &\int_{\mathbb{R}^4} dy'\, dt\, \gamma_{ij}(y'-t) g(t) \left\{\varphi * \check{\rho}(t) - \varphi * \check{\rho}(y') - \varphi * \nabla\check{\rho}(t)\cdot(t-y')\mathbf{1}_{|\mathbf{y'}-\mathbf{t}|\leq \mathbf{1}}\right\} \\
= \quad &\int_{\mathbb{R}^2} dt g(t) \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \varphi * \check{\rho}.
\end{aligned}$$

• The second formula is then easily deduced from the first one: using the fact that $\nabla\check{\rho}(x) = -\nabla\rho(-x) = -\widetilde{\nabla\rho}(x)$, we infer

$$\begin{aligned}
\int_{\mathbb{R}^2} \nabla\varphi \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \rho * g &= \int_{\mathbb{R}^2} g\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \check{\rho} * \nabla\varphi \\
&= \int_{\mathbb{R}^2} g\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \nabla\check{\rho} * \varphi \\
&= -\int_{\mathbb{R}^2} g\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \widetilde{\nabla\rho} * \varphi \\
&= -\int_{\mathbb{R}^2} \varphi\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right] \nabla\rho * g.
\end{aligned}$$

□

We are now ready to state the main result of this section:

Corollary 5.18. *Let $v_0 \in \mathbb{K}$. Then there exists a unique solution u of* (5.4) *such that $u|_{x_3=0} = v_0$ and*

$$\begin{aligned}
\forall a > 0, \quad & \sup_{k\in\mathbb{Z}^2} \int_{k+[0,1]^2} \int_0^a |u(x_h, x_3)|^2 dx_3\, dx_h < \infty, \\
\exists q \in \mathbb{N}^*, \quad & \sup_{k\in\mathbb{Z}^2} \int_{k+[0,1]^2} \int_1^\infty |\nabla^q u(x_h, x_3)|^2 dx_3\, dx_h < \infty.
\end{aligned} \tag{5.23}$$

Remark 5.19. As in Proposition 5.14, the integer q in the two results above is explicit and does not depend on v_0 (one can take $q = 4$ for instance).

Proof of Corollary 5.18. Uniqueness. Let u be a solution of (5.4) satisfying (5.23) and such that $u|_{x_3=0} = 0$. We use the same type of proof as in Proposition 5.2 (see also [GVM10]). Using a Poincaré inequality near the boundary $x_3 = 0$, we have

$$\sup_{k\in\mathbb{Z}^2} \int_{k+[0,1]^2} \int_0^\infty |\nabla^q u(x_h, x_3)|^2 dx_3\, dx_h < \infty.$$

Hence $u \in \mathcal{C}(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^2))$ and we can take the Fourier transform of u with respect to the horizontal variable. The rest of the proof is identical to the one of Proposition 5.2. The equations in (5.6) are meant in the sense of tempered distributions in x_3, and in the sense of distributions in x_3, which is enough to perform all calculations.
Existence. For all $x_h \in \mathbb{R}^2$, let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ on $B(x_h, 1)$. Then we set

$$\begin{aligned}
u(x) &= \mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi) \begin{pmatrix} \widehat{\chi v_{0,h}}(\xi) \\ \widehat{\nabla \cdot (\chi V_h)} \end{pmatrix} \exp(-\lambda_k x_3)\right)(x) \\
&+ \sum_{k=1}^3 \mathcal{I}[M_k^1] f_k(\cdot, x_3) * ((1-\chi)v_{0,h})(x) \\
&+ \sum_{k=1}^3 \mathcal{I}[N_k^1] f_k(\cdot, x_3) * ((1-\chi)V_h)(x) \\
&+ \varphi_{HF} * \begin{pmatrix} (1-\chi)v_{0,h} \\ \nabla \cdot ((1-\chi)V_h) \end{pmatrix}(x) \\
&+ \psi_1 * ((1-\chi)v_{0,h})(x) + \psi_2 * ((1-\chi)V_h)(x).
\end{aligned} \tag{5.24}$$

We first claim that this formula does not depend on the choice of the function χ: indeed, let $\chi_1,\ \chi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi_i \equiv 1$ on $B(x_h, 1)$. Then, since $\chi_1 - \chi_2 = 0$ on $B(x_h, 1)$ and $\chi_1 - \chi_2$ is compactly supported, we may write

$$\begin{aligned}
&\sum_{k=1}^3 \mathcal{I}[M_k^1] f_k(\cdot, x_3) * ((\chi_1 - \chi_2) v_{0,h}) + \psi_1 * ((\chi_1 - \chi_2) v_{0,h}) \\
= \;& \mathcal{F}^{-1}\left(\sum_{k=1}^3 \phi(\xi) M_k \widehat{(\chi_1 - \chi_2) v_{0,h}} \exp(-\lambda_k x_3) \right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^3 \mathcal{I}[N_k^1] f_k(\cdot, x_3) * ((\chi_1 - \chi_2) V_h) + \psi_2 * ((\chi_1 - \chi_2) V_h) \\
= \;& \mathcal{F}^{-1}\left(\sum_{k=1}^3 \phi(\xi) N_k \widehat{(\chi_1 - \chi_2) V_h} \exp(-\lambda_k x_3) \right) \\
= \;& \mathcal{F}^{-1}\left(\sum_{k=1}^3 \phi(\xi) L_k e_3 \mathcal{F}\left(\nabla \cdot (\chi_1 - \chi_2) V_h \right) \exp(-\lambda_k x_3) \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\varphi_{HF} * \begin{pmatrix} (\chi_1 - \chi_2) v_{0,h} \\ \nabla \cdot ((\chi_1 - \chi_2) V_h) \end{pmatrix} \\
= \;& \mathcal{F}^{-1}\left(\sum_{k=1}^3 (1 - \phi(\xi)) L_k \begin{pmatrix} \widehat{(\chi_1 - \chi_2) v_{0,h}} \\ \nabla \cdot \widehat{((\chi_1 - \chi_2) V_h)} \end{pmatrix} \exp(-\lambda_k x_3) \right).
\end{aligned}$$

Gathering all the terms, we find that the two definitions coincide. Moreover, u satisfies (5.23) (we refer to the proof of Proposition 5.14 for the derivation of such estimates: notice that the proof of Proposition 5.14 only uses local integrability properties of v_0).

There remains to prove that u is a solution of the Stokes system, which is not completely trivial due to the complexity of the representation formula. We start by deriving a duality formula: we claim that for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2)^3$, for all $x_3 > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^2} u(x_h, x_3) \cdot \eta(x_h)\, dx_h \;=\; & \int_{\mathbb{R}^2} v_{0,h}(x_h) \cdot \mathcal{F}^{-1}\left(\sum_{k=1}^3 \left(\overline{{}^t L_k} \hat{\eta}(\xi) \right)_h \exp(-\bar{\lambda}_k x_3) \right) \\
& - \int_{\mathbb{R}^2} V_h(x_h) \cdot \mathcal{F}^{-1}\left(\sum_{k=1}^3 i\xi \left(\overline{{}^t L_k} \hat{\eta}(\xi) \right)_3 \exp(-\bar{\lambda}_k x_3) \right).
\end{aligned} \tag{5.25}$$

To that end, in (5.24), we may choose a function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ on the set

$$\{x \in \mathbb{R}^2,\ d(x, \operatorname{Supp} \eta) \le 1\}.$$

We then transform every term in (5.24). We have, according to the Parseval formula

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\left(\frac{\widehat{\chi v_{0,h}}(\xi)}{\widehat{\nabla\cdot(\chi V_h)}(\xi)}\right)\exp(-\lambda_k x_3)\right)\cdot\eta \\
=\ & \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\sum_{k=1}^3 \overline{\hat\eta(\xi)}\cdot L_k(\xi)\left(\frac{\widehat{\chi v_{0,h}}(\xi)}{\widehat{\nabla\cdot(\chi V_h)}(\xi)}\right)\exp(-\lambda_k x_3)\,d\xi \\
=\ & \int_{\mathbb{R}^2}\chi v_{0,h}\mathcal{F}^{-1}\left(\sum_{k=1}^3\left(\overline{{}^tL_k}\hat\eta(\xi)\right)_h\exp(-\bar\lambda_k x_3)\right) \\
-\ & \int_{\mathbb{R}^2}(1-\chi)V_h\cdot\mathcal{F}^{-1}\left(\sum_{k=1}^3 i\xi\left(\overline{{}^tL_k}\hat\eta(\xi)\right)_3\exp(-\bar\lambda_k x_3)\right).
\end{aligned}$$

Using standard convolution results, we have

$$\int_{\mathbb{R}^2}\psi_1 * ((1-\chi)v_{0,h})\eta = \int_{\mathbb{R}^2}(1-\chi)v_{0,h}\,{}^t\check\psi_1 * \eta.$$

The terms with ψ_2 and φ_{HF} are transformed using identical computations. Concerning the term with $\mathcal{I}[M_k^1]$, we use Lemma 5.16, from which we infer that

$$\int_{\mathbb{R}^2}\mathcal{I}\left[M_k^1\right]f_k * ((1-\chi)v_{0,h})\eta = \int_{\mathbb{R}^2}(1-\chi)v_{0,h}\mathcal{I}\left[{}^tM_k^1\right]\check f_k * \eta.$$

Notice also that by definition of M_k^1, $\widecheck{M_k^1} = M_k^1$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^2}\psi_1 * ((1-\chi)v_{0,h})\eta \quad &+ \quad \sum_{k=1}^3\int_{\mathbb{R}^2}\mathcal{I}\left[M_k^1\right]f_k * ((1-\chi)v_{0,h})\eta \\
&= \int_{\mathbb{R}^2}(1-\chi)v_{0,h}\cdot\mathcal{F}^{-1}\left(\sum_{k=1}^3 {}^t\left(\check L_k e_1 \quad \check L_k e_2\right)\hat\eta\check\phi(\xi)\exp(-\check\lambda_k x_3)\right).
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2}\psi_2 * ((1-\chi)V_h\eta + \sum_{k=1}^3\int_{\mathbb{R}^2}\mathcal{I}\left[N_k^1\right]f_k * ((1-\chi)V_h)\eta \\
=\ & \int_{\mathbb{R}^2}(1-\chi)V_h\cdot\mathcal{F}^{-1}\left(\sum_{k=1}^3\xi\,{}^t\left(i\check L_k e_3\right)\hat\eta\check\phi(\xi)\exp(-\check\lambda_k x_3)\right).
\end{aligned}$$

Now, we recall that if $v_0 \in H^{1/2}(\mathbb{R}^2)\cap\mathbb{K}$ is real-valued, then so is the solution u of (5.4). Therefore, in Fourier space,

$$\overline{\hat u(\cdot,x_3)} = \check{\hat u}(\cdot,x_3) \quad \forall x_3 > 0.$$

We infer in particular that

$$\sum_{k=1}^3\check L_k\exp(-\check\lambda_k x_3) = \sum_{k=1}^3\bar L_k\exp(-\bar\lambda_k x_3).$$

Gathering all the terms, we obtain (5.25).

Now, let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^2 \times (0,\infty))^3$ such that $\nabla \cdot \zeta = 0$, and $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2 \times (0,\infty))$. We seek to prove that

$$\int_{\mathbb{R}^3_+} u\,(-\Delta\zeta - e_3 \times \zeta) = 0 \tag{5.26}$$

as well as

$$\int_{\mathbb{R}^3_+} u \cdot \nabla\eta = 0. \tag{5.27}$$

Using (5.25), we infer that

$$\begin{aligned}&\int_{\mathbb{R}^3_+} u\,(-\Delta\zeta - e_3 \times \zeta)\\ = \;& \int_0^\infty \int_{\mathbb{R}^2} v_{0,h}\mathcal{F}^{-1}\left(\sum_{k=1}^3 \overline{\mathcal{M}_k(\xi)}\hat{\zeta}(\xi)\exp(-\bar{\lambda}_k x_3)\right)\\ + \;& \int_0^\infty \int_{\mathbb{R}^2} V_h\mathcal{F}^{-1}\left(\sum_{k=1}^3 \overline{\mathcal{N}_k(\xi)}\hat{\zeta}(\xi)\exp(-\bar{\lambda}_k x_3)\right),\end{aligned}$$

where

$$\mathcal{M}_k := (|\xi|^2 - \lambda_k^2)\,{}^tM_k + {}^tM_k\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \mathcal{N}_k := (|\xi|^2 - \lambda_k^2)\,{}^tN_k + {}^tN_k\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

According to Lemma 5.7,

$$\mathcal{M}_k = \begin{pmatrix} i\xi_1 q_{k,1} & i\xi_2 q_{k,1} & -\lambda_k q_{k,1}\\ i\xi_1 q_{k,2} & i\xi_2 q_{k,2} & -\lambda_k q_{k,2}\end{pmatrix}$$

so that, since $i\xi \cdot \hat{\zeta}_h + \partial_3\hat{\zeta}_3 = 0$,

$$\overline{\mathcal{M}_k(\xi)}\hat{\zeta}(\xi, x_3) = (\partial_3\hat{\zeta}_3 - \bar{\lambda}_k\hat{\zeta}_3)\begin{pmatrix}\bar{q}_{k,1}\\ \bar{q}_{k,2}\end{pmatrix}.$$

Integrating in x_3, we find that

$$\int_0^\infty \overline{\mathcal{M}_k(\xi)}\hat{\zeta}(\xi, x_3)\exp(-\bar{\lambda}_k x_3)dx_3 = 0.$$

Similar arguments lead to

$$\int_0^\infty \int_{\mathbb{R}^2} V_h\mathcal{F}^{-1}\left(\sum_{k=1}^3 \overline{\mathcal{N}_k(\xi)}\hat{\zeta}(\xi, x_3)\exp(-\bar{\lambda}_k x_3)\right) = 0$$

and to the divergence-free condition (5.27). □

5.2.2 The Dirichlet to Neumann operator for the Stokes-Coriolis system

We now define the Dirichlet to Neumann operator for the Stokes-Coriolis system with boundary data in $\mathbb{K}$. We start by deriving its expression for a boundary data $v_0 \in H^{1/2}(\mathbb{R}^2)$ satisfying (5.5), for which we consider the unique solution u of (5.4) in $\dot{H}^1(\mathbb{R}^3_+)$. We recall that u is defined in Fourier space by (5.11). The corresponding pressure term is given by

$$\hat{p}(\xi, x_3) = \sum_{k=1}^3 A_k(\xi)\frac{|\xi|^2 - \lambda_k(\xi)^2}{\lambda_k(\xi)}\exp(-\lambda_k(\xi)x_3).$$

The Dirichlet to Neumann operator is then defined by

$$\mathrm{DN}\, v_0 := -\partial_3 u|_{x_3=0} + p|_{x_3=0} e_3.$$

Consequently, in Fourier space, the Dirichlet to Neumann operator is given by

$$\widehat{\mathrm{DN}\, v_0}(\xi) = \sum_{k=1}^{3} A_k(\xi) \begin{pmatrix} \frac{i}{|\xi|^2}(-\lambda_k^2 \xi + (|\xi|^2 - \lambda_k^2)^2 \xi^\perp) \\ \frac{|\xi|^2}{\lambda_k} \end{pmatrix} =: M_{SC}(\xi)\hat{v}_0(\xi), \tag{5.28}$$

where $M_{SC} \in \mathcal{M}_{3,3}(\mathbb{C})$. Using the notations of the previous paragraph, we have

$$M_{SC} = \sum_{k=1}^{3} \lambda_k L_k + e_3\, {}^t q_k.$$

Let us first review a few useful properties of the Dirichlet to Neumann operator:

Proposition 5.20.

– *Behaviour at large frequencies: when* $|\xi| \gg 1$,

$$M_{SC}(\xi) = \begin{pmatrix} |\xi| + \dfrac{\xi_1^2}{|\xi|} & \dfrac{\xi_1\xi_2}{|\xi|} & i\xi_1 \\ \dfrac{\xi_1\xi_2}{|\xi|} & |\xi| + \dfrac{\xi_2^2}{|\xi|} & i\xi_2 \\ -i\xi_1 & -i\xi_2 & 2|\xi| \end{pmatrix} + O(|\xi|^{1/3}).$$

– *Behaviour at small frequencies: when* $|\xi| \ll 1$,

$$M_{SC}(\xi) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 & \dfrac{i(\xi_1+\xi_2)}{|\xi|} \\ 1 & 1 & \dfrac{i(\xi_2-\xi_1)}{|\xi|} \\ \dfrac{i(\xi_2-\xi_1)}{|\xi|} & \dfrac{-i(\xi_1+\xi_2)}{|\xi|} & \dfrac{\sqrt{2}}{|\xi|} - 1 \end{pmatrix} + O(|\xi|).$$

– *The horizontal part of the Dirichlet to Neumann operator, denoted by* DN_h, *maps* $H^{1/2}(\mathbb{R}^2)$ *into* $H^{-1/2}(\mathbb{R}^2)$.

– *Let* $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ *such that* $\phi(\xi) = 1$ *for* $|\xi| \le 1$. *Then*

$$\begin{aligned} (1-\phi(D))\,\mathrm{DN}_3 &: H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2), \\ D\phi(D)\,\mathrm{DN}_3, |D|\phi(D)\,\mathrm{DN}_3 &: L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \end{aligned}$$

where, classically, $a(D)$ *denotes the operator defined in Fourier space by*

$$\widehat{a(D)u} = a(\xi)\hat{u}(\xi)$$

for $a \in \mathcal{C}(\mathbb{R}^2)$, $u \in L^2(\mathbb{R}^2)$.

Remark 5.21. For $|\xi| \gg 1$, the Dirichlet to Neumann operator for the Stokes-Coriolis system has the same expression, at main order, as the one of the Stokes system. This can be easily understood since at large frequencies, the rotation term in the system (5.6) can be neglected in front of $|\xi|^2\hat{u}$, and therefore the system behaves roughly as the Stokes system.

Proof. The first two points follow from the expression (5.28) together with the asymptotic expansions in Lemma 5.5. Since they are lengthy but straightforward calculations, we postpone them to the Appendix 5.A.

The horizontal part of the Dirichlet to Neumann operator satisfies

$$\begin{aligned}|\widehat{\mathrm{DN}_h\, v_0}(\xi)| &= O(|\xi|\,|\hat{v}_0(\xi)|) \quad \text{for } |\xi| \gg 1,\\ |\widehat{\mathrm{DN}_h\, v_0}(\xi)| &= O(|\hat{v}_0(\xi)|) \quad \text{for } |\xi| \ll 1.\end{aligned}$$

Therefore, if $\int_{\mathbb{R}^2}(1+|\xi|^2)^{1/2}|\hat{v}_0(\xi)|^2\, d\xi < \infty$, we deduce that

$$\int_{\mathbb{R}^2}(1+|\xi|^2)^{-1/2}|\widehat{\mathrm{DN}_h\, v_0}(\xi)|^2\, d\xi < \infty.$$

Hence $\mathrm{DN}_h : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$.

In a similar way,

$$|\widehat{\mathrm{DN}_3\, v_0}(\xi)| = O(|\xi|\,|\hat{v}_0(\xi)|) \quad \text{for } |\xi| \gg 1,$$

so that if $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ is such that $\phi(\xi) = 1$ for ξ in a neighbourhood of zero, there exists a constant C such that

$$\left|(1-\phi(\xi))\widehat{\mathrm{DN}_3\, v_0}(\xi)\right| \le C|\xi|\,|\hat{v}_0(\xi)| \quad \forall \xi \in \mathbb{R}^2.$$

Therefore $(1-\phi(D))\,\mathrm{DN}_3 : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$.

The vertical part of the Dirichlet to Neumann operator, however, is singular at low frequencies. This is consistent with the singularity observed in $L_1(\xi)$ for ξ close to zero. More precisely, for ξ close to zero, we have

$$\widehat{\mathrm{DN}_3\, v_0}(\xi) = \frac{1}{|\xi|}\hat{v}_{0,3} + O(|\hat{v}_0(\xi)|).$$

Consequently, for all $\xi \in \mathbb{R}^2$

$$\left|\xi\phi(\xi)\widehat{\mathrm{DN}_3\, v_0}(\xi)\right| \le C|\hat{v}_0(\xi)|. \qquad \square$$

Following [GVM10], we now extend the definition of the Dirichlet to Neumann operator to functions which are not square integrable in $\mathbb{R}^2$, but rather locally uniformly integrable. There are several differences with [GVM10]: first, the Fourier multiplier associated with DN is not homogeneous, even at the main order. Therefore its kernel (the inverse Fourier transform of the multiplier) is not homogeneous either, and, in general, does not have the same decay as the kernel of Stokes system. Moreover, the singular part of the Dirichlet to Neumann operator for low frequencies prevents us from defining DN on $H^{1/2}_{uloc}$. Hence we will define DN on $\mathbb{K}$ only (see also Corollary 5.18).

Let us briefly recall the definition of the Dirichlet to Neumann operator for the Stokes system (see [GVM10]), which we denote by DN_S [2]. The Fourier multiplier of DN_S is

$$M_S(\xi) := \begin{pmatrix} |\xi| + \dfrac{\xi_1^2}{|\xi|} & \dfrac{\xi_1\xi_2}{|\xi|} & i\xi_1 \\ \dfrac{\xi_1\xi_2}{|\xi|} & |\xi| + \dfrac{\xi_2^2}{|\xi|} & i\xi_2 \\ -i\xi_1 & -i\xi_2 & 2|\xi| \end{pmatrix}.$$

2. In [GVM10], D. Gérard-Varet and N. Masmoudi consider the Stokes system in $\mathbb{R}^2_+$ and not $\mathbb{R}^3_+$, but this part of their proof does not depend on the dimension.

The corresponding kernel, denoted by K_S (i.e. the inverse Fourier transform of M_S in $\mathcal{S}'$) is homogeneous of order -3, and therefore

$$|K_S(t)| \le \frac{C}{|t|^3}.$$

Hence DN_S is defined on $H^{1/2}_{uloc}$ in the following way: for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ on the set $\{t \in \mathbb{R}^2, d(t, \operatorname{Supp}\varphi) \le 1\}$. Then

$$\langle \mathrm{DN}_S\, u, \varphi\rangle_{\mathcal{D}',\mathcal{D}} := \langle \mathcal{F}^{-1}\left(M_S \widehat{\chi u}\right), \varphi\rangle_{H^{-1/2},H^{1/2}} + \int_{\mathbb{R}^2} K_S * ((1-\chi)u)\cdot\varphi.$$

The assumption on χ ensures that there is no singularity in the last integral, while the decay of K_S ensures its convergence.

We wish to adopt a similar method here, but a few precautions must be taken because of the singularities at low frequencies, in the spirit of the representation formula (5.24). Hence, before defining the action of DN on $\mathbb{K}$, let us decompose the Fourier multiplier associated with DN. We have

$$M_{SC}(\xi) = M_S(\xi) + \phi(\xi)(M_{SC} - M_S)(\xi) + (1-\phi)(\xi)(M_{SC} - M_S)(\xi).$$

Concerning the third term, we have the following result, which is a straightforward consequence of Proposition 5.20 and Appendix 5.B:

Lemma 5.22. *As $|\xi| \to \infty$, there holds*

$$\nabla_\xi^\alpha (M_{SC} - M_S)(\xi) = O\left(|\xi|^{\frac{1}{3}-|\alpha|}\right)$$

for $\alpha \in \mathbb{N}^2$, $0 \le |\alpha| \le 3$.

We deduce from Lemma 5.22 that $\nabla^\alpha\left[(1-\phi(\xi))(M_{SC} - M_S)(\xi)\right] \in L^1(\mathbb{R}^2)$ for all $\alpha \in \mathbb{N}^2$ with $|\alpha| = 3$, so that it follows from lemma 5.35 that there exists a constant $C > 0$ such that

$$\left|\mathcal{F}^{-1}\left[(1-\phi(\xi))(M_{SC} - M_S)(\xi)\right](t)\right| \le \frac{C}{|t|^3}.$$

There remains to decompose $\phi(\xi)(M_{SC} - M_S)(\xi)$. As in Proposition 5.14, the multipliers which are homogeneous of order one near $\xi = 0$ are treated separately. Note that since the last column and the last line of M_{SC} act on horizontal divergences (see Proposition 5.23), we are interested in multipliers homogeneous of order zero in $M_{SC,3i}, M_{SC,i3}$ for $i = 1, 2$, and homogeneous of order -1 in $M_{SC,33}$. In the following, we set

$$\bar{M}_h := \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \bar{M} := \begin{pmatrix} \bar{M}_h & 0 \\ 0 & 0 \end{pmatrix},$$
$$V_1 := \frac{i\sqrt{2}}{2|\xi|}\begin{pmatrix} \xi_1 + \xi_2 \\ \xi_1 - \xi_2 \end{pmatrix}, \quad V_2 := \frac{i\sqrt{2}}{2|\xi|}\begin{pmatrix} -\xi_1 + \xi_2 \\ -\xi_1 - \xi_2 \end{pmatrix}.$$

We decompose $M_{SC} - M_S$ near $\xi = 0$ as

$$\phi(\xi)(M_{SC} - M_S)(\xi) = \bar{M} + \phi(\xi)\begin{pmatrix} M_1 & V_1 \\ {}^tV_2 & |\xi|^{-1} \end{pmatrix} - (1-\phi(\xi))\bar{M} + \phi(\xi)M^{rem},$$

where $M_1 \in \mathcal{M}_2(\mathbb{C})$ only contains homogeneous and nonpolynomial terms of order one, and M^{rem}_{ij} contains either polynomial terms or remainder terms which are $o(|\xi|)$ for ξ close to zero if $1 \leq i, j \leq 2$. Looking closely at the expansions for λ_k in a neighbourhood of zero (see (5.65)) and at the calculations in paragraph 5.A.4, we infer that M_{ij}^{rem} contains either polynomial terms or remainder terms of order $O(|\xi|^2)$ if $1 \leq i, j \leq 2$. We emphasize that the precise expression of M^{rem} is not needed in the following, although it can be computed by pushing forward the expansions of Appendix 5.A. In a similar fashion, $M_{i,3}^{rem}$ and $M_{3,i}^{rem}$ contain constant terms and remainder terms of order $O(|\xi|)$ for $i = 1, 2$, $M_{3,3}^{rem}$ contains remainder terms of order $O(1)$. As a consequence, if we define the low-frequency kernels $K_i^{rem} : \mathbb{R}^2 \to \mathcal{M}_2(\mathbb{C})$ for $1 \leq i \leq 4$ by

$$\begin{aligned}
K_1^{rem} &:= \mathcal{F}^{-1}\left(\phi \begin{pmatrix} M_{11}^{rem} & M_{12}^{rem} \\ M_{21}^{rem} & M_{22}^{rem} \end{pmatrix}\right), \\
K_2^{rem} &:= \mathcal{F}^{-1}\left(\phi \begin{pmatrix} M_{13}^{rem} \\ M_{23}^{rem} \end{pmatrix} i \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}\right), \\
K_3^{rem} &:= \mathcal{F}^{-1}\left(-i\phi(\xi)\xi \begin{pmatrix} M_{31}^{rem} & M_{32}^{rem} \end{pmatrix}\right), \\
K_4^{rem} &:= \mathcal{F}^{-1}\left(\phi(\xi) M_{33}^{rem} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}\right)
\end{aligned}$$

we have, for $1 \leq i \leq 4$ (see Lemmas 5.33 and 5.36)

$$|K_i^{rem}(x_h)| \leq \frac{C}{|x_h|^3} \quad \forall x_h \in \mathbb{R}^2.$$

We also set

$$M_{HF}^{rem} := \mathcal{F}^{-1}\left(-(1-\phi)\bar{M} + (1-\phi)(M_{SC} - M_S)\right),$$

which also satisfies

$$|M_{HF}^{rem}(x_h)| \leq \frac{C}{|x_h|^3} \quad \forall x_h \in \mathbb{R}^2.$$

There remains to define the kernels homogeneous of order one besides M_1. We set

$$\begin{aligned}
M_2 &:= V_1 i \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}, \\
M_3 &:= -i\xi\, {}^tV_2, \\
M_4 &:= \frac{1}{|\xi|} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix},
\end{aligned}$$

so that M_1, M_2, M_3, M_4 are 2×2 real valued matrices whose coefficients are linear combinations of $\frac{\xi_i \xi_j}{|\xi|}$. In the end, we will work with the following decomposition for the matrix M_{SC}, where the treatment of each of the terms has been explained above:

$$M_{SC} = M_S + \bar{M} + (1-\phi)(M_{SC} - M_S - \bar{M}) + \phi \begin{pmatrix} M_1 & V_1 \\ {}^tV_2 & |\xi|^{-1} \end{pmatrix} + \phi M^{rem}.$$

We are now ready to extend the definition of the Dirichlet to Neumann operator to functions in $\mathbb{K}$: in the spirit of Proposition 5.14-Corollary 5.18, we derive a representation formula for functions in $\mathbb{K} \cap H^{1/2}(\mathbb{R}^2)^3$, which still makes sense for functions in $\mathbb{K}$:

Proposition 5.23. *Let $\varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}^2\right)^3$ such that $\varphi_3 = \nabla_h \cdot \Phi_h$ for some $\Phi_h \in \mathcal{C}_0^\infty\left(\mathbb{R}^2\right)$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ on the set*

$$\{x \in \mathbb{R}^2,\ d(x, \operatorname{Supp} \varphi \cup \operatorname{Supp} \Phi_h) \leq 1\}.$$

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_\xi^2)$ such that $\phi(\xi) = 1$ if $|\xi| \leq 1$, and let $\rho := \mathcal{F}^{-1}\phi$.

– *Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ such that $v_{0,3} = \nabla_h \cdot V_h$. Then*

$$\begin{aligned}
\langle \mathrm{DN}(v_0), \varphi\rangle_{\mathcal{D}',\mathcal{D}} &= \langle \mathrm{DN}_S(v_0), \varphi\rangle_{\mathcal{D}',\mathcal{D}} + \int_{\mathbb{R}^2} \varphi \cdot \bar{M} v_0 \\
&+ \left\langle \mathcal{F}^{-1}\left((1-\phi)\left(M_{SC} - M_S - \bar{M}\right)\widehat{\chi v_0}\right), \varphi\right\rangle_{H^{-1/2},H^{1/2}} \\
&+ \int_{\mathbb{R}^2} \varphi \cdot M_{HF}^{rem} * ((1-\chi)v_0) \\
&+ \left\langle \mathcal{F}^{-1}\left(\phi\left(M^{rem} + \begin{pmatrix} M_1 & V_1 \\ {}^tV_2 & |\xi|^{-1}\end{pmatrix}\right)\begin{pmatrix}\widehat{\chi v_{0,h}} \\ i\xi \cdot \widehat{\chi V_h}\end{pmatrix}\right), \varphi\right\rangle_{H^{-1/2},H^{1/2}} \\
&+ \int_{\mathbb{R}^2} \varphi_h \cdot \{\mathcal{I}[M_1](\rho * (1-\chi)v_{0,h}) + K_1^{rem} * ((1-\chi)v_{0,h})\} \\
&+ \int_{\mathbb{R}^2} \varphi_h \cdot \{\mathcal{I}[M_2](\rho * (1-\chi)V_h) + K_2^{rem} * ((1-\chi)V_h)\} \\
&+ \int_{\mathbb{R}^2} \Phi_h \cdot \{\mathcal{I}[M_3](\rho * (1-\chi)v_{0,h}) + K_3^{rem} * ((1-\chi)v_{0,h})\} \\
&+ \int_{\mathbb{R}^2} \Phi_h \cdot \{\mathcal{I}[M_4](\rho * (1-\chi)V_h) + K_4^{rem} * ((1-\chi)V_h)\}.
\end{aligned}$$

– *The above formula still makes sense when $v_0 \in \mathbb{K}$, which allows us to extend the definition of* DN *to* $\mathbb{K}$.

Remark 5.24. Notice that if $v_0 \in \mathbb{K}$ and $\varphi \in \mathbb{K}$ with $\varphi_3 = \nabla_h \cdot \Phi_h$, and if φ, Φ_h have compact support, then the right-hand side of the formula in Proposition 5.23 still makes sense. Therefore $\mathrm{DN}\, v_0$ can be extended into a linear form on the set of functions in $\mathbb{K}$ with compact support. In this case, we will denote it by

$$\langle \mathrm{DN}(v_0), \varphi\rangle,$$

without specifying the functional spaces.

The proof of the Proposition 5.23 is very close to the one of Proposition 5.14 and Corollary 5.18, and therefore we leave it to the reader.

The goal is now to link the solution of the Stokes-Coriolis system in $\mathbb{R}^3_+$ with $v_0 \in \mathbb{K}$ and $\mathrm{DN}(v_0)$. This is done through the following lemma:

Lemma 5.25. *Let $v_0 \in \mathbb{K}$, and let u be the unique solution of* (5.4) *with $u|_{x_3=0} = v_0$, given by Corollary 5.18.*

Let $\varphi \in \mathcal{C}_0^\infty(\bar{\mathbb{R}}^3_+)^3$ such that $\nabla \cdot \varphi = 0$. Then

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi = \langle \mathrm{DN}(v_0), \varphi|_{x_3=0}\rangle.$$

In particular, if $v_0 \in \mathbb{K}$ with $v_{0,3} = \nabla_h \cdot V_h$ and if v_0, V_h have compact support, then

$$\langle \mathrm{DN}(v_0), v_0\rangle \geq 0.$$

Remark 5.26. If $\varphi \in \mathcal{C}_0^\infty\left(\overline{\mathbb{R}^3_+}\right)^3$ is such that $\nabla \cdot \varphi = 0$, then in particular

$$\begin{aligned}
\varphi_{3|x_3=0}(x_h) &= -\int_0^\infty \partial_3 \varphi_3(x_h, z)\, dz \\
&= \int_0^\infty \nabla_h \cdot \varphi_h(x_h, z) = \nabla_h \cdot \Phi_h
\end{aligned}$$

for $\Phi_h := \int_0^\infty \varphi_h(\cdot, z) dz \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. In particular $\varphi|_{x_3=0}$ is a suitable test function for Proposition 5.23.

Proof. The proof relies on two duality formulas in the spirit of (5.25), one for the Stokes-Coriolis system and the other for the Dirichlet to Neumann operator. We claim that if $v_0 \in \mathbb{K}$, then on the one hand

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi = \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left({}^t\bar{M}_{SC}(\xi) \hat{\varphi}|_{x_3=0}(\xi) \right) \tag{5.29}$$

and on the other hand, for any $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2)^3$ such that $\eta_3 = \nabla_h \cdot \theta_h$ for some $\theta_h \in \mathcal{C}_0^\infty(\mathbb{R}^2)^2$,

$$\langle \mathrm{DN}(v_0), \eta \rangle_{\mathcal{D}', \mathcal{D}} = \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left({}^t\bar{M}_{SC}(\xi) \hat{\eta}(\xi) \right). \tag{5.30}$$

Applying formula (5.30) with $\eta = \varphi|_{x_3=0}$ then yields the desired result. Once again, the proofs of (5.29), (5.30) are close to the one of (5.25). From (5.25), one has

$$\begin{aligned}
\int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi &= -\int_{\mathbb{R}^3_+} u \cdot e_3 \times \varphi \\
&= -\int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\int_0^\infty \sum_{k=1}^3 \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k e_3 \times \hat{\varphi} \right) \\
&= \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\int_0^\infty \sum_{k=1}^3 \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{\varphi} \right).
\end{aligned}$$

Moreover, we deduce from the representation formula for u and from Lemma 5.16 a representation formula for ∇u: we have

$$\begin{aligned}
\nabla u(x) &= \mathcal{F}^{-1} \left(\sum_{k=1}^3 \exp(-\lambda_k x_3) L_k(\xi) \begin{pmatrix} \widehat{\chi v_{0,h}} \\ \nabla \cdot (\chi V_h) \end{pmatrix} \begin{pmatrix} i\xi_1 & i\xi_2 & -\lambda_k \end{pmatrix} \right)(x) \\
&+ \sum_{k=1}^3 \mathcal{I}[M_k^1] \nabla f_k(\cdot, x_3) * ((1-\chi) v_{0,h})(x) \\
&+ \sum_{k=1}^3 \mathcal{I}[N_k^1] \nabla f_k(\cdot, x_3) * ((1-\chi) V_h)(x) \\
&+ \nabla \varphi_{HF} * \begin{pmatrix} (1-\chi) v_{0,h}(\xi) \\ \nabla \cdot ((1-\chi) V_h) \end{pmatrix} \\
&+ \nabla \psi_1 * ((1-\chi) v_{0,h})(x) + \nabla \psi_2 * ((1-\chi) V_h)(x).
\end{aligned}$$

Then, proceeding exactly as in the proof of Corollary 5.18, we infer that

$$\begin{aligned}
\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi &= \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\sum_{k=1}^3 \int_0^\infty |\xi|^2 \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k \hat{\varphi}(\xi, x_3) dx_3 \right) \\
&- \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\sum_{k=1}^3 \int_0^\infty \bar{\lambda}_k \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k \partial_3 \hat{\varphi}(\xi, x_3) dx_3 \right).
\end{aligned}$$

Integrating by parts in x_3, we obtain

$$\int_0^\infty \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k \partial_3 \hat{\varphi}(\xi, x_3) dx_3 = \bar{\lambda}_k \int_0^\infty \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k \hat{\varphi}(\xi, x_3) dx_3 - {}^t\bar{L}_k \hat{\varphi}|_{x_3=0}(\xi).$$

Gathering the terms, we infer

$$\begin{aligned}\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi &= \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\int_0^\infty \sum_{k=1}^3 \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{P}_k \hat{\varphi} \right) \\ &+ \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\sum_{k=1}^3 \bar{\lambda}_k\, {}^t\bar{L}_k \hat{\varphi}|_{x_3=0} \right),\end{aligned}$$

where

$$\begin{aligned} P_k &:= (|\xi|^2 - \lambda_k^2) L_k + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_k \\ &= -\begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -\lambda_k \end{pmatrix} \begin{pmatrix} q_{k,1} & q_{k,2} & q_{k,3} \end{pmatrix} \end{aligned}$$

according to Lemma 5.7. Therefore, since φ is divergence-free, we have

$${}^t\bar{P}_k \hat{\varphi} = (-\partial_3 \hat{\varphi}_3 + \bar{\lambda}_k \hat{\varphi}_3) \begin{pmatrix} \bar{q}_{k,1} \\ \bar{q}_{k,2} \\ \bar{q}_{k,3} \end{pmatrix},$$

so that eventually, after integrating by parts once more in x_3,

$$\begin{aligned} &\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi \\ = &\int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left(\left[\sum_{k=1}^3 \bar{\lambda}_k\, {}^t L_k + \begin{pmatrix} \bar{q}_{k,1} \\ \bar{q}_{k,2} \\ \bar{q}_{k,3} \end{pmatrix} {}^t e_3 \right] \hat{\varphi}|_{x_3=0} \right) \\ = &\int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1} \left({}^t\bar{M}_{SC} \hat{\varphi}|_{x_3=0} \right). \end{aligned}$$

The derivation of (5.30) is very similar to the one of (5.25) and therefore we skip its proof. □

We conclude this paragraph with some estimates on the Dirichlet to Neumann operator:

Lemma 5.27. *There exists a positive constant C such that the following property holds. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)^3$ such that $\varphi_3 = \nabla_h \cdot \Phi_h$ for some $\Phi_h \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, and let $v_0 \in \mathbb{K}$ with $v_{0,3} = \nabla_h \cdot V_h$. Let $R \geq 1$ and $x_0 \in \mathbb{R}^2$ such that*

$$\operatorname{Supp} \varphi \cup \operatorname{Supp} \Phi_h \subset B(x_0, R).$$

Then

$$|\langle \mathrm{DN}(v_0), \varphi \rangle_{\mathcal{D}',\mathcal{D}}| \leq CR \left(\|\varphi\|_{H^{1/2}(\mathbb{R}^2)} + \|\Phi_h\|_{H^{1/2}(\mathbb{R}^2)} \right) \left(\|v_0\|_{H^{1/2}_{uloc}} + \|V_h\|_{H^{1/2}_{uloc}} \right).$$

Moreover, if $v_0, V_h \in H^{1/2}(\mathbb{R}^2)$, then

$$|\langle \mathrm{DN}(v_0), \varphi \rangle_{\mathcal{D}',\mathcal{D}}| \leq C \left(\|\varphi\|_{H^{1/2}(\mathbb{R}^2)} + \|\Phi_h\|_{H^{1/2}(\mathbb{R}^2)} \right) \left(\|v_0\|_{H^{1/2}} + \|V_h\|_{H^{1/2}} \right).$$

Proof. The second inequality is classical and follows from the Fourier definition of the Dirichlet to Neumann operator. We therefore focus on the first inequality, for which we use the representation formula of Proposition 5.23.

We consider a truncation function χ such that $\chi \equiv 1$ on $B(x_0, R+1)$ and $\chi \equiv 0$ on $B(x_0, R+2)^c$, and such that $\|\nabla^\alpha \chi\|_\infty \le C_\alpha$, with C_α independent of R, for all $\alpha \in \mathbb{N}$. We must evaluate three different types of term:

▷ Terms of the type

$$\int_{\mathbb{R}^2} K * ((1-\chi)v_0) \cdot \varphi,$$

where K is a matrix such that $|K(x)| \le C|x|^{-3}$ for all $x \in \mathbb{R}^2$ (of course, we include in the present discussion all the variants involving V_h and Φ_h). These terms are bounded by

$$\begin{aligned}
& C \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|t|^3} |1-\chi(x-t)|\, |v_0(x-t)|\, |\varphi(x)|\, dx\, dt \\
\le\ & C \int_{\mathbb{R}^2} dx\, |\varphi(x)| \left(\int_{|t|\ge 1} \frac{|v_0(x-t)|^2}{|t|^3} dt \right)^{1/2} \left(\int_{|t|\ge 1} \frac{1}{|t|^3} dt \right)^{1/2} \\
\le\ & C \|v_0\|_{L^2_{uloc}} \|\varphi\|_{L^1} \\
\le\ & CR \|v_0\|_{L^2_{uloc}} \|\varphi\|_{L^2}.
\end{aligned}$$

▷ Terms of the type

$$\int_{\mathbb{R}^2} \varphi_h \cdot \mathcal{I}[M]((1-\chi)v_{0,h}) * \rho,$$

where M is a 2×2 matrix whose coefficients are linear combinations of $\xi_i\xi_j/|\xi|$. Using Lemma 5.11 and Remark 5.12, these terms are bounded by

$$C\|\varphi\|_{L^1} \|v_0\|_{L^2_{uloc}} \|(1+|\cdot|^2)\rho\|_{L^2}^{1/2} \|(1+|\cdot|^2)\nabla^2\rho\|_{L^2}^{1/2}.$$

Using Plancherel's Theorem, we have (up to a factor 2π)

$$\begin{gathered}
\|(1+|\cdot|^2)\rho\|_{L^2} = \|(1-\Delta)\phi\|_{L^2(\mathbb{R}^2)} \le C, \\
\|(1+|\cdot|^2)\nabla^2\rho\|_{L^2} = \|(1-\Delta)|\cdot|^2\phi\|_{L^2(\mathbb{R}^2)} \le C,
\end{gathered}$$

so that eventually

$$\left| \int_{\mathbb{R}^2} \varphi_h \cdot \mathcal{I}[M]((1-\chi)v_{0,h}) * \rho \right| \le C\|\varphi\|_{L^1} \|v_0\|_{L^2_{uloc}} \le CR\|v_0\|_{L^2_{uloc}} \|\varphi\|_{L^2}.$$

▷ Terms of the type

$$\langle \mathcal{F}^{-1}\left(M(\xi)\widehat{\chi v_0}(\xi)\right), \varphi\rangle_{H^{-1/2},H^{1/2}} \text{ and } \int_{\mathbb{R}^2} \varphi \cdot \bar{M} v_0$$

where $M(\xi)$ is some kernel such that $\mathrm{Op}(M) : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$ and $\bar{M}$ is a constant matrix.

All these terms are bounded by

$$C\|\chi v_0\|_{H^{1/2}(\mathbb{R}^2)} \|\varphi\|_{H^{1/2}(\mathbb{R}^2)}.$$

In fact, the trickiest part of the Lemma is to prove that

$$\|\chi v_0\|_{H^{1/2}(\mathbb{R}^2)} \le CR\|v_0\|_{H^{1/2}_{uloc}}. \tag{5.31}$$

To that end, we recall that

$$\|\chi v_0\|^2_{H^{1/2}(\mathbb{R}^2)} = \|\chi v_0\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|(\chi v_0)(x) - (\chi v_0)(y)|^2}{|x-y|^3} dx\, dy.$$

Clearly,

$$\|\chi v_0\|_{L^2(\mathbb{R}^2)} \le \|\chi\|_\infty \|v_0\|_{L^2(B(x_0,R+2))} \le CR\|u\|_{L^2_{uloc}}.$$

We then notice that the integrand of the second term is identically zero if $x, y \in B(x_0, R+2)^c$. We then decompose the domain $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$\begin{aligned}
\mathbb{R}^2 \times \mathbb{R}^2 &= B(x_0,R) \times B(x_0,R) \\
&\cup\ (B(x_0,R+2) \setminus B(x_0,R)) \times B(x_0,R) \cup B(x_0,R) \times (B(x_0,R+2)\setminus B(x_0,R)) \\
&\cup\ B(x_0,R+2)^c \times B(x_0,R+2)^c
\end{aligned}$$

and we evaluate the corresponding integral on each of the subdomains.

First, by definition of χ and of the $H^{1/2}_{uloc}$ norm

$$\int_{B(x_0,R)\times B(x_0,R)} \frac{|(\chi v_0)(x) - (\chi v_0)(y)|^2}{|x-y|^3} dx\, dy = \|v_0\|^2_{\dot H^{1/2}(B(x_0,R))} \le CR^2 \|v_0\|^2_{H^{1/2}_{uloc}}.$$

Moreover, by symmetry arguments, it is easily checked that

$$\begin{aligned}
&\int_{(B(x_0,R+2)\setminus B(x_0,R))\times B(x_0,R)} \frac{|(\chi v_0)(x) - (\chi v_0)(y)|^2}{|x-y|^3} dx\, dy \\
=\ &\int_{B(x_0,R)\times (B(x_0,R+2)\setminus B(x_0,R))} \frac{|(\chi v_0)(x) - (\chi v_0)(y)|^2}{|x-y|^3} dx\, dy,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{(B(x_0,R+2)\setminus B(x_0,R))\times B(x_0,R)} \frac{|(\chi v_0)(x) - (\chi v_0)(y)|^2}{|x-y|^3} dx\, dy \\
\le\ &\int_{(B(x_0,R+2)\setminus B(x_0,R))\times B(x_0,R)} |\chi(y)|^2 \frac{|v_0(x) - v_0(y)|^2}{|x-y|^3} dx\, dy \\
&+ \int_{(B(x_0,R+2)\setminus B(x_0,R))\times B(x_0,R)} |v_0(x)|^2 \frac{|\chi(x) - \chi(y)|^2}{|x-y|^3} dx\, dy \\
\le\ & CR^2 \|v_0\|^2_{H^{1/2}_{uloc}} + C\|\chi\|^2_{W^{1,\infty}} \|v_0\|^2_{L^2(B(x_0,R+2)\setminus B(x_0,R))} \int_{|z|\le 2R+2} \frac{dz}{|z|} \\
\le\ & CR^2 \|v_0\|^2_{H^{1/2}_{uloc}}.
\end{aligned}$$

Gathering all the terms, we obtain (5.31). This concludes the proof of the Lemma. □

5.2.3 Presentation of the new system

We now come to our main concern in this paper, which is to prove the existence of weak solutions to the linear system of rotating fluids in the bumpy half-space (5.1). There are two features which make this problem particularly difficult. Firstly, the fact that the bottom is now bumpy rather than flat prevents us from the use of the Fourier transform in the tangential direction. Secondly, as the domain Ω is unbounded, it is not possible to rely on Poincaré type inequalities. We face this problem using an idea of [GVM10]. It consists in defining a problem equivalent to (5.1) yet posed in the bounded channel Ω^b, by the mean of a transparent boundary condition at the interface $\Sigma = \{x_3 = 0\}$, namely

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p &= 0 \quad \text{in } \Omega^b, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega^b, \\ u|_\Gamma &= u_0, \\ -\partial_3 u + p e_3 &= \mathrm{DN}(u|_{x_3=0}) \quad \text{on } \Sigma. \end{cases} \tag{5.32}$$

In the system above and throughout the rest of the paper, we assume without any loss of generality that $\sup \omega < 0$, $\inf \omega \geq -1$. Notice that thanks to assumption (5.3), we have

$$\begin{aligned} u_{3|x_3=0}(x_h) &= u_{0,3}(x_h) - \int_{\omega(x_h)}^0 \nabla_h \cdot u_h(x_h, z)\, dz \\ &= u_{0,3}(x_h) - \nabla_h \omega \cdot u_{0,h}(x_h) \\ &\quad -\nabla_h \cdot \int_{\omega(x_h)}^0 u_h(x_h, z)\, dz \\ &= \nabla_h \cdot \left(U_h(x_h) - \int_{\omega(x_h)}^0 u_h(x_h, z)\, dz \right), \end{aligned}$$

so that $u_{3|x_3=0}$ satisfies the assumptions of Proposition 5.23.

Let us start by explaining the meaning of (5.32):

Definition 5.28. A function $u \in H^1_{uloc}(\Omega^b)$ is a solution of (5.32) if it satisfies the bottom boundary condition $u|_\Gamma = u_0$ in the trace sense, and if, for all $\varphi \in \mathcal{C}_0^\infty\left(\overline{\Omega_b}\right)$ such that $\nabla \cdot \varphi = 0$ and $\varphi|_\Gamma = 0$, there holds

$$\int_{\Omega^b} (\nabla u \cdot \nabla \varphi + e_3 \times u \cdot \varphi) = -\langle \mathrm{DN}(u|_{x_3=0}), \varphi|_{x_3=0} \rangle_{\mathcal{D}', \mathcal{D}}.$$

Remark 5.29. Notice that if $\varphi \in \mathcal{C}_0^\infty\left(\overline{\Omega_b}\right)$ is such that $\nabla \cdot \varphi = 0$ and $\varphi|_\Gamma = 0$, then

$$\varphi_{3|x_3=0} = \nabla_h \cdot \Phi_h, \text{ where } \Phi_h(x_h) := -\int_{\omega(x_h)}^0 \varphi_h(x_h, z) dz \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

Therefore φ is an admissible test function for Proposition 5.23.

We then have the following result, which is the Stokes-Coriolis equivalent of [GVM10, Proposition 9], and which follows easily from Lemma 5.25 and Corollary 5.18:

Proposition 5.30. *Let $u_0 \in L^2_{uloc}(\mathbb{R}^2)$ satisfying* (5.3)*, and assume that $\omega \in W^{1,\infty}(\mathbb{R}^2)$.*

– *Let (u, p) be a solution of* (5.1) *in Ω such that $u \in H^1_{loc}(\Omega)$ and*

$$\forall a > 0, \quad \sup_{l \in \mathbb{Z}^2} \int_{l + [0,1]^2} \int_{\omega(x_h)}^a (|u|^2 + |\nabla u|^2) < \infty,$$
$$\sup_{l \in \mathbb{Z}^2} \int_{l + [0,1]^2} \int_1^\infty |\nabla^q u|^2 < \infty,$$

for some $q \in \mathbb{N}$, $q \geq 1$.

Then $u|_{\Omega^b}$ is a solution of (5.32)*, and for $x_3 > 0$, u is given by* (5.24)*, with $v_0 := u|_{x_3=0} \in \mathbb{K}$.*

– *Conversely, let* $u^- \in H^1_{uloc}(\Omega^b)$ *be a solution of* (5.32), *and let* $v_0 := u^-|_{x_3=0} \in \mathbb{K}$. *Consider the function* $u^+ \in H^1_{loc}(\mathbb{R}^3_+)$ *defined by* (5.24). *Then, setting*

$$u(x) := \begin{cases} u^-(x) & \text{if } \omega(x_h) < x_3 < 0, \\ u^+(x) & \text{if } x_3 > 0, \end{cases}$$

the function $u \in H^1_{loc}(\Omega)$ *is such that*

$$\forall a > 0, \quad \sup_{l \in \mathbb{Z}^2} \int_{l+[0,1]^2} \int_{\omega(x_h)}^a (|u|^2 + |\nabla u|^2) < \infty,$$
$$\sup_{l \in \mathbb{Z}^2} \int_{l+[0,1]^2} \int_1^\infty |\nabla^q u|^2 < \infty,$$

for some $q \in \mathbb{N}$ *sufficiently large, and is a solution of* (5.1).

As a consequence, we work with the system (5.32) from now on. In order to have a homogeneous Poincaré inequality in Ω^b, it is convenient to lift the boundary condition on Γ, so as to work with a homogeneous Dirichlet boundary condition. Therefore, we define $V = (V_h, V_3)$ by

$$V_h := u_{0,h}, \quad V_3 := u_{0,3} - \nabla_h \cdot u_{0,h} \left(x_3 - \omega(x_h)\right).$$

Notice that $V|_{x_3=0} \in \mathbb{K}$ thanks to (5.3), and that V is divergence free. By definition, the function

$$\tilde{u} := u - V \mathbf{1}_{\mathbf{x} \in \mathbf{\Omega}^\mathbf{b}}$$

is a solution of

$$\begin{cases} -\Delta \tilde{u} + e_3 \times \tilde{u} + \nabla \tilde{p} &= f & \text{in } \Omega^b, \\ \operatorname{div} \tilde{u} &= 0 & \text{in } \Omega^b, \\ \tilde{u}|_\Gamma &= 0, & \\ -\partial_3 \tilde{u} + \tilde{p} e_3 &= \mathrm{DN}(\tilde{u}|_{x_3=0^-}) + F, & \text{on } \Sigma \times \{0\} \end{cases} \tag{5.33}$$

where

$$f := \Delta V - e_3 \times V = \Delta_h V - e_3 \times V,$$
$$F := \mathrm{DN}(V|_{x_3=0}) + \partial_3 V|_{x_3=0}.$$

Notice that thanks to the regularity assumptions on u_0 and ω, we have, for all $l \in \mathbb{N}$ and for all $\varphi \in \mathcal{C}_0^\infty(\overline{\Omega^b})^3$ with $\operatorname{Supp} \varphi \subset ((-l, l)^2 \times (-1, 0)) \cap \overline{\Omega^b}$,

$$|\langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}}| \leq Cl(\|u_{0,h}\|_{H^2_{uloc}} + \|u_{0,3}\|_{H^1_{uloc}}) \|\varphi\|_{H^1(\Omega^b)} . \tag{5.34}$$

where the constant C depends only on $\|\omega\|_{W^{1,\infty}}$. In a similar fashion, if $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)^3$ is such that $\varphi_3 = \nabla_h \cdot \Phi_h$ for some $\Phi_h \in \mathcal{C}_0^\infty(\mathbb{R}^2)^2$, and if $\operatorname{Supp} \varphi, \operatorname{Supp} \Phi_h \subset B(x_0, l)$, then according to Lemma 5.27,

$$|\langle F, \varphi \rangle_{\mathcal{D}', \mathcal{D}}| \leq Cl(\|u_{0,h}\|_{H^2_{uloc}} + \|u_{0,3}\|_{H^1_{uloc}} + \|U_h\|_{H^{1/2}_{uloc}}) \left(\|\varphi\|_{H^{1/2}(\mathbb{R}^2)} + \|\Phi_h\|_{H^{1/2}(\mathbb{R}^2)}\right) . \tag{5.35}$$

5.2.4 Strategy of the proof

From now on, we drop the $\tilde{}$ in (5.33) so as to lighten the notation.

• In order to prove the existence of solutions of (5.33) in $H^1_{uloc}(\Omega)$, we truncate horizontally the domain Ω, and we derive uniform estimates on the solutions of the Stoke-Coriolis system in the truncated domains. More precisely, we introduce, for all $n \in \mathbb{N}$, $k \in \mathbb{N}$,

$$\begin{aligned} \Omega_n &:= \Omega^b \cap \{x \in \mathbb{R}^3,\ |x_1| \le n,\ x_2 \le n\},\\ \Omega_{k,k+1} &:= \Omega_{k+1} \setminus \Omega_k,\\ \Sigma_n &:= \{(x_h, 0) \in \mathbb{R}^3,\ |x_1| \le n,\ x_2 \le n\},\\ \Sigma_{k,k+1} &:= \Sigma_{k+1} \setminus \Sigma_k. \end{aligned}$$

We consider the Stokes-Coriolis system in Ω_n, with homogeneous boundary conditions on the lateral boundaries

$$\left\{ \begin{aligned} -\Delta u_n + e_3 \times u_n + \nabla p_n &= f, && x \in \Omega_n\\ \nabla \cdot u_n &= 0, && x \in \Omega_n\\ u_n &= 0, && x \in \Omega^b \setminus \Omega_n\\ -\partial_3 u_n + p_n e_3|_{x_3=0} &= \mathrm{DN}\left(u_n|_{x_3=0}\right) + F, && x \in \Sigma_n. \end{aligned} \right. \tag{5.36}$$

Notice that the transparent boundary condition involving the Dirichlet to Neumann operator only makes sense if $u_n|_{x_3=0}$ is defined on the whole plane Σ (and not merely on Σ_n), due to the non-locality of the operator DN. This accounts for the condition $u_n|_{\Omega^b\setminus\Omega_n} = 0$.

Taking u_n as a test function in (5.36), we get a first energy estimate on u_n

$$\begin{aligned} &\|\nabla u_n\|^2_{L^2(\Omega^b)}\\ = \ &\underbrace{-\left\langle \mathrm{DN}\left(u_n|_{x_3=0}\right), u_n|_{x_3=0}\right\rangle}_{\le 0} - \langle F, u_n|_{x_3=0}\rangle + \langle f, u_n\rangle \qquad (5.37)\\ \le \ & Cn \left(\left\| u_{n,h}|_{x_3=0}\right\|_{H^{1/2}(\Sigma_n)} + \left\| \int_{\omega(x_h)}^0 u_{n,h}(x_h, z')dz' \right\|_{H^{1/2}(\Sigma_n)} \right) + Cn \|u_n\|_{H^1(\Omega_n)}\\ \le \ & Cn \|u_n\|_{H^1(\Omega_n)}, \end{aligned}$$

where the constant C depends only on $\|u_0\|_{H^2_{uloc}}$ and $\|\omega\|_{W^{1,\infty}}$. This implies, thanks to the Poincaré inequality,

$$E_n := \int_\Omega \nabla u_n \cdot \nabla u_n \le C_0 n^2. \tag{5.38}$$

The existence of u_n in $H^1(\Omega^b)$ follows. Uniqueness is a consequence of equality (5.37) with $F = 0$ and $f = 0$.

In order to prove the existence of u, we will derive H^1_{uloc} estimates on u_n, uniform with respect to n. Then, passing to the limit in (5.36) and in the estimates, we deduce the existence of a solution of (5.33) in $H^1_{uloc}(\Omega^b)$. In order to obtain H^1_{uloc} estimates on u_n, we follow the strategy of Gérard-Varet and Masmoudi in [GVM10], which is inspired from the work of Ladyzhenskaya and Solonnikov [LS80]. We work with the energies

$$E_k := \int_{\Omega_k} \nabla u_n \cdot \nabla u_n. \tag{5.39}$$

The goal is to prove an inequality of the type

$$E_k \le C\left(k^2 + (E_{k+1} - E_k)\right), \quad \forall k \in \{m, \dots\ n\}, \tag{5.40}$$

where $m \in \mathbb{N}$ is a large, but fixed integer (independent of n) and C is a constant depending only on $\|\omega\|_{W^{1,\infty}}$ and $\|u_{0,h}\|_{H^2_{uloc}}$, $\|u_{0,3}\|_{H^1_{uloc}}$, $\|U_h\|_{H^{1/2}_{uloc}}$. Then, by backwards induction on k, we deduce that

$$E_k \le Ck^2 \quad \forall k \in \{m, \dots\ n\}$$

so that E_m, in particular, is bounded, uniformly in n. Since the derivation of the energy estimates is invariant by translation in the horizontal variable, we infer that for all $n \in \mathbb{N}$,

$$\sup_{c \in \mathcal{C}_m} \int_{(c\times(-1,0))\cap\Omega^b} |\nabla u_n|^2 \leq C$$

where

$$\mathcal{C}_m := \left\{c, \text{ square of edge of length } m \text{ contained in } \Sigma_n \text{ with vertices in } \mathbb{Z}^2\right\}. \tag{5.41}$$

Hence the uniform H^1_{uloc} bound on u_n is proved. As a consequence, by a diagonal argument, we can extract a subsequence $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ such that $u_{\psi(n)} \rightharpoonup u$ weakly in $H^1(\Omega_k)$ and $u_{\psi(n)}|_{x_3=0} \rightharpoonup u|_{x_3=0}$ weakly in $H^{1/2}(\Sigma_k)$ for all $k \in \mathbb{N}$. Of course, u is a solution of the Stokes-Coriolis system in Ω^b, and $u \in H^1_{uloc}(\Omega^b)$. Looking closely at the representation formula in Proposition 5.23, we infer that

$$\langle \mathrm{DN}\, u_{\psi(n)}|_{x_3=0}, \varphi\rangle_{\mathcal{D}',\mathcal{D}} \overset{n\to\infty}{\longrightarrow} \langle \mathrm{DN}\, u|_{x_3=0}, \varphi\rangle_{\mathcal{D}',\mathcal{D}}$$

for all admissible test functions φ. For instance,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi M_{HF}^{rem} * (1-\chi)\left(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0}\right)\\
=\ &\int_{\mathbb{R}^2} dx \int_{|t|\leq k} dt \varphi(x) M_{HF}^{rem}(x-t)(1-\chi)\left(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0}\right)(t)\\
+\ &\int_{\mathbb{R}^2} dx \int_{|t|\geq k} dt \varphi(x) M_{HF}^{rem}(x-t)(1-\chi)\left(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0}\right)(t).
\end{aligned}$$

For all k, the first integral vanishes as $n \to \infty$ as a consequence of the weak convergence in $L^2(\Sigma_k)$. As for the second integral, let $R > 0$ such that $\operatorname{Supp}\varphi \subset B_R$, and let $k \geq R+1$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} dx \int_{|t|\geq k} dt \varphi(x) M_{HF}^{rem}(x-t)((1-\chi)\left(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0}\right)(t)\\
&C\int_{\mathbb{R}^2} dx \int_{|t|\geq k} dt |\varphi(x)| \frac{1}{|x-t|^3}\left(\left|u_{\psi(n)}|_{x_3=0}(t)\right| + \left|u|_{x_3=0}(t)\right|\right)\\
\leq\ & C\int_{\mathbb{R}^2} dx |\varphi(x)| \left(\int_{|t|\geq k} \frac{1}{|x-t|^3} dt\right)^{1/2} \left(\int_{|x-t|\geq 1} \frac{dt}{|x-t|^3}(|u|_{x_3=0}|^2 + \left|u_{\psi(n)}|_{x_3=0}\right|^2)\right)^{1/2}\\
\leq\ & C\left(\left\|u|_{x_3=0}\right\|_{L^2_{uloc}} + \sup_n \left\|u_n|_{x_3=0}\right\|_{L^2_{uloc}}\right) \int_{\mathbb{R}^2} dx |\varphi(x)| \left(\int_{|t|\geq k} \frac{1}{|x-t|^3} dt\right)^{1/2}\\
\leq\ & C\left(\left\|u|_{x_3=0}\right\|_{L^2_{uloc}} + \sup_n \left\|u_n|_{x_3=0}\right\|_{L^2_{uloc}}\right) \|\varphi\|_{L^1} (k-R)^{-1/2}.
\end{aligned}$$

Hence the second integral vanishes as $k \to \infty$ uniformly in n. We infer that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \varphi M_{HF}^{rem} * ((1-\chi)(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0}) = 0.$$

Therefore u is a solution of (5.33).

The final induction inequality we will be much more complicated than (5.40), and the proof will also be more involved than the one of [GVM10]. However, the general scheme will be very close to the one described above.

• Concerning uniqueness of solutions of (5.33), we use the same type of energy estimates as above. Once again, we give in the present paragraph a very rough idea of the computations, and we refer to section 5.4 for all details. When $f = 0$ and $F = 0$, the energy estimates (5.40) become

$$E_k \le C(E_{k+1} - E_k),$$

and therefore

$$E_k \le rE_{k+1}$$

with $r := C/(1+C) \in (0,1)$. Hence, by induction,

$$E_1 \le r^{k-1}E_k \le Cr^{k-1}k^2$$

for all $k \ge 1$, since u is assumed to be bounded in $H^1_{uloc}(\Omega^b)$. Letting $k \to \infty$, we deduce that $E_1 = 0$. Since all estimates are invariant by translation in x_h, we obtain that $u = 0$.

5.3 Estimates in the rough channel

This section is devoted to the proof of energy estimates of the type (5.40) for solutions of the system (5.36), which eventually lead to the existence of a solution of (5.33).

The goal is to prove that for some $m \ge 1$ sufficiently large (but independent of n), E_m is bounded uniformly in n, which automatically implies the boundedness of u_n in $H^1_{uloc}\left(\Omega^b\right)$. We reach this objective in two steps:

– We prove a Saint-Venant estimate: we claim that there exists a constant $C_1 > 0$ uniform in n such that for all $m \in \mathbb{N} \setminus \{0\}$, for all $k \in \mathbb{N}$, $k \ge m$,

$$E_k \le C_1 \left[k^2 + E_{k+m+1} - E_k + \frac{k^4}{m^5} \sup_{j \ge m+k} \frac{E_{j+m} - E_j}{j} \right]. \tag{5.42}$$

The crucial fact is that C_1 is independent of n, k and m.

– This estimate allows to deduce the bound in $H^1_{uloc}(\Omega)$ via a non trivial induction argument.

Let us first explain the induction, assuming that (5.42) holds. The proof of (5.42) is postponed to the subsection 5.3.2.

5.3.1 Induction

We aim at deducing from (5.42) that there exists $m \in \mathbb{N} \setminus \{0\}$, $C > 0$ such that for all $n \in \mathbb{N}$,

$$\int_{\Omega_m} \nabla u_n \cdot \nabla u_n \le C. \tag{5.43}$$

The proof of this uniform bound is divided into two points:

– Firstly, we deduce from (5.42), by downward induction on k, that there exist positive constants C_2, C_3, m_0, depending only on $\|\omega\|_{W^{1,\infty}}$ and $\|u_{0,h}\|_{H^2_{uloc}}$, $\|u_{0,3}\|_{H^1_{uloc}}$, $\|U_h\|_{H^{1/2}_{uloc}}$, such that for all (k, m) such that $k \ge C_3 m$ and $m \ge m_0$,

$$E_k \le C_2 \left[k^2 + m^3 + \frac{k^4}{m^5} \sup_{j \ge m+k} \frac{E_{j+m} - E_j}{j} \right]. \tag{5.44}$$

Let us insist on the fact that C_2 and C_3 are independent of n, k, m. They will be adjusted in the course of the induction argument (see (5.49)).

– Secondly, we notice that (5.44) yields the bound we are looking for, choosing $k = \lfloor C_3 m \rfloor + 1$ and m large enough.

• We thus start with the proof of (5.44), assuming that (5.42) holds.

First, notice that thanks to (5.38), (5.44) is true for $k \geq n$ as soon as $C_2 \geq C_0$, remembering that $u_n = 0$ on $\Omega^b \backslash \Omega_n$. We then assume that (5.44) holds for $n, n-1, \dots\ k+1$, where k is an integer such that $k \geq C_3 m$ (further conditions on C_2, C_3 will be derived at the end of the induction argument, see (5.48)).

We prove (5.44) at the rank k by contradiction. Hence, assume that (5.44) does not hold at the rank k, so that

$$E_k > C_2 \left[k^2 + m^3 + \frac{k^4}{m^5} \sup_{j \geq m+k} \frac{E_{j+m} - E_j}{j} \right]. \tag{5.45}$$

Then, the induction assumption implies

$$\begin{aligned} & E_{k+m+1} - E_k \\ \leq\ & C_2 \left[(k+m+1)^2 - k^2 + \frac{(k+m+1)^4 - k^4}{m^5} \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right] \\ \leq\ & C_2 \left[2k(m+1) + (m+1)^2 + 80 \frac{k^3}{m^4} \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right]. \end{aligned} \tag{5.46}$$

Above, we have used the following inequality, which holds for all $k \geq m \geq 1$

$$\begin{aligned} (k+m+1)^4 - k^4 &= 4k^3(m+1) + 6k^2(m+1)^2 + 4k(m+1)^3 + (m+1)^4 \\ &\leq 8mk^3 + 6k^2 \times 4m^2 + 4k \times 8m^3 + 16m^4 \\ &\leq 80mk^3. \end{aligned}$$

Using (5.45), (5.42) and (5.46), we get

$$\begin{aligned} & C_2 \left[k^2 + m^3 + \frac{k^4}{m^5} \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right] \\ <\ & E_k \\ \leq\ & C_1 \left[k^2 + 2C_2 k(m+1) + C_2(m+1)^2 + \left(80 C_2 \frac{k^3}{m^4} + \frac{k^4}{m^5} \right) \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right]. \end{aligned} \tag{5.47}$$

The constants C_0, $C_1 > 0$ are fixed and depend only on $\|\omega\|_{W^{1,\infty}}$ and $\|u_{0,h}\|_{H^2_{uloc}}$, $\|u_{0,3}\|_{H^1_{uloc}}$, $\|U_h\|_{H^{1/2}_{uloc}}$ (cf. (5.38) for the definition of C_0). We choose $m_0 > 1$, $C_2 > C_0$ and $C_3 \geq 1$ depending only on C_0 and C_1 so that

$$\left\{ \begin{array}{l} k \geq C_3 m \\ \text{and } m \geq m_0 \end{array} \right. \quad \text{implies} \quad \left\{ \begin{array}{l} C_2(k^2 + m^3) > C_1 \left[k^2 + 2C_2 k(m+1) + C_2(m+1)^2 \right] \\ \text{and } C_2 \frac{k^4}{m^5} \geq C_1 \left(80 C_2 \frac{k^3}{m^4} + \frac{k^4}{m^5} \right). \end{array} \right. \tag{5.48}$$

One can easily check that it suffices to choose C_2, C_3 and m_0 so that

$$\begin{aligned} C_2 &> \max(2C_1, C_0), \\ (C_2 - C_1) C_3 &> 80 C_1 C_2, \\ \forall m \geq m_0, \quad (C_2 C_1 + C_1)(m+1)^2 &< m^3. \end{aligned} \tag{5.49}$$

Plugging (5.48) into (5.47), we reach a contradiction. Therefore (5.44) is true at the rank k. By induction, (5.44) is proved for all $m \geq m_0$ and for all $k \geq C_3 m$.

• It follows from (5.44), choosing $k = \lfloor C_3 m \rfloor + 1$, that there exists a constant $C > 0$, depending only on C_0, C_1, C_2, C_3, and therefore only on $\|\omega\|_{W^{1,\infty}}$ and on Sobolev-Kato norms on u_0 and U_h, such that for all $m \geq m_0$,

$$E_{\lfloor m/2 \rfloor} \leq E_{\lfloor C_3 m \rfloor + 1} \leq C \left[m^3 + \frac{1}{m} \sup_{j \geq \lfloor C_3 m \rfloor + m + 1} \frac{E_{j+m} - E_j}{j} \right]. \tag{5.50}$$

Let us now consider the set $\mathcal{C}_m$ defined by (5.41) for an even integer m. As $\mathcal{C}_m$ is finite, there exists a square c in $\mathcal{C}_m$, which maximizes

$$\left\{ \|u_n\|_{H^1(\Omega_c)}, c \in \mathcal{C}_m \right\}$$

where $\Omega_c = \left\{ x \in \Omega^b, x_h \in c \right\}$. We then shift u_n in such a manner that c is centered at 0. We call $\tilde{u}_n$ the shifted function. It is still compactly supported, yet not in Ω_n but in Ω_{2n},

$$\int_{\Omega_{2n}} |\nabla \tilde{u}_n|^2 = \int_{\Omega_n} |\nabla u_n|^2 \quad \text{and} \quad \int_{\Omega_{m/2}} |\nabla \tilde{u}_n|^2 = \int_{\Omega_c} |\nabla u_n|^2 .$$

Analogously to E_k, we define $\widetilde{E}_k$. Since the arguments leading to the derivation of energy estimates are invariant by horizontal translation, and all constants depend only on Sobolev norms on u_0, U_h and ω, we infer that (5.50) still holds when E_k is replaced by $\widetilde{E}_k$. On the other hand, recall that $\widetilde{E}_{m/2}$ maximizes $\|\tilde{u}_n\|^2_{H^1(\Omega_c)}$ on the set of squares of edge length m. Moreover, in the set $\Sigma_{j+m} \setminus \Sigma_j$ for $j \geq 1$, there are at most $4(j+m)/m$ squares of edge length m. As a consequence, we have, for all $j \in \mathbb{N}^*$,

$$\widetilde{E}_{j+m} - \widetilde{E}_j \leq 4 \frac{j+m}{m} \widetilde{E}_{m/2},$$

so that (5.50) written for $\tilde{u}_n$ becomes

$$\begin{aligned} \widetilde{E}_{m/2} &\leq C \left[m^3 + \frac{1}{m^2} \left(\sup_{j \geq (C_3+1)m} 1 + \frac{m}{j} \right) \widetilde{E}_{m/2} \right] \\ &\leq C \left[m^3 + \frac{1}{m^2} \widetilde{E}_{m/2} \right]. \end{aligned}$$

This estimate being uniform in $m \in \mathbb{N}$ provided $m \geq m_0$, we can take m large enough and get

$$\widetilde{E}_{m/2} \leq C \frac{m^3}{1 - C\frac{1}{m^2}},$$

so that eventually there exists $m \in \mathbb{N}$ such that

$$\sup_{c \in \mathcal{C}_m} \|u_n\|^2_{H^1((c \times (-1,0) \cap \Omega^b))} \leq C \frac{m^3}{1 - C\frac{1}{m^2}}.$$

This means exactly that u_n is uniformly bounded in $H^1_{uloc}(\Omega^b)$. Existence follows, as explained in paragraph 5.2.4.

5.3.2 Saint-Venant estimate

This part is devoted to the proof of (5.42). We carry out a Saint-Venant estimate on the system (5.36), focusing on having constants uniform in n as explained in the section 5.2.4. The preparatory work of the section 5.2.1 and 5.2.2 allows us to focus on very few issues. The main problem is the non-locality of the Dirichlet to Neumann operator, which at first sight does not seem to be compatible with getting estimates independent of the size of the support of u_n.

Let $n \in \mathbb{N} \setminus \{0\}$ be fixed. Let also $\varphi \in \mathcal{C}_0^\infty(\Omega^b)$ such that

$$\nabla \cdot \varphi = 0, \qquad \varphi = 0 \text{ on } \Omega^b \setminus \Omega_n, \qquad \varphi|_{x_3=\omega(x_h)} = 0. \tag{5.51}$$

Remark 5.29 states that such a function φ is an appropriate test function for (5.36). In the spirit of Definition 5.28, we are led to the following weak formulation:

$$\begin{aligned}\int_{\Omega^b} \nabla u_n \cdot \nabla \varphi &+ \int_{\Omega^b} u_{n,h}^\perp \cdot \varphi_h \\ &= -\left\langle \mathrm{DN}\left(u_n|_{x_3=0^-}\right), \varphi|_{x_3=0^-}\right\rangle_{\mathcal{D}',\mathcal{D}} - \left\langle F, \varphi|_{x_3=0^-}\right\rangle_{\mathcal{D}',\mathcal{D}} + \langle f, \varphi\rangle_{\mathcal{D}',\mathcal{D}} \end{aligned}\tag{5.52}$$

Thanks to the representation formula for DN in Proposition 5.23, and to the estimates (5.34) for f and (5.35) for F, the weak formulation (5.52) still makes sense for $\varphi \in H^1(\Omega^b)$ satisfying (5.51).

In the sequel we drop the subscripts n. Note that all constants appearing in the inequalities below are uniform in n. However, one should be aware that E_k defined by (5.39) *depends on n.* Furthermore, we denote $u|_{x_3=0^-}$ by v_0.

In order to estimate E_k, we introduce a smooth cutoff function $\chi_k = \chi_k(y_h)$ supported in Σ_{k+1} and identically equal to 1 on Σ_k. We carry out energy estimates on the system (5.36). Remember that a test function has to meet the conditions (5.51). We therefore choose

$$\begin{aligned}\varphi &= \begin{pmatrix} \varphi_h \\ \nabla\cdot\Phi_h \end{pmatrix} := \begin{pmatrix} \chi_k u_h \\ -\nabla_h \cdot \left(\chi_k \int_{\omega(x_h)}^z u_h(x_h, z')dz'\right) \end{pmatrix} \in H^1(\Omega^b), \\ &= \chi_k u - \begin{pmatrix} 0 \\ \nabla_h \chi_k(x_h) \cdot \int_{\omega(x_h)}^z u_h(x_h, z')dz' \end{pmatrix}\end{aligned}$$

which can be readily checked to satisfy (5.51). Notice that this choice of test function is different from the one of [GVM10], which is merely $\chi_k u$. Aside from being a suitable test function for (5.36), the function φ has the advantage of being divergence free, so that there will be no need to estimate commutator terms stemming from the pressure.

Plugging φ in the weak formulation (5.52), we get

$$\begin{aligned}\int_\Omega \chi_k |\nabla u|^2 &= -\int_\Omega \nabla u \cdot (\nabla \chi_k)\, u + \int_\Omega \nabla u_3 \cdot \nabla \left(\nabla_h \chi_k(x_h) \cdot \int_{\omega(x_h)}^z u_h(x_h, z')dz'\right) \\ &\quad - \left\langle \mathrm{DN}(v_0), \varphi|_{x_3=0^-}\right\rangle - \left\langle F, \varphi|_{x_3=0^-}\right\rangle + \langle f, \varphi\rangle. \end{aligned}\tag{5.53}$$

Before coming to the estimates, we state an easy bound on Φ_h and φ

$$\|\Phi_h\|_{H^1(\Omega^b)} + \|\varphi\|_{H^{1/2}(\Omega^b)} + \|\Phi_h|_{x_3=0}\|_{H^{1/2}(\mathbb{R}^2)} + \|\varphi|_{x_3=0}\|_{H^{1/2}(\mathbb{R}^2)} \le C E_{k+1}^{\frac{1}{2}}. \tag{5.54}$$

As we have recourse to Lemma 5.27 to estimate some terms in (5.53), we use (5.54) repeatedly in the sequel, sometimes with slight changes.

We have to estimate each of the terms appearing in (5.53). The most difficult term is the one involving the Dirichlet to Neumann operator, because of the non-local feature of the latter: although v_0 is supported in Σ_n, $\mathrm{DN}(v_0)$ is not in general. However, each term in (5.53), except $-\left\langle \mathrm{DN}(v_0), \varphi|_{x_3=0^-}\right\rangle$, is local, and hence very easy to bound. Let us sketch the estimates of the local terms. For the first term, we simply use the Cauchy-Schwarz and the Poincaré inequalities:

$$\left|\int_\Omega \nabla u \cdot (\nabla \chi_k)\, u\right| \leq C \left(\int_{\Omega_{k,k+1}} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_{k,k+1}} |u|^2\right)^{\frac{1}{2}} \leq C\,(E_{k+1} - E_k)\,.$$

In the same fashion, using (5.54), we find that the second term is bounded by

$$\begin{aligned}
&\left|\int_\Omega \nabla u_3 \cdot \nabla \left(\nabla_h \chi_k(x_h) \cdot \int_{\omega(x_h)}^z u_h(x_h, z')dz'\right) dx_h dz\right| \\
\leq\ & \int_\Omega |\nabla u_3|\, |\nabla \nabla_h \chi_k(x_h)| \int_{\omega(x_h)}^z |u_h(x_h, z')|\, dz' dx_h dz \\
&+ \int_\Omega |\nabla_h u_3|\, |\nabla_h \chi_k(x_h)| \int_{\omega(x_h)}^z |\nabla_h u_h(x_h, z')|\, dz' dx_h dz \\
&+ \int_\Omega |\partial_3 u_3 \nabla_h \chi_k(x_h) \cdot u_h(x_h, z)|\, dx_h dz \\
\leq\ & C\,(E_{k+1} - E_k)\,.
\end{aligned}$$

We finally bound the two last terms in (5.53) using (5.54), and (5.35) or (5.34):

$$\begin{aligned}
|\langle F, \varphi|_{x_3=0^-}\rangle| &\leq C(k+1) \left[\|\chi_k u_h|_{x_3=0^-}\|_{H^{1/2}(\mathbb{R}^2)} + \left\|\nabla_h \cdot \left(\chi_k \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right)\right\|_{H^{1/2}(\mathbb{R}^2)}\right] \\
&\leq C(k+1) \left[E_{k+1}^{\frac{1}{2}} + (E_{k+1} - E_k)^{\frac{1}{2}}\right] \leq C(k+1) E_{k+1}^{1/2}, \\
|\langle f, \varphi\rangle| &\leq (k+1) E_{k+1}^{\frac{1}{2}}.
\end{aligned}$$

The last term to handle is $-\left\langle \mathrm{DN}_h\,(v_0)\,, \varphi|_{x_3=0^-}\right\rangle$. The issue of the non-locality of the Dirichlet to Neumann operator is already present for the Stokes system. Again, we attempt to adapt the ideas of [GVM10]. So as to handle the large scales of $\mathrm{DN}(v_0)$, we are led to introduce the auxiliary parameter $m \in \mathbb{N}^*$, which appears in (5.42). We decompose v_0 into

$$\begin{aligned}
v_0 = &\begin{pmatrix} \chi_k v_{0,h} \\ -\nabla_h \cdot \left(\chi_k \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix} + \begin{pmatrix} (\chi_{k+m} - \chi_k)\, v_{0,h} \\ -\nabla_h \cdot \left((\chi_{k+m} - \chi_k) \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix} \\
&+ \begin{pmatrix} (1 - \chi_{k+m})\, v_{0,h} \\ -\nabla_h \cdot \left((1 - \chi_{k+m}) \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix}.
\end{aligned}$$

The truncations on the vertical component of v_0 are put inside the horizontal divergence, in order to apply the Dirichlet to Neumann operator to functions in $\mathbb{K}$.

The term corresponding to the truncation of v_0 by χ_k, namely

$$\begin{aligned}
&-\left\langle \mathrm{DN} \begin{pmatrix} \chi_k v_{0,h} \\ -\nabla_h \cdot \left(\chi_k \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix}, \begin{pmatrix} \varphi_h|_{x_3=0^-} \\ \nabla_h \cdot \Phi_h|_{x_3=0^-} \end{pmatrix} \right\rangle \\
= &-\left\langle \mathrm{DN} \begin{pmatrix} \chi_k v_{0,h} \\ -\nabla_h \cdot \left(\chi_k \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix}, \begin{pmatrix} \chi_k v_{0,h} \\ -\nabla_h \cdot \left(\chi_k \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix} \right\rangle
\end{aligned}$$

is negative by positivity of the operator DN (see Lemma 5.25). For the term corresponding to the truncation by $\chi_{k+m} - \chi_k$ we resort to Lemma 5.27 and (5.54). This yields

$$\begin{aligned}
&\left|\left\langle \mathrm{DN} \begin{pmatrix} (\chi_{k+m} - \chi_k)\, v_{0,h} \\ -\nabla_h \cdot \left((\chi_{k+m} - \chi_k) \int_{\omega(x_h)}^0 u_h(x_h, z')dz'\right) \end{pmatrix}, \begin{pmatrix} \varphi_h|_{x_3=0^-} \\ \nabla_h \cdot \Phi_h|_{x_3=0^-} \end{pmatrix} \right\rangle\right| \\
\leq\ & C\,(E_{k+m+1} - E_k)^{\frac{1}{2}}\, E_{k+1}^{\frac{1}{2}}.
\end{aligned}$$

However, the estimate of Lemma 5.27 is not refined enough to address the large scales independently of n. For the term

$$\left\langle \mathrm{DN} \begin{pmatrix} (1-\chi_{k+m})\, v_{0,h} \\ -\nabla_h \cdot \left((1-\chi_{k+m}) \int_{\omega(x_h)}^0 u_h(x_h,z')dz' \right) \end{pmatrix}, \begin{pmatrix} \varphi_h|_{x_3=0^-} \\ \nabla_h \cdot \Phi_h|_{x_3=0^-} \end{pmatrix} \right\rangle,$$

we must have a closer look at the representation formula given in Proposition 5.23. Let

$$\tilde v_0 := \begin{pmatrix} (1-\chi_{k+m})\, v_{0,h} \\ -\nabla_h \cdot \left((1-\chi_{k+m}) \int_{\omega(x_h)}^0 u_h(x_h,z')dz' \right) \end{pmatrix} = \begin{pmatrix} (1-\chi_{k+m})\, v_{0,h} \\ -\nabla_h \cdot \tilde V_h \end{pmatrix}.$$

We take $\chi := \chi_{k+1}$ in the formula of Proposition 5.23. If $m \geq 2$, $\operatorname{Supp} \chi_{k+1} \cap \operatorname{Supp}(1-\chi_{k+m}) = \emptyset$, so that the formula of Proposition 5.23 becomes [3]

$$\begin{aligned} \langle \mathrm{DN}\, \tilde v_0, \varphi \rangle = & \int_{\mathbb{R}^2} \varphi|_{x_3=0^-} \cdot K_S * \tilde v_0 + \int_{\mathbb{R}^2} \varphi|_{x_3=0^-} \cdot M_{HF}^{rem} * \tilde v_0 \\ & + \int_{\mathbb{R}^2} \varphi_{h|x_3=0^-} \cdot \{ \mathcal{I}[M_1] \left(\rho * \tilde v_{0,h}\right) + K_1^{rem} * \tilde v_{0,h} \} \\ & + \int_{\mathbb{R}^2} \varphi_{h|x_3=0^-} \cdot \left\{ \mathcal{I}[M_2] \left(\rho * \tilde V_h\right) + K_2^{rem} * \tilde V_h \right\} \\ & + \int_{\mathbb{R}^2} \Phi_{h|x_3=0^-} \cdot \{ \mathcal{I}[M_3] \left(\rho * \tilde v_{0,h}\right) + K_3^{rem} * \tilde v_{0,h} \} \\ & + \int_{\mathbb{R}^2} \Phi_{h|x_3=0^-} \cdot \left\{ \mathcal{I}[M_4] \left(\rho * \tilde V_h\right) + K_4^{rem} * \tilde V_h \right\}. \end{aligned}$$

Thus, we have two types of terms to estimate:
- On the one hand are the convolution terms with the kernels K_S, M_{HF}^{rem}, and K_i^{rem} for $1 \leq i \leq 4$, which all decay like $\frac{1}{|x_h|^3}$.
- On the other hand are the terms involving $\mathcal{I}[M_i]$ for $1 \leq i \leq 4$.

For the first ones, we rely on the following nontrivial estimate:

Lemma 5.31. *For all $k \geq m$,*

$$\left\| \tilde v_0 * \frac{1}{|\cdot|^3} \right\|_{L^2(\Sigma_{k+1})} \leq C \frac{k^{\frac32}}{m^2} \left(\sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right)^{\frac12}. \tag{5.55}$$

This estimate still holds with $\tilde V_h$ in place of $\tilde v_0$.

For the second ones, we have recourse to:

Lemma 5.32. *For all $k \geq m$, for all $1 \leq i,\ j \leq 2$,*

$$\left\| \mathcal{I} \left[\frac{\xi_i \xi_j}{|\xi|} \right] \left(\rho * \tilde v_{0,h}\right) \right\|_{L^2(\Sigma_{k+1})} \leq C \frac{k^2}{m^{\frac52}} \left(\sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right)^{\frac12}. \tag{5.56}$$

This estimate still holds with $\tilde V_h$ in place of $v_{0,h}$.

3. Here, we use in a crucial (but hidden) way the fact that the zero order terms at low frequencies are constant. Indeed, such terms are local, so that

$$\int_{\mathbb{R}^2} \varphi|_{x_3=0^-} \cdot \bar M \tilde v_0 = 0.$$

We postpone the proofs of these two key lemmas to section 5.3.3. Applying repeatedly Lemma 5.31 and Lemma 5.32 together with the estimates (5.54), we are finally led to the estimate

$$E_k \leq C\left((k+1)E_{k+1}^{\frac{1}{2}} + (E_{k+1}-E_k) + E_{k+1}^{\frac{1}{2}}\left(E_{k+m+1}-E_k\right)^{\frac{1}{2}} + \frac{k^2}{m^{\frac{5}{2}}}E_{k+1}^{\frac{1}{2}}\left(\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\right)^{\frac{1}{2}}\right),$$

for all $k \geq m \geq 1$. Now, since E_k is increasing in k, we have

$$E_{k+1} \leq E_k + (E_{k+m+1}-E_k).$$

Using Young's inequality, we infer that for all $\nu > 0$, there exists a constant C_ν such that

$$E_k \leq \nu E_k + C_\nu\left(E_{k+m+1}-E_k + \frac{k^4}{m^5}\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\right).$$

Choosing $\nu < 1$, inequality (5.42) follows.

Inequality (5.42) then follows easily from Young's inequality and from the fact that E_k is increasing in k.

5.3.3 Proof of the key lemmas

It remains to establish the estimates (5.55) and (5.56). The proofs are quite technical, but similar ideas and tools are used in the two proofs.

Proof of Lemma 5.31. We use an idea of Gérard-Varet and Masmoudi (see [GVM10]) to treat the large scales: we decompose the set $\Sigma \setminus \Sigma_{k+m}$ as

$$\Sigma \setminus \Sigma_{k+m} = \bigcup_{j=1}^{\infty} \Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}.$$

On every set $\Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$, we bound the L^2 norm of $\tilde{v}_0$ by $E_{k+m(j+1)} - E_{k+mj}$. Let us stress here a technical difference with the work of Gérard-Varet and Masmoudi: since Σ has dimension two, the area of the set $\Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$ is of order $(k+mj)m$. In particular, we expect $E_{k+m(j+1)} - E_{k+mj} \sim (k+mj)m\|u\|^2_{H^1_{uloc}}$ to grow with j. Thus we work with the quantity

$$\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j},$$

which we expect to be bounded uniformly in n, k, rather than with $\sup_{j\geq k+m}(E_{j+m}-E_j)$.

Now, applying the Cauchy-Schwarz inequality yields for $\eta > 0$

$$\int_{\Sigma_{k+1}} dy\left(\int_{\mathbb{R}^2}\frac{1}{|y-t|^3}\tilde{v}_0(t)dt\right)^2 \leq C\int_{\Sigma_{k+1}} dy\int_{\Sigma\setminus\Sigma_{k+m}}\frac{|t|}{|y-t|^{3+2\eta}}dt\int_{\Sigma\setminus\Sigma_{k+m}}\frac{|\tilde{v}_0(t)|^2}{|t||y-t|^{3-2\eta}}dt.$$

The role of the division by the $|t|$ factor in the second integral is precisely to force the apparition of the quantities $(E_{j+m}-E_j)/j$. More precisely, for $y\in\Sigma_{k+1}$ and $m\geq 1$,

$$\begin{aligned}\int_{\Sigma\setminus\Sigma_{k+m}}\frac{|\tilde v_0(t)|^2}{|t||y-t|^{3-2\eta}}dt &= \sum_{j=1}^\infty\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}}\frac{|\tilde v_0(t)|^2}{|t||y-t|^{3-2\eta}}dt\\ &\leq C\sum_{j=1}^\infty(E_{k+m(j+1)}-E_{k+mj})\frac{1}{(k+mj)|mj+k-|y|_\infty|^{3-2\eta}}\\ &\leq C\left(\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\right)\sum_{j=1}^\infty\frac{1}{|mj+k-|y|_\infty|^{3-2\eta}}\\ &\leq C_\eta\frac{1}{m}\frac{1}{|m+k-|y|_\infty|^{2-2\eta}}\left(\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\right),\end{aligned}$$

where $|x|_\infty:=\max(|x_1|,|x_2|)$ for $x\in\mathbb{R}^2$. A simple rescaling yields

$$\begin{aligned}&\int_{\Sigma_{k+1}}\int_{\Sigma\setminus\Sigma_{k+m}}\frac{|t|}{|y-t|^{3+2\eta}|m+k-|y|_\infty|^{2-2\eta}}dtdy\\ =&\int_{\Sigma_{1+\frac1k}}\int_{\Sigma\setminus\Sigma_{1+\frac mk}}\frac{|t|}{|y-t|^{3+2\eta}\left|1+\frac mk-|y|_\infty\right|^{2-2\eta}}dtdy.\end{aligned}$$

Let us assume that $k\geq m\geq 2$ and take $\eta\in\left]\frac12,1\right[$. We decompose $\Sigma\setminus\Sigma_{1+\frac mk}$ as $\Sigma\setminus\Sigma_2\cup\Sigma_2\setminus\Sigma_{1+\frac mk}$. On the one hand, since $|t-y|\geq C|t-y|_\infty\geq C(|t|_\infty-|y|_\infty)\geq C(|t|_\infty-3/2)$,

$$\int_{\Sigma_{1+\frac1k}}\int_{\Sigma\setminus\Sigma_2}\frac{|t|}{|y-t|^{3+2\eta}\left|1+\frac mk-|y|_\infty\right|^{2-2\eta}}dtdy\leq C\int_{\Sigma_{1+\frac1k}}\frac{dy}{\left|1+\frac mk-|y|_\infty\right|^{2-2\eta}}.$$

Decomposing $\Sigma_{1+\frac1k}$ into elementary annular regions of the type $\Sigma_{r+dr}\setminus\Sigma_r$, on which $|y|_\infty\simeq r$, we infer that the right-hand side of the above inequality is bounded by

$$\begin{aligned}&C\int_0^{1+\frac1k}\frac{r}{\left|1+\frac mk-r\right|^{2-2\eta}}dr\leq C\int_0^{1+\frac1k}\frac{dr}{\left|r+\frac{m-1}{k}\right|^{2-2\eta}}\\ &\leq C_\eta\left(\left(1+\frac mk\right)^{2\eta-1}-\left(\frac{m-1}{k}\right)^{2\eta-1}\right)\leq C_\eta.\end{aligned}$$

On the other hand, $y\in\Sigma_{1+\frac1k}$ implies $\left|1+\frac mk-|y|_\infty\right|\geq\frac{m-1}{k}$, so

$$\begin{aligned}&\int_{\Sigma_{1+\frac1k}}\int_{\Sigma_2\setminus\Sigma_{1+\frac mk}}\frac{|t|}{|y-t|^{3+2\eta}\left|1+\frac mk-|y|_\infty\right|^{2-2\eta}}dtdy\\ \leq\;& C\left(\frac{k}{m-1}\right)^{2-2\eta}\int_{\Sigma_{1+\frac1k}}dy\int_{\Sigma_2\setminus\Sigma_{1+\frac mk}}\frac{dt}{|t-y|^{3+2\eta}}\\ \leq\;& C\left(\frac{k}{m-1}\right)^{2-2\eta}\int_{\substack{X\in\mathbb{R}^2\\ \frac{m-1}{k}\leq|X|\leq C}}\frac{dX}{|X|^{3+2\eta}}\leq C_\eta\left(\frac km\right)^3.\end{aligned}$$

Gathering these bounds leads to (5.55). □

Proof of Lemma 5.32. As in the preceding proof, the overall strategy is to decompose

$$(1-\chi_{k+m})v_{0,h}=\sum_{j=1}^\infty(\chi_{k+m(j+1)}-\chi_{k+mj})v_{0,h}.$$

In the course of the proof, we introduce some auxiliary parameters, whose meaning we explain. We cannot use Lemma 5.11 as such, because we will need a much finer estimate. We therefore rely on the splitting (5.21) with $K := \frac{m}{2}$. An important property is the fact that $\rho := \mathcal{F}^{-1}\phi$ belongs to the Schwartz space $\mathcal{S}\left(\mathbb{R}^2\right)$ of rapidly decreasing functions.

As in the proof of Lemma 5.11, for $K = m/2$ and $x \in \Sigma_{k+1}$, we have

$$|\,\mathrm{A}(x)| \le Cm\|\nabla^2\rho * ((1-\chi_{k+m}v_{0,h}))\|_{L^\infty(\Sigma_{k+1+\frac{m}{2}})},$$

and for all $\alpha > 0$, for all $y \in \Sigma_{k+1+\frac{m}{2}}$,

$$\begin{aligned}\left|\nabla^2\rho * (1-\chi_{k+m})v_{0,h}(y)\right| &\le \int_{\Sigma\setminus\Sigma_{k+m}} \left|\nabla^2\rho(y-t)\right| |v_{0,h}(t)|dt \\ &\le \left(\int_{\Sigma\setminus\Sigma_{k+m}} \left|\nabla^2\rho(y-t)\right|^2 |t|^\alpha dt\right)^{1/2} \left(\int_{\Sigma\setminus\Sigma_{k+m}} \frac{|v_{0,h}(t)|^2}{|t|^\alpha} dt\right)^{1/2}.\end{aligned}$$

Yet, on the one hand, for $\alpha > 2$,

$$\begin{aligned}\int_{\Sigma\setminus\Sigma_{k+m}} \frac{|v_{0,h}(t)|^2}{|t|^\alpha}dt &= \sum_{j=1}^\infty \int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} \frac{|v_{0,h}(t)|^2}{|t|^\alpha}dt \\ &\le \left(\sup_{j\ge k+m} \frac{E_{j+m}-E_j}{j}\right)\sum_{j=1}^\infty \frac{1}{(k+mj)^{\alpha-1}} \\ &\le C\frac{1}{m}\frac{1}{(k+m)^{\alpha-2}}\left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right).\end{aligned}$$

On the other hand, $y \in \Sigma_{k+1+\frac{m}{2}}$ and $t \in \Sigma \setminus \Sigma_{k+m}$ implies $|y-t| \ge \frac{m}{2} - 1$, and for all $\beta > 0$

$$\begin{aligned}&\int_{\Sigma\setminus\Sigma_{k+m}} \left|\nabla^2\rho(y-t)\right|^2 |t|^\alpha dt \\ \le\ & C\int_{\Sigma\setminus\Sigma_{k+m}} \left|\nabla^2\rho(y-t)\right|^2 (|y-t|^\alpha + |y|^\alpha)dt \\ \le\ & C\left(\left(k+1+\frac{m}{2}\right)^\alpha \int_{|s|\ge\frac{m}{2}-1} \left|\nabla^2\rho(s)\right|^2 ds + \int_{|s|\ge\frac{m}{2}-1}\left|\nabla^2\rho(s)\right|^2 |s|^\alpha ds\right).\end{aligned}$$

Now, since $\rho \in \mathcal{S}(\mathbb{R}^2)$, for all $\beta > 0, \alpha > 0$ there exists a constant $C_{\alpha,\beta}$ such that

$$\int_{|s|\ge\frac{m}{2}-1} (1+|s|^\alpha)\left|\nabla^2\rho(s)\right|^2 ds \le C_\beta m^{-2\beta}.$$

The role of auxiliary parameter β is to "eat" the powers of k in order to get a Saint-Venant estimate for which the induction procedure of section 5.3.1 works. Gathering the latter bounds, we obtain for $k \ge m$

$$\|\,\mathrm{A}\,\|_{L^\infty(\Sigma_{k+1})} \le C_\beta k m^{-\beta}\left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right)^{1/2}. \tag{5.57}$$

The second term in (5.21) is even simpler to estimate. One ends up with

$$\|\,\mathrm{B}\,\|_{L^\infty(\Sigma_{k+1})} \le C_\beta k m^{-\beta}\left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right)^{1/2}. \tag{5.58}$$

Therefore A and B satisfy the desired estimate, since

$$\| \mathrm{A} \|_{L^2(\Sigma_{k+1})} \leq Ck \| \mathrm{A} \|_{L^\infty(\Sigma_{k+1})}, \quad \| \mathrm{B} \|_{L^2(\Sigma_{k+1})} \leq Ck \| \mathrm{B} \|_{L^\infty(\Sigma_{k+1})}.$$

The last integral in (5.21) is more intricate, because it is a convolution integral. Moreover, $\rho * (1-\chi_{k+m}) v_{0,h}(y)$ is no longer supported in $\Sigma \setminus \Sigma_{k+m}$. The idea is to "invert" the variables y and t, i.e. to replace the kernel $|x-y|^{-3}$ by $|x-t|^{-3}$. Indeed, we have, for all $x,\ y,\ t \in \mathbb{R}^2$,

$$\left| \frac{1}{|x-y|^3} - \frac{1}{|x-t|^3} \right| \leq \frac{C|y-t|}{|x-y||x-t|^3} + \frac{C|y-t|}{|x-y|^3|x-t|}. \tag{5.59}$$

We decompose the integral term accordingly. We obtain, using the fast decay of ρ,

$$\begin{aligned}
&\int_{|x-y|\geq m/2} dy \frac{1}{|x-y|^3} |\rho * ((1-\chi_{k+m}) v_{0,h})(y)| \\
\leq\ & C \int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt \frac{1}{|x-t|^3} |\rho(y-t)||v_{0,h}(t)| \\
&+ C \int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt \frac{|y-t|}{|x-y|^3|x-t|} |\rho(y-t)||v_{0,h}(t)| \\
&+ C \int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt \frac{|y-t|}{|x-y||x-t|^3} |\rho(y-t)||v_{0,h}(t)| \\
\leq\ & C \int_{\Sigma\setminus\Sigma_{k+m}} dt \frac{1}{|x-t|^3} |v_{0,h}(t)| \\
&+ C \int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt \frac{|y-t|}{|x-y|^3|x-t|} |\rho(y-t)||v_{0,h}(t)|.
\end{aligned}$$

The first term in the right hand side above can be addressed thanks to Lemma 5.31. We focus on the second term. As above, we use the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_{\Sigma\setminus\Sigma_{k+m}} \frac{|y-t|\,|\rho(y-t)|}{|x-t|} |v_{0,h}(t)| dt \\
\leq\ & \sum_{j=1}^\infty \int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} \frac{|y-t|\,|\rho(y-t)|}{|x-t|} |v_{0,h}(t)| dt \\
\leq\ & \left(\sup_{j\geq k+m} \frac{E_{m+j}-E_j}{j} \right)^{\frac12} \sum_{j=1}^\infty \frac{1}{k+mj-|x|_\infty} \left(\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2 |\rho(y-t)|^2 |t| dt \right)^{\frac12}.
\end{aligned}$$

The idea is to use the fast decay of ρ so as to bound the integral over $\Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$. However, $\sum_{j=1}^\infty \frac{1}{k+mj-|x|} = \infty$, so that we also need to recover some decay with respect to j in this integral. For $t \in \Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$,

$$1 \leq \frac{|t|-|x|_\infty}{k+mj-|x|_\infty} \leq \frac{|t|}{k+mj-|x|_\infty},$$

so that for all $\eta > 0$,

$$\begin{aligned}
&\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2|\rho(y-t)|^2|t|dt \\
\leq\ & \frac{1}{(k+mj-|x|_\infty)^{2\eta}}\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2|\rho(y-t)|^2|t|^{1+2\eta}dt \\
\leq\ & \frac{C}{(k+mj-|x|_\infty)^{2\eta}}\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2(|y-t|^{1+2\eta}+|y|^{1+2\eta})|\rho(y-t)|^2dt \\
\leq\ & \frac{C_\eta}{(k+mj-|x|_\infty)^{2\eta}}(1+|y-x|^{1+2\eta}+|x|^{1+2\eta})).
\end{aligned}$$

Summing in j, we have as before

$$\sum_{j=1}^\infty \frac{1}{(k+mj-|x|_\infty)^{1+\eta}} \leq \frac{C_\eta}{m(k+m-|x|_\infty)^\eta} \leq \frac{C_\eta}{m^{1+\eta}}$$

so that for $0<\eta<\frac{1}{2}$, one finally obtains

$$\begin{aligned}
&\int_{|x-y|\geq\frac{m}{2}} dy \int_{\Sigma\setminus\Sigma_{k+m}} \frac{|y-t|\,|\rho(y-t)|}{|x-y|^3|x-t|}|v_{0,h}(t)|dt \\
\leq\ & Cm^{-1-\eta}\left(\sup_{j\geq k+m}\frac{E_{m+j}-E_j}{j}\right)^{\frac{1}{2}}\int_{|x-y|\geq\frac{m}{2}}\left[|x-y|^{-\frac{5}{2}+\eta}+|x|^{\frac{1}{2}}|x-y|^{-3+\eta}\right]dy \\
\leq\ & Cm^{-\frac{3}{2}}\left[1+\left(\frac{k}{m}\right)^{\frac{1}{2}}\right]\left(\sup_{j\geq k+m}\frac{E_{k+j}-E_j}{j}\right)^{\frac{1}{2}}.
\end{aligned}$$

Gathering all the terms, and using one again the fact that

$$\|F\|_{L^2(\Sigma_{k+1})} \leq Ck\|F\|_{L^\infty(\Sigma_{k+1})} \quad \forall F\in L^\infty(\Sigma_{k+1}),$$

we infer that for all $k\geq m$,

$$\|\,\mathrm{C}\,\|_{L^2(\Sigma_{k+1})} \leq C\frac{k^{\frac{3}{2}}}{m^2}\left(\sup_{j\geq k+m}\frac{E_{k+j}-E_j}{j}\right)^{\frac{1}{2}}.$$

Lemma 5.32 is thus proved. □

5.4 Uniqueness

This section is devoted to the proof of uniqueness of solutions of (5.33). Therefore we consider the system (5.33) with $f=0$ and $F=0$, and we intend to prove that the solution u is identically zero.

Following the notations of the previous section, we set

$$E_k := \int_{\Omega_k}\nabla u\cdot\nabla u.$$

We can carry out the same estimates than those of paragraph 5.3.2 and get a constant $C_1>0$ such that for all $m\in\mathbb{N}$, for all $k\geq m$,

$$E_k \leq C_1\left(E_{k+m+1}-E_k+\frac{k^4}{m^5}\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\right). \tag{5.60}$$

Let m a positive even integer and $\varepsilon > 0$ be fixed. Analogously to paragraph 5.3.1, the set $\mathcal{C}_m$ is defined by

$$\mathcal{C}_m := \left\{c, \text{ square of edge of length } m \text{ with vertices in } \mathbb{Z}^2\right\}.$$

Note that the situation is not quite the same as in paragraph 5.3.1 since this set is infinite. The values of $E_c := \int_{\Omega_c} |\nabla u|^2$, when $c \in \mathcal{C}_m$ are bounded by $m^2 \|u\|^2_{H^1_{uloc}(\Omega^b)}$, so the following supremum exists

$$\mathcal{E}_m := \sup_{c\in\mathcal{C}_m} E_c < \infty,$$

but may not be attained. Therefore for $\varepsilon > 0$, we choose a square $c \in \mathcal{C}_m$ such that $\mathcal{E}_m - \varepsilon \le E_c \le \mathcal{E}_m$. As in paragraph 5.3.1, up to a shift we can always assume that c is centered in 0.

From (5.60), we retrieve, for all $m, k \in \mathbb{N}$ with $k \ge m$,

$$E_k \le \frac{C_1}{C_1+1} E_{k+m+1} + \frac{C_1}{C_1+1}\frac{k^4}{m^5} \sup_{j\ge k+m} \frac{E_{j+m} - E_j}{j}.$$

Again, the conclusion $E_k = 0$ would be very easy to get if there were no second term in the right hand side taking into account the large scales due to the non local operator DN.

An induction argument then implies that for all $r \in \mathbb{N}$,

$$E_k \le \left(\frac{C_1}{C_1+1}\right)^r E_{k+r(m+1)} + \sum_{r'=0}^{r-1} \left(\frac{C_1}{C_1+1}\right)^{r'+1} \frac{(k+r'(m+1))^4}{m^5} \sup_{j\ge k+m} \frac{E_{j+m} - E_j}{j}. \tag{5.61}$$

Now, for $\kappa := \ln\left(\frac{C_1}{C_1+1}\right) < 0$ and for $k \in \mathbb{N}$ large enough, the function $x \mapsto \exp(\kappa(x+1))(k+x(m+1))^4$ is decreasing on $(-1,\infty)$, so that

$$\begin{aligned}
\sum_{r'=0}^{r-1} \left(\frac{C_1}{C_1+1}\right)^{r'+1} \frac{(k+r'(m+1))^4}{m^5} &\le \sum_{r'=0}^{\infty} \left(\frac{C_1}{C_1+1}\right)^{r'+1} \frac{(k+r'(m+1))^4}{m^5} \\
&\le \frac{1}{m^5} \int_{-1}^{\infty} \exp\left(\kappa(x+1)\right)\left(k+x(m+1)\right)^4 dx \\
&\le C\frac{k^5}{m^6} \int_{-\frac{m+1}{k}}^{\infty} \exp\left(\frac{\kappa k}{m+1} u\right) (1+u)^4 \, du \\
&\le C\frac{k^5}{m^6}
\end{aligned}$$

since $k/(m+1) \ge 1/2$ as soon as $k \ge m \ge 1$. Therefore, we conclude from (5.61) for $k = m$ that for all $r \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{E}_m - \varepsilon \le E_m &\le \left(\frac{C_1}{C_1+1}\right)^r E_{m+r(m+1)} + \frac{C}{m} \sup_{j\ge 2m} \frac{E_{j+m} - E_j}{j} \\
&\le \left(\frac{C_1}{C_1+1}\right)^r (r+1)^2(m+1)^2 \|u\|^2_{H^1_{uloc}} + 4\frac{C}{m} \sup_{j\ge 2m} \frac{j+m}{jm} \mathcal{E}_m \\
&\le \left(\frac{C_1}{C_1+1}\right)^r (r+1)^2(m+1)^2 \|u\|^2_{H^1_{uloc}} + \frac{C}{m^2}\mathcal{E}_m.
\end{aligned}$$

Since the constants are uniform in m, we have for m sufficiently large and for all $\varepsilon > 0$,

$$\mathcal{E}_m \le C\left[\left(\frac{C_1}{C_1+1}\right)^r (r+1)^2(m+1)^2 + \varepsilon\right],$$

which letting $r \to \infty$ and $\varepsilon \to 0$ gives $\mathcal{E}_m = 0$. The latter holds for all m large enough, and thus we have $u = 0$.

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5.A Proof of Lemmas 5.4 and 5.5

This section is devoted to the proofs of Lemma 5.4, which gives a formula for the determinant of M, and Lemma 5.5, containing the low and high frequency expansions of the main functions we work with, namely λ_k and A_k. As A_1, A_2, A_3 can be expressed in terms of the eigenvalues λ_k solution to (5.8), it is essential to begin by stating some properties of the latter. Usual properties on the roots of polynomials entail that the eigenvalues satisfy

$$\begin{aligned}
&\mathcal{R}(\lambda_k) > 0 \text{ for } k = 1,2,3, \quad \lambda_1 \in]0,\infty[, \quad \lambda_2 = \overline{\lambda_3},\\
&-(\lambda_1\lambda_2\lambda_3)^2 = -|\xi|^6, \quad \lambda_1\lambda_2\lambda_3 = |\xi|^3,\\
&(|\xi|^2 - \lambda_1^2)(|\xi|^2 - \lambda_2^2)(|\xi|^2 - \lambda_3^2) = |\xi|^2,\\
&\frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} = \frac{\lambda_k}{|\xi|^2 - \lambda_k^2}
\end{aligned} \tag{5.62}$$

and can be computed exactly

$$\lambda_1^2(\xi) = |\xi|^2 + \left(\frac{-|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac12}}{2}\right)^{\frac13} - \left(\frac{|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac12}}{2}\right)^{\frac13}, \tag{5.63a}$$

$$\lambda_2^2(\xi) = |\xi|^2 + j\left(\frac{-|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac12}}{2}\right)^{\frac13} - j^2\left(\frac{|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac12}}{2}\right)^{\frac13}, \tag{5.63b}$$

$$\lambda_3^2(\xi) = |\xi|^2 + j^2\left(\frac{-|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac12}}{2}\right)^{\frac13} - j\left(\frac{|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac12}}{2}\right)^{\frac13}. \tag{5.63c}$$

5.A.1 Expansion of the eigenvalues λ_k

The expansions below follow directly from the exact formulas (5.63). In high frequencies, that is for $|\xi| \gg 1$, we have

$$\lambda_1^2 = |\xi|^2\left(1 - |\xi|^{-\frac43} + O\left(|\xi|^{-\frac83}\right)\right), \quad \lambda_1 = |\xi| - \tfrac12|\xi|^{-\frac13} + O\left(|\xi|^{-\frac53}\right), \tag{5.64a}$$

$$\lambda_2^2 = |\xi|^2\left(1 - j^2|\xi|^{-\frac43} + O\left(|\xi|^{-\frac83}\right)\right), \quad \lambda_2 = |\xi| - \tfrac{j^2}{2}|\xi|^{-\frac13} + O\left(|\xi|^{-\frac53}\right), \tag{5.64b}$$

$$\lambda_3^2 = |\xi|^2\left(1 - j|\xi|^{-\frac43} + O\left(|\xi|^{-\frac83}\right)\right), \quad \lambda_3 = |\xi| - \tfrac{j}{2}|\xi|^{-\frac13} + O\left(|\xi|^{-\frac53}\right). \tag{5.64c}$$

In low frequencies, that is for $|\xi| \ll 1$, we have

$$\left(|\xi|^4 + \frac{4}{27}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{27}}\left[1 + \frac{27}{8}|\xi|^4 + O\left(|\xi|^8\right)\right],$$

$$\left(\frac{-|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac{1}{2}}}{2}\right)^{\frac{1}{3}} = \frac{1}{\sqrt{3}} - \frac{1}{2}|\xi|^2 - \frac{\sqrt{3}}{8}|\xi|^4 + O\left(|\xi|^6\right),$$

$$\left(\frac{|\xi|^2 + \left(|\xi|^4 + \frac{4}{27}\right)^{\frac{1}{2}}}{2}\right)^{\frac{1}{3}} = \frac{1}{\sqrt{3}} + \frac{1}{2}|\xi|^2 - \frac{\sqrt{3}}{8}|\xi|^4 + O\left(|\xi|^6\right),$$

from which we deduce

$$\lambda_2^2 = i + \frac{3}{2}|\xi|^2 - \frac{3}{8}i|\xi|^4 + O(|\xi|^6), \quad \lambda_2 = e^{i\frac{\pi}{4}}\left(1 - \tfrac{3}{4}i|\xi|^2 + \tfrac{3}{32}|\xi|^4 + O(|\xi|^6)\right), \tag{5.65a}$$

$$\lambda_3^2 = -i + \frac{3}{2}|\xi|^2 + \frac{3}{8}i|\xi|^4 + O(|\xi|^6), \quad \lambda_3 = e^{-i\frac{\pi}{4}}\left(1 + \tfrac{3}{4}i|\xi|^2 + \tfrac{3}{32}|\xi|^4 + O(|\xi|^6)\right). \tag{5.65b}$$

Since $\lambda_1\lambda_2\lambda_3 = |\xi|^3$, we infer that

$$\lambda_1 = |\xi|^3 + O(|\xi|^7).$$

5.A.2 Expansion of A_1, A_2 and A_3

Let us recall that $A_k = A_k(\xi)$, $k = 1, \dots 3$, solve the linear system

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \frac{\left(|\xi|^2-\lambda_1^2\right)^2}{\lambda_1} & \frac{\left(|\xi|^2-\lambda_2^2\right)^2}{\lambda_2} & \frac{\left(|\xi|^2-\lambda_3^2\right)^2}{\lambda_3} \end{pmatrix}}_{=:M(\xi)} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \widehat{v_{0,3}} \\ i\xi \cdot \widehat{v_{0,h}} \\ -i\xi^\perp \cdot \widehat{v_{0,h}} \end{pmatrix}.$$

The exact computation of A_k is not necessary. For the record, note however that A_k can be written in the form of a quotient

$$A_k = \frac{P\left(\xi_1, \xi_2, \lambda_1, \lambda_2, \lambda_3\right)}{Q\left(|\xi|, \lambda_1, \lambda_2, \lambda_3\right)} \tag{5.66}$$

where P is a polynomial with complex coefficients and

$$Q := \det(M) = \left(\lambda_1 - \lambda_2\right)\left(\lambda_2 - \lambda_3\right)\left(\lambda_3 - \lambda_1\right)\left(|\xi| + \lambda_1 + \lambda_2 + \lambda_3\right). \tag{5.67}$$

This formula for $\det(M)$ is shown using the relations (5.62)

$$\begin{aligned}\det(M) &= \frac{\lambda_2^2\left(|\xi|^2 - \lambda_3^2\right)^2 - \lambda_3^2\left(|\xi|^2 - \lambda_2^2\right)^2}{\lambda_2\lambda_3} - \frac{\lambda_1^2\left(|\xi|^2 - \lambda_3^2\right)^2 - \lambda_3^2\left(|\xi|^2 - \lambda_1^2\right)^2}{\lambda_1\lambda_3} \\ &\qquad + \frac{\lambda_1^2\left(|\xi|^2 - \lambda_2^2\right)^2 - \lambda_2^2\left(|\xi|^2 - \lambda_1^2\right)^2}{\lambda_1\lambda_2} \\ &= |\xi|\left(\lambda_1\left(\lambda_2^2 - \lambda_3^2\right) - \lambda_2\left(\lambda_1^2 - \lambda_3^2\right) + \lambda_3\left(\lambda_1^2 - \lambda_2^2\right)\right) \\ &\qquad + \lambda_2\lambda_3\left(\lambda_3^2 - \lambda_2^2\right) - \lambda_1\lambda_3\left(\lambda_3^2 - \lambda_1^2\right) + \lambda_1\lambda_2\left(\lambda_2^2 - \lambda_1^2\right) \\ &= \left(\lambda_1 - \lambda_2\right)\left(\lambda_2 - \lambda_3\right)\left(\lambda_3 - \lambda_1\right)\left(|\xi| + \lambda_1 + \lambda_2 + \lambda_3\right).\end{aligned}$$

This proves (5.67), and thus lemma 5.4.

We now concentrate on the expansions of $M(\xi)$ for $|\xi| \gg 1$ and $|\xi| \ll 1$.

High frequency expansion

At high frequencies, it is convenient to work with the quantities B_1, B_2, B_3 introduced in (5.15). Indeed, inserting the expansions (5.64) into the system (5.10) yields

$$\begin{aligned}
B_1 &= \widehat{v_{0,3}},\\
|\xi| B_1 - \frac{1}{2}|\xi|^{-1/3} B_2 + O(|\xi|^{-5/3}|A|) &= i\xi \cdot \widehat{v_{0,h}},\\
|\xi|^{1/3} B_3 + O(|\xi|^{-1}|A|) &= -i\xi^\perp \cdot \widehat{v_{0,h}}.
\end{aligned}$$

Of course A and B are of the same order, so that the above system becomes

$$\begin{aligned}
B_1 &= \hat{v}_{0,3},\\
B_2 = 2|\xi|^{1/3}(|\xi|\widehat{v_{0,3}} - i\xi \cdot \widehat{v_{0,h}}) &+ O(|\xi|^{-4/3}|B|),\\
B_3 = -i|\xi|^{-1/3}\xi^\perp \cdot \widehat{v_{0,h}} &+ O(|\xi|^{-4/3}|B|).
\end{aligned}$$

We infer immediately that $|B| = O(|\xi|^{4/3}|\widehat{v_0}|)$, and therefore the result of Lemma 5.5 follows.

Low frequency expansion

At low frequencies, we invert M thanks to the adjugate matrix formula

$$M^{-1}(\xi) = \frac{1}{\det(M(\xi))}\left[\mathrm{Cof}(M(\xi))\right]^T.$$

We have

$$\frac{(|\xi|^2 - \lambda_2^2)^2}{\lambda_2} = \frac{e^{i\pi}(1 + O(|\xi|^2))}{e^{i\pi/4}(1 + O(|\xi|^2))} = -e^{-i\pi/4} + O(|\xi|^2) = \overline{\frac{(|\xi|^2 - \lambda_3^2)^2}{\lambda_3}}.$$

Hence,

$$M(\xi) = \begin{pmatrix} 1 & 1 & 1 \\ O\left(|\xi|^3\right) & e^{i\frac{\pi}{4}} + O\left(|\xi|^2\right) & e^{-i\frac{\pi}{4}} + O\left(|\xi|^2\right) \\ |\xi| + O\left(|\xi|^5\right) & -e^{-i\frac{\pi}{4}} + O\left(|\xi|^2\right) & -e^{i\frac{\pi}{4}} + O\left(|\xi|^2\right) \end{pmatrix}$$

and

$$\mathrm{Cof}(M) = \begin{pmatrix} -2i & |\xi| e^{-i\frac{\pi}{4}} & -|\xi| e^{i\frac{\pi}{4}} \\ \sqrt{2} i & -e^{i\frac{\pi}{4}} - |\xi| & e^{-i\frac{\pi}{4}} + |\xi| \\ -\sqrt{2} i & -e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \end{pmatrix} + O\left(|\xi|^2\right).$$

We deduce that

$$\begin{aligned}
M^{-1}(\xi) &= -\frac{1}{2i\left(1 + \frac{\sqrt{2}}{2}|\xi| + O\left(|\xi|^2\right)\right)}\left[\mathrm{Cof}(M(\xi))\right]^T \\
&= \begin{pmatrix} 1 - \frac{\sqrt{2}}{2}|\xi| & -\frac{\sqrt{2}}{2}\left[1 - \frac{\sqrt{2}}{2}|\xi|\right] & +\frac{\sqrt{2}}{2}\left[1 - \frac{\sqrt{2}}{2}|\xi|\right] \\ \frac{e^{i\frac{\pi}{4}}}{2}|\xi| & -\frac{1}{2i}\left[-e^{i\frac{\pi}{4}} - \left(1 - \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}}\right)|\xi|\right] & -\frac{e^{i\frac{\pi}{4}}}{2}\left[1 - \frac{\sqrt{2}}{2}|\xi|\right] \\ \frac{e^{-i\frac{\pi}{4}}}{2}|\xi| & -\frac{1}{2i}\left[e^{-i\frac{\pi}{4}} + \left(1 - \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}}\right)|\xi|\right] & -\frac{e^{-i\frac{\pi}{4}}}{2}\left[1 - \frac{\sqrt{2}}{2}|\xi|\right] \end{pmatrix} + O\left(|\xi|^2\right).
\end{aligned}$$

Finally,

$$A_1 = \left(1 - \frac{\sqrt{2}}{2}|\xi|\right)\widehat{v_{0,3}} - \frac{\sqrt{2}}{2} i\left(\xi + \xi^\perp\right)\cdot \widehat{v_{0,h}} + O\left(|\xi|^2\,|\widehat{v_0}|\right), \tag{5.68a}$$

$$A_2 = \frac{e^{i\frac{\pi}{4}}}{2}|\xi|\widehat{v_{0,3}} + \frac{1}{2}e^{i\frac{\pi}{4}}\xi\cdot\widehat{v_{0,h}} - \frac{1}{2}e^{-i\frac{\pi}{4}}\xi^\perp\cdot\widehat{v_{0,h}} + O\left(|\xi|^2\,|\widehat{v_0}|\right), \tag{5.68b}$$

$$A_3 = \frac{e^{-i\frac{\pi}{4}}}{2}|\xi|\widehat{v_{0,3}} - \frac{1}{2}e^{-i\frac{\pi}{4}}\xi\cdot\widehat{v_{0,h}} + \frac{1}{2}e^{i\frac{\pi}{4}}\xi^\perp\cdot\widehat{v_{0,h}} + O\left(|\xi|^2\,|\widehat{v_0}|\right). \tag{5.68c}$$

5.A.3 Low frequency expansion for L_1, L_2 and L_3

For the sake of completeness, we sketch the low frequency expansion of L_1 in detail. We recall that

$$L_k(\xi)\widehat{v_0}(\xi) = \begin{pmatrix} \frac{i}{|\xi|^2}(-\lambda_k\xi + \frac{(|\xi|^2-\lambda_k^2)^2}{\lambda_k}\xi^\perp) \\ 1 \end{pmatrix} A_k(\xi)$$

Hence, for $|\xi| \ll 1$,

$$L_1(\xi) = \begin{pmatrix} \frac{i}{|\xi|}\xi^\perp + O(|\xi|^2) \\ 1 \end{pmatrix}\begin{pmatrix} -\frac{i\sqrt{2}}{2}(\xi_1-\xi_2) & -\frac{i\sqrt{2}}{2}(\xi_1+\xi_2) & 1-\frac{\sqrt{2}}{2}|\xi| \end{pmatrix} + O(|\xi|^2)$$

which yields (5.18). The calculations for L_2 and L_3 are completely analogous.

5.A.4 The Dirichlet to Neumann operator

Let us recall the expression of the operator DN in Fourier space:

$$\widehat{\mathrm{DN}(v^0)} = \sum_{k=1}^{3}\begin{pmatrix} \frac{i}{|\xi|^2}\left[\left(|\xi|^2-\lambda_k^2\right)^2\xi^\perp - \lambda_k^2\xi\right] \\ \lambda_k + \frac{|\xi|^2-\lambda_k^2}{\lambda_k}\end{pmatrix} A_k \tag{5.69}$$

$$= \begin{pmatrix} -i\widehat{v_3^0}(\xi)\xi \\ i\xi\cdot\widehat{v_h^0}(\xi)\end{pmatrix} + \sum_{k=1}^{3}\begin{pmatrix} \frac{i}{|\xi|^2}\left[\left(|\xi|^2-\lambda_k^2\right)^2\xi^\perp + \left(|\xi|^2-\lambda_k^2\right)\xi\right] \\ \frac{|\xi|^2-\lambda_k^2}{\lambda_k}\end{pmatrix} A_k. \tag{5.70}$$

High frequency expansion

Using the exact formula (5.70) for $\widehat{\mathrm{DN}\, v_0}$ together with the expansions (5.64) and (5.13), we get for the high frequencies

$$\widehat{\mathrm{DN}\, v_0} = \begin{pmatrix} -i\widehat{v_3^0}(\xi)\xi \\ i\xi\cdot\widehat{v_h^0}(\xi)\end{pmatrix} + \begin{pmatrix} \frac{i}{|\xi|^2}\left((|\xi|^{4/3}B_3 + O(|\xi|^{4/3}|\widehat{v_0}|))\xi^\perp + (|\xi|^{2/3}B_2 + O(|\xi|^{2/3}|\widehat{v_0}|))\xi\right) \\ |\xi|^{-1/3}B_2 + O(|\xi|^{-1/3}|\widehat{v_0}|)\end{pmatrix} \tag{5.71}$$

$$= \begin{pmatrix} |\xi|\widehat{v_h^0} + \frac{\xi\cdot\widehat{v_h^0}}{|\xi|}\xi + i\widehat{v_3^0}\xi \\ 2|\xi|\widehat{v_3^0} - i\xi\cdot\widehat{v_h^0}\end{pmatrix} + O\left(|\xi|^{\frac{1}{3}}|\widehat{v_0}|\right).$$

Low frequency expansion

For $|\xi| \ll 1$, using (5.69), (5.65) and (5.68) leads to

$$
\begin{aligned}
&\widehat{\mathrm{DN}_h\, v_0} \\
&= \frac{i}{2|\xi|^2} \sum_{\pm} \left(-\xi^\perp \mp i\xi + O(|\xi|^3)\right) \left(e^{\pm i\pi/4}|\xi|\widehat{v_{0,3}} \pm e^{\pm i\pi/4}\xi \cdot \widehat{v_{0,h}} \mp e^{\mp i\pi/4}\xi^\perp \cdot \widehat{v_{0,h}} + O(|\xi|^2|\widehat{v_0}|)\right) \qquad (5.72a)\\
&= \frac{\sqrt{2}i}{2}\frac{\xi - \xi^\perp}{|\xi|}\widehat{v_{0,3}} + \frac{\sqrt{2}}{2}(\widehat{v_{0,h}} + \widehat{v_{0,h}}^\perp) + O(|\xi||\widehat{v_0}|). \qquad (5.72b)
\end{aligned}
$$

For the vertical component of the operator DN, we have in low frequencies

$$
\begin{aligned}
\widehat{\mathrm{DN}_3\, v_0} &= i\xi \cdot \widehat{v_{0,h}} + \left(\frac{1}{|\xi|} + O\left(|\xi|\right)\right) A_1(\xi) - \left(e^{i\frac{\pi}{4}} + O\left(|\xi|^2\right)\right) A_2(\xi) - \left(e^{-i\frac{\pi}{4}} + O\left(|\xi|^2\right)\right) A_3(\xi) \\
&= \frac{\widehat{v_{0,3}}}{|\xi|} - \frac{\sqrt{2}}{2}\widehat{v_{0,3}} - \frac{\sqrt{2}i}{2}\frac{\xi \cdot \widehat{v_{0,h}} + \xi^\perp \cdot \widehat{v_{0,h}}}{|\xi|} + O\left(|\xi|\, |\widehat{v_0}|\right). \qquad (5.72c)
\end{aligned}
$$

5.B Lemmas for the remainder terms

The goal of this section is to prove that the various remainder terms encountered throughout the paper decay like $|x|^{-3}$. To that end, we introduce the algebra

$$
\begin{aligned}
E := \Big\{ f \in \mathcal{C}([0,\infty),\mathbb{R}), \exists \mathcal{A} \subset \mathbb{R} \text{ finite},\ \exists r_0 > 0,\ f(r) = \sum_{\alpha\in\mathcal{A}} r^\alpha f_\alpha(r)\ \forall r \in [0, r_0), \\
\text{where } \forall \alpha \in \mathcal{A},\ f_\alpha : \mathbb{R} \to \mathbb{R} \text{ is analytic in } B(0, r_0) \Big\}. \qquad (5.73)
\end{aligned}
$$

We then have the following result:

Lemma 5.33. *Let $\varphi \in \mathcal{S}'(\mathbb{R}^2)$.*

- *Assume that* $\operatorname{Supp}\hat\varphi \subset B(0,1)$*, and that $\hat\varphi(\xi) = f(|\xi|)$ for ξ in a neighbourhood of zero, with $f \in E$ and $f(r) = O(r^\alpha)$ for some $\alpha > 1$. Then $\varphi \in L^\infty_{loc}(\mathbb{R}^2 \setminus \{0\})$ and there exists a constant C such that*

$$|\varphi(x)| \le \frac{C}{|x|^3} \quad \forall x \in \mathbb{R}^2.$$

- *Assume that* $\operatorname{Supp}\hat\varphi \subset \mathbb{R}^2 \setminus B(0,1)$*, and that $\hat\varphi(\xi) = f(|\xi|^{-1})$ for $|\xi| > 1$, with $f \in E$ and $f(r) = O(r^\alpha)$ for some $\alpha > -1$. Then $\varphi \in L^\infty_{loc}(\mathbb{R}^2 \setminus \{0\})$ and there exists a constant C such that*

$$|\varphi(x)| \le \frac{C}{|x|^3} \quad \forall x \in \mathbb{R}^2.$$

We prove the Lemma in several steps: we first give some properties of the algebra E. We then compute the derivatives of order 3 of functions of the type $f(|\xi|)$ and $f(|\xi|^{-1})$. Eventually, we explain the link between the bounds in Fourier space and in the physical space.

Properties of the algebra E

Lemma 5.34. – *E is stable by differentiation.*

– *Let $f \in E$ with $f(r) = \sum_{\alpha \in \mathcal{A}} r^\alpha f_\alpha(r)$, and let $\alpha_0 \in \mathbb{R}$. Assume that*

$$f(r) = O(r^{\alpha_0})$$

for r in a neighbourhood of zero. Then

$$\inf\{\alpha \in \mathcal{A}, f_\alpha(0) \neq 0\} \geq \alpha_0.$$

– *Let $f \in E$, and let $\alpha_0 \in \mathbb{R}$ such that*

$$f(r) = O(r^{\alpha_0})$$

for r in a neighbourhood of zero. Then

$$f'(r) = O(r^{\alpha_0 - 1})$$

for $0 < r \ll 1$.

Proof. The first point simply follows from the chain rule and the fact that if f_α is analytic in $B(0, r_0)$, then so is f'_α. Concerning the second point, notice that we can always choose the set $\mathcal{A}$ and the functions f_α so that

$$f(r) = r^{\alpha_1} f_{\alpha_1}(r) + \cdots + r^{\alpha_s} f_{\alpha_s}(r),$$

where $\alpha_1 < \cdots < \alpha_s$ and f_{α_i} is analytic in $B(0, r_0)$ with $f_{\alpha_i}(0) \neq 0$. Therefore

$$f(r) \sim r^{\alpha_1} f_{\alpha_1}(0) \text{ as } r \to 0,$$

so that $r^{\alpha_1} = O(r^{\alpha_0})$. It follows that $\alpha_1 \geq \alpha_0$. Using the same expansion, we also obtain

$$f'(r) = \sum_{i=1}^{s} \alpha_i r^{\alpha_i - 1} f_{\alpha_i}(r) + r^{\alpha_i} f'_{\alpha_i}(r) = O(r^{\alpha_1 - 1}).$$

Since $r^{\alpha_1} = O(r^{\alpha_0})$, we infer eventually that $f'(r) = O(r^{\alpha_0 - 1})$. □

Differentiation formulas

Now, since we wish to apply the preceding Lemma to functions of the type $f(|\xi|)$, or $f(|\xi|^{-1})$, where $f \in E$, we need to have differentiation formulas for such functions. Tedious but easy computations yield, for $\varphi \in \mathcal{C}^3(\mathbb{R})$,

$$\begin{aligned} \partial^3_{\xi_i} f(|\xi|) &= \left(3\frac{\xi_i^3}{|\xi|^5} - 3\frac{\xi_i}{|\xi|^3} \right) f'(|\xi|) \\ &+ \left(3\frac{\xi_i}{|\xi|^2} - \frac{\xi_i^3}{|\xi|^4} \right) f''(|\xi|) \\ &+ \frac{\xi_i^3}{|\xi|^3} f'^{(3)}(|\xi|) \end{aligned}$$

and

$$\begin{aligned}\partial^3_{\xi_i} f(|\xi|^{-1}) &= \left(9\frac{\xi_i}{|\xi|^5} - 11\frac{\xi_i^3}{|\xi|^7}\right) f'(|\xi|^{-1}) \\ &+ \left(3\frac{\xi_i}{|\xi|^6} - 7\frac{\xi_i^3}{|\xi|^8}\right) f''(|\xi|^{-1}) \\ &+ \frac{\xi_i^3}{|\xi|^9} f'^{(3)}(|\xi|^{-1})\end{aligned}$$

In particular, if $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is such that $\varphi(\xi) = f(|\xi|)$ for ξ in a neighbourhood of zero, where $f \in E$ is such that $f(r) = O(r^\alpha)$ for r close to zero, we infer that

$$|\partial^3_{\xi_1}\varphi(\xi)| + |\partial^3_{\xi_2}\varphi(\xi)| = O(|\xi|^{\alpha-3})$$

for $|\xi| \ll 1$. In a similar fashion, if $\varphi(\xi) = f(|\xi|^{-1})$ for ξ in a neighbourhood of zero, where $f \in E$ is such that $f(r) = O(r^\alpha)$ for r close to zero, we infer that

$$|\partial^3_{\xi_1}\varphi(\xi)|+|\partial^3_{\xi_2}\varphi(\xi)| = O\left(|\xi|^{-4}(|\xi|^{-1})^{-\alpha-1} + |\xi|^{-5}(|\xi|^{-1})^{-\alpha-2}|\xi|^{-6}(|\xi|^{-1})^{-\alpha-3}\right) = O(|\xi|^{\alpha-3}).$$

Moments of order 3 in the physical space

Lemma 5.35. *Let $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ such that $\partial^3_{\xi_1}\varphi, \partial^3_{\xi_2}\varphi \in L^1(\mathbb{R}^2)$. Then, for all $x_h \in \mathbb{R}^2 \setminus \{0\}$,*

$$\left|\mathcal{F}^{-1}(\varphi)(x_h)\right| \le C\frac{C}{|x_h|^3}.$$

Proof. The proof follows from the formula

$$x_h^\alpha \mathcal{F}^{-1}(\varphi) = i\mathcal{F}^{-1}(\nabla^\alpha_\xi \varphi)$$

for all $\alpha \in \mathbb{N}^2$ such that $|\alpha| = 3$. When $\varphi \in \mathcal{S}(\mathbb{R}^2)$, the formula is a consequence of standard properties of the Fourier transform. It is then extended to $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ by duality. □

The result of Lemma 5.33 then follows easily. There only remains to explain how we can apply it to the functions in the present paper. To that end, we first notice that for all $k \in \{1, 2, 3\}$, λ_k is a function of $|\xi|$ only, say $\lambda_k = f_k(|\xi|)$. In a similar fashion,

$$L_k(\xi) = G^0_k(|\xi|) + \xi_1 G^1_k(|\xi|) + \xi_2 G^2_k(|\xi|).$$

We then claim the following result:

Lemma 5.36.
- *For all $k \in \{1, 2, 3\}$, $j \in \{0, 1, 2\}$, the functions f_k, G^j_k, as well as*

$$r \mapsto f_k(r^{-1}),\ r \mapsto G^j_k(r^{-1}) \tag{5.74}$$

  all belong to E.
- *For ξ in a neighbourhood of zero,*

$$\begin{aligned}M^{rem}_k &= P_k(\xi) + \sum_{1\le i,j,\le 2} \xi_i\xi_j a^{ij}_k(|\xi|) + \xi \cdot b_k(|\xi|), \\ N^{rem}_k &= Q_k(\xi) + \sum_{1\le i,j,\le 2} \xi_i\xi_j c^{ij}_k(|\xi|) + \xi \cdot d_k(|\xi|),\end{aligned}$$

  where P_k, Q_k are polynomials, and $a^{ij}_k, c^{i,j}_k \in E$, $b_k,\ d_k \in E^2$ with $b_k(r),\ d_k(r) = O(r)$ for r close to zero.

– *There exists a function $m \in E$ such that*

$$(M_{SC} - M_S)(\xi) = m(|\xi|^{-1})$$

for $|\xi| \gg 1$.

The lemma can be easily proved using the formulas (5.63) together with the Maclaurin series for functions of the type $x \mapsto (1+x)^s$ for $s \in \mathbb{R}$.

5.C Fourier multipliers supported in low frequencies

This appendix is concerned with the proof of Lemma 5.8, which is a slight variant of a result by Droniou and Imbert [DI06] on integral formulas for the fractional laplacian. Notice that this corresponds to the operator $\mathcal{I}[|\xi|] = \mathcal{I}\left[\frac{\xi_1^2+\xi_2^2}{|\xi|}\right]$. We recall that $g \in \mathcal{S}(\mathbb{R}^2)$, $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $\rho := \mathcal{F}^{-1}\zeta \in \mathcal{S}(\mathbb{R}^2)$. Then, for all $x \in \mathbb{R}^2$,

$$\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|}\right) * \mathcal{F}^{-1}\left(\xi_i\xi_j\zeta(\xi)\hat{g}(\xi)\right)(x).$$

As explained in [DI06], the function $|\xi|^{-1}$ is locally integrable in $\mathbb{R}^2$ and therefore belongs to $\mathcal{S}'(\mathbb{R}^2)$. Its inverse Fourier transform is a radially symmetric distribution with homogeneity $-2+1=-1$. Hence there exists a constant C_I such that

$$\mathcal{F}^{-1}\left(\frac{1}{|\xi|}\right) = \frac{C_I}{|x|}.$$

We infer that

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) &= \frac{C_I}{|\cdot|} * \partial_{ij}(\rho * g)\\ &= C_I\int_{\mathbb{R}^2}\frac{1}{|x-y|}\partial_{ij}(\rho*g)(y)dy\\ &= C_I\int_{\mathbb{R}^2}\frac{1}{|y|}\partial_{ij}(\rho*g)(x+y)dy.\end{aligned}$$

The idea is to put the derivatives ∂_{ij} on the kernel $\frac{1}{|y|}$ through integrations by parts. As such it is not possible to realize this idea. Indeed, $y \mapsto \partial_i\left(\frac{1}{|y|}\right)\partial_j(\rho*g)(x+y)$ is not integrable in the vicinity of 0. In order to compensate for this lack of integrability, we consider an even function $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $0 \le \theta \le 1$ and $\theta = 1$ on $B(0,K)$, and we introduce the auxiliary function

$$U_x(y) := \rho * g(x+y) - \rho * g(x) - \theta(y)\,(y\cdot\nabla)\,\rho * g(x)$$

which satisfies

$$|U_x(y)| \le C|y|^2, \qquad |\nabla_y U_x(y)| \le C|y|, \tag{5.75}$$

for y close to 0. Then, for all $y \in \mathbb{R}^2$,

$$\partial_{y_i}\partial_{y_j}U_x = \partial_{y_i}\partial_{y_j}\rho * g(x+y) - \left(\partial_{y_i}\partial_{y_j}\theta\right)(y\cdot\nabla)\rho*g(x) - \left(\partial_{y_j}\theta\right)\partial_{x_i}\rho*g(x) - \left(\partial_{y_i}\theta\right)\partial_{x_j}\rho*g(x)$$

where

$$y \mapsto -\left(\partial_{y_i}\partial_{y_j}\theta\right)(y\cdot\nabla)\rho*g(x) - \left(\partial_{y_j}\theta\right)\partial_{x_i}\rho*g(x) - \left(\partial_{y_i}\theta\right)\partial_{x_j}\rho*g(x)$$

is an odd function. Therefore, for all $\varepsilon > 0$,

$$\int_{\varepsilon<|y|<\varepsilon^{-1}} \frac{1}{|y|}\partial_{ij}(\rho * g)(x+y)dy = \int_{\varepsilon\le|y|\le\frac{1}{\varepsilon}} \frac{1}{|y|}\partial_{y_i}\partial_{y_j}U_x(y)dy.$$

A first integration by parts yields

$$\begin{aligned}
&\int_{\varepsilon\le|y|\le\frac{1}{\varepsilon}} \frac{1}{|y|}\partial_{y_i}\partial_{y_j}\rho * g(x+y)dy \\
=\ &\int_{\varepsilon\le|y|\le\frac{1}{\varepsilon}} \frac{1}{|y|}\partial_{y_i}\partial_{y_j}U_x(y)dy \\
=\ &\int_{|y|=\varepsilon} \frac{1}{|y|}\partial_{y_j}U_x(y)n_i(y)dy + \int_{|y|=\frac{1}{\varepsilon}} \frac{1}{|y|}\partial_{y_j}U_x(y)n_i(y)dy + \int_{\varepsilon\le|y|\le\frac{1}{\varepsilon}} \frac{y_i}{|y|^3}\partial_{y_j}U_x(y)dy.
\end{aligned}$$

The first boundary integral vanishes as $\varepsilon \to 0$ because of (5.75), and the second thanks to the fast decay of $\rho * g \in \mathcal{S}(\mathbb{R}^2)$. Another integration by parts leads to

$$\begin{aligned}
&\int_{\varepsilon\le|y|\le\frac{1}{\varepsilon}} \frac{y_i}{|y|^3}\partial_{y_j}U_x(y)dy \\
=\ &\int_{|y|=\varepsilon} \frac{y_i}{|y|^3}U_x(y)n_j(y)dy + \int_{|y|=\frac{1}{\varepsilon}} \frac{y_i}{|y|^3}U_x(y)n_j(y)dy + \int_{\varepsilon\le|y|\le\frac{1}{\varepsilon}} \left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)U_x(y)dy \\
&\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^2} \left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)U_x(y)dy,
\end{aligned}$$

where

$$\partial_{y_i}\partial_{y_j}\frac{1}{|y|} = -\frac{\delta_{ij}}{|y|^3} + 3\frac{y_iy_j}{|y|^5}, \qquad \left|\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right| \le \frac{C}{|y|^3},$$

and the boundary terms vanish because of (5.75) and the fast decay of U_x. Therefore, for all $x \in \mathbb{R}^2$,

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) = C_I\int_{\mathbb{R}^2}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)U_x(y)dy \\
=\ & C_I\int_{\mathbb{R}^2}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)\left[\rho * g(x+y) - \rho * g(x) - \theta(y)\left(y\cdot\nabla\right)\rho * g(x)\right]dy \\
=\ & C_I\int_{B(0,K)}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)\left[\rho * g(x+y) - \rho * g(x) - y\cdot\nabla\rho * g(x)\right]dy \\
+\ & C_I\int_{\mathbb{R}^2\setminus B(0,K)}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)\left[\rho * g(x+y) - \rho * g(x)\right]dy \\
-\ & C_I\int_{\mathbb{R}^2\setminus B(0,K)}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)\theta(y)\left(y\cdot\nabla\right)\rho * g(x)dy.
\end{aligned}$$

The last integral is zero as $y \mapsto \theta(y)\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)y$ is odd. We then perform a last change of variables by setting $y' = x + y$, and we obtain

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) \\
=\ & -\int_{|x-y'|\le K}\gamma_{ij}(x-y')\left\{\rho * g(y') - \rho * g(x) - (y'-x)\nabla\rho * g(x)\right\}dy' \\
& -\int_{|x-y'|\le K}\gamma_{ij}(x-y')\left\{\rho * g(y') - \rho * g(x)\right\}dy'.
\end{aligned}$$

This terminates the proof of Lemma 5.8.

Chapter 6

Newtonian limit for weakly viscoelastic fluid flows

Joint work with Didier Bresch

Contents

Abstract This chapter addresses the low Weissenberg asymptotic analysis (newtonian limit) of some macroscopic models of viscoelastic fluid flows in the framework of global weak solutions. It gathers a few preliminary results obtained in collaboration with Didier Bresch. Our mid-term goal is to have a better understanding of the effect of a small amount of elasticity on the dynamics of non-newtonian fluid flows with weak regularity. We investigate the convergence of the corotational Johnsson-Segalman and the FENE-P models. Relying on a priori bounds coming from energy or free energy estimates, we first study the weak convergence toward the Navier-Stokes system. We then turn to the main focus of our paper, i.e. the strong convergence. The novelty of our work is to address these issues thanks to relative entropy estimates, which ask for the introduction of some corrector terms. We obtain a convergence result for the corotational system. Using a similar relative entropy method and carrying out far more intricate calculations, we are about to obtain a similar result for the FENE-P system. This work is still in progress, and

will be submitted soon. We also take into account the presence of defect measures in the initial data, uniform with respect to the Weissenberg number, and prove that they do not perturb the newtonian limit of the corotational system.

6.1 Introduction

This work is concerned with viscoelastic fluid flows, which have an elastic behaviour in short times, and a viscous one in large times. Such non-newtonian fluids are ubiquitous: glaciers, Earth's mantle, dough, paint, solutions of polymers. They have a complex dynamic. For instance, phenomena such as the rod climbing effect, the tubeless siphon effect and die swell can be observed in polymeric liquids. In order to get an insight into the physics of viscoelastic fluid flows, the reader is refered to [Ren00, LBL09, Ott05, Osw05].

Because of elasticity, viscoelastic fluids remember their history, which means that the dynamic of the flow at a given time depends on the past. This is in strong constrast with newtonian fluids (i.e. purely viscous fluids). The viscoelastic relaxation time is roughly the time on which the flow remembers the past. The dimensionless number, which compares the viscoelastic relaxation time to a time scale relevant to the fluid flow, is the Weissenberg (or Deborah number) We. The bigger We, the more important is the elasticity with respect to the viscosity.

The purpose of our paper is to face a problem raised by J.-C. Saut in his recent review article [Sau12]: the mathematical study of the newtonian limit of models from non-newtonian fluid mechanics, that is to say the limit We $\to 0$. We focus on some macroscopic models of polymeric viscoelastic fluid flows. The works presented here are a first step toward a better understanding of the effect of a small amount of elasticity on the newtonian dynamic of a fluid with weak regularity.

6.1.1 Macroscopic models of viscoelastic fluid flows

All macro-macro models we consider here are the coupling of a momentum equation on the incompressible velocity $u = u(t,x) \in \mathbb{R}^d$ and an equation for the symmetric stress tensor $\tau = \tau(t,x) \in M_d(\mathbb{R})$ (or a symmetric structure tensor $A = A(t,x) \in M_d(\mathbb{R})$, which has a microscopic meaning). In the sequel, we concentrate on two models: namely the corotational Johnson-Segalman model

$$\left\{\begin{aligned} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau,\\ \nabla\cdot u &= 0,\\ \mathrm{We}\left(\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau\right) + \tau &= 2\omega D(u),\end{aligned}\right. \tag{6.1}$$

and the FENE-P model

$$\left\{\begin{aligned} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau,\\ \nabla\cdot u &= 0,\\ \tau &= \tfrac{(b+d)\omega}{b}\tfrac{1}{\mathrm{We}}\left(\tfrac{A}{1-\frac{\mathrm{Tr}\,A}{b}} - \mathrm{I}\right),\\ \partial_t A + u\cdot\nabla A - \nabla u A - A\left(\nabla u\right)^T + \tfrac{1}{\mathrm{We}}\tfrac{A}{1-\frac{\mathrm{Tr}\,A}{b}} &= \tfrac{1}{\mathrm{We}}\,\mathrm{I}\,.\end{aligned}\right. \tag{6.2}$$

Notice that these systems are posed in $\Omega \subset \mathbb{R}^d$ a bounded domain, $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. We recall that $D(u) := \frac{\nabla u + (\nabla u)^T}{2}$ is the deformation tensor and that $W(u) := \frac{\nabla u - (\nabla u)^T}{2}$ is the vorticity tensor. The quantity $\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau$ (resp. $\partial_t A + u\cdot\nabla A - \nabla u A - A\left(\nabla u\right)^T$) is known as the corotational (resp. upper convected) derivative of τ (resp. A).

In addition, we assume that u satisfies a noslip boundary condition on $\partial\Omega$. There is no condition for τ (nor A) on the boundary. We start from the initial conditions:

$$u(0,\cdot) := u_0, \qquad \tau(0,\cdot) := \tau_0, \qquad A(0,\cdot) := A_0.$$

We consider only the case of Jeffrey fluids, for which $0 < \omega < 1$, in the framework of global in time weak solutions. The case $\omega = 1$ turns out to be much more complicated (like Euler in comparison to Navier-Stokes).

Corotational model A simple a priori energy estimate on (6.1) leads to

$$\omega \|u(t,\cdot)\|^2_{L^2(\Omega)} + 2\omega(1-\omega)\int_0^t \|\nabla u\|^2_{L^2(\Omega)} + \frac{\mathrm{We}}{2}\|\tau(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|\tau\|^2_{L^2(\Omega)} \leq \omega \|u_0\|^2_{L^2(\Omega)} + \frac{\mathrm{We}}{2}\|\tau_0\|^2_{L^2(\Omega)}. \quad (6.3)$$

This decay of energy is a consequence of the algebraic identity

$$(\tau W(u) - W(u)\tau) : \tau = 0.$$

Although it is convenient from a mathematical viewpoint and greatly simplifies the analysis of the system, (6.3) points out some drawbacks of the corotational model. Indeed, as underlined in [WH98], this decay of energy is not relevant from a physical viewpoint. Furthermore, as noticed in [Ren00, Chapter 3], the corotational model is unable to predict some behaviours, such as the rod climbing effect.

The existence of weak solutions to the corotational model (6.1) for $d = 2$ or 3 is due to P.-L. Lions and N. Masmoudi [LM00]. The starting point of their analysis is the inequality (6.3). They intensively rely on the use of defect measures to pass to the limit in the product $\tau_n W(u_n)$, where (u_n, τ_n) is an approximated smooth solution to (6.1). Note that the regularity provided by (6.3) is barely $\tau \in L^\infty((0,\infty); L^2)$ and $\nabla u \in L^2((0,\infty); L^2)$. The key of the proof is the control of τ in $L^\infty((0,T); L^q)$ for a $q > 2$ and $0 < T < \infty$.

The initial velocity field is taken in the space $I_{p,q} \subset W^{-1,q}$: for all $1 < p, q < \infty$

$$I_{p,q} := \left\{ u_0 \in W^{-1,q};\ \left\| A_q^{-\frac{1}{2}} u_0 \right\|_{L^q} + \left(\int_0^\infty \left\| A_q e^{-tA_q} A_q^{-\frac{1}{2}} u_0 \right\|_{L^q} dt \right)^{\frac{1}{p}} < \infty \right\}, \qquad (6.4)$$

where $A_q := P_q \Delta$ is the Stokes operator with domain

$$D(A_q) := \left\{ u \in L^{q,\sigma};\ \nabla^2 u \in L^q,\ u|_{\partial\Omega} = 0 \right\},$$

P_q being the Helmholtz projector on $L^{q,\sigma}$ with domain L^q. The space $L^{q,\sigma}$ is the closure of the space $\{v \in C_0^\infty(\Omega) : \nabla \cdot v = 0\}$ in L^q. For more properties about the space $I_{p,q}$ and the Stokes operator, we refer to [GS91].

In more details, their existence result reads:

Result A (P.-L. Lions, N. Masmoudi). *There exists a global weak solution (u,τ) of (6.1) satisfying the energy inequality (6.3) such that for all $0 < T < \infty$,*

$$\nabla u \in L^p((0,T); L^q) \quad \textit{and} \quad \tau \in C^0([0,\infty); L^q),$$

provided that $\tau_0 \in L^q$ and $u_0 \in I_{p,q}$,

– *for some $2 < q < +\infty$, $1 < p < +\infty$, if $d = 2$,*
– *and for some $2 < q \leq 3$, $1 < p \leq \frac{q}{2q-3}$, if $d = 3$.*

FENE-P model This model is one of the many closure approximations of the microscopic FENE (Finite Extensible Nonlinear Elastic) dumbbel model (see [DLY05a, DLY05b]). Its low computational costs, compared to micro-macro models, and its acceptable predictions make it a widely used model for numerical simulation of viscoelastic fluid flows. However, as pointed out in [Keu97], it does not capture all the physics of the microscopic model. Note that the parameter b in (6.2) relates to the extensibility of the elastic dumbbels at the microscopic scale.

The energy is replaced by a non-trivial free energy (or entropy), which has been known from physicists since the work of L. E. Wedgewood and R. B. Bird [WB88] (see also [WH98, Ott05]). D. Hu and T. Lelièvre in [HL07] have recently rediscovered this entropy and showed that it decays in time:

$$\begin{aligned}
&\frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Omega)} + (1-\omega)\int_0^t \|\nabla u\|^2_{L^2(\Omega)} \\
&\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega \left[-\ln(\det A) - b\ln\left(1-\frac{\mathrm{Tr}(A)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](t) \\
&\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}^2}\int_0^t\int_\Omega \left[\frac{\mathrm{Tr}\,A}{\left(1-\frac{\mathrm{Tr}\,A}{b}\right)^2} - \frac{2d}{1-\frac{\mathrm{Tr}\,A}{b}} + \mathrm{Tr}\left(A^{-1}\right)\right] \\
&\leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega \left[-\ln(\det A_0) - b\ln\left(1-\frac{\mathrm{Tr}(A_0)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right]
\end{aligned} \tag{6.5}$$

Based on this decay, N. Masmoudi [Mas11] has achieved an existence result for the system (6.2). The fundamental point is that the decay of the entropy (6.5) yields a control of the $L^2((0,\infty)\times\Omega)$ norm of τ.

Result B (N. Masmoudi)**.** *Let $d \geq 2$. Assume that $u_0 \in L^2$ is a divergence free vector field and that $A_0 = A_0(x)$ is a symmetric positive definite matrix with $\mathrm{Tr}\,A_0 < b$ and such that*

$$\int_\Omega \left[-\ln(\det A_0) - b\ln\left(1-\frac{\mathrm{Tr}\,A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right] < \infty.$$

Then, there exists a global weak solution (u, A, τ) to (6.2) satisfying (6.5), such that

$$u \in L^\infty\left((0,\infty);L^2\right) \cap L^\infty\big((0,\infty);\dot H^1\big),\ \ A \in L^\infty((0,\infty)\times\Omega) \ \text{ and } \tau \in L^2((0,\infty)\times\Omega).$$

6.1.2 Outline of our results

The existence theorems of global weak solutions open the way to the asymptotic analysis at low Weissenberg number. From a formal perspective, it is easy to see that the velocity field u of the non-newtonian fluid model (6.1) converges toward a solution u^0 of the Navier-Stokes system

$$\begin{cases} \partial_t u^0 + u^0\cdot\nabla u^0 - \Delta u^0 + \nabla p^0 &= 0, \quad \Omega, \\ \nabla\cdot u^0 &= 0, \quad \Omega, \\ u^0 &= 0, \quad \partial\Omega. \end{cases} \tag{6.6}$$

Notice that the noslip condition is compatible with the limit, so that no boundary layers are involved in this limit (at least at the leading order in We). That is why, we state our results for Ω a bounded domain, $\Omega = \mathbb{T}^d$ or $\Omega = \mathbb{R}^d$ without making any difference.

As far as we know, the newtonian limit of non-newtonian fluids has only been studied in the context of strong solutions. First results in the direction of a better understanding of this limit have been reached by J.-C. Saut in [Sau86] for Maxwell type flows (no diffusion term in the momentum equation) in the linear régime. The only other result we are aware of is the one of L. Molinet and R. Talhouk [MT08] for strong solutions of Johnson-Segalman systems (including the corotational and the Oldroyd-B systems). For these models no energy of the type of (6.3) is available in general, so they rely on a splitting in low and high frequencies at a cut-off frequency depending on We.

The originality of our work is to address the newtonian limit in the framework of weak solutions relying only on energy (or free energy) methods. Thus our results do not ask for more smoothness than the natural regularity available.

The first logical step in our study of the limit, is to investigate the weak convergence. For the corotational and the FENE-P models, we easily obtain the weak convergence toward the Navier-Stokes system. An estimate on a relative entropy involving higher-order corrector terms makes it then possible to achieve a strong convergence result on the corotational system. Using a similar relative entropy method and carrying out far more intricate calculations, we are about to obtain a similar result for the FENE-P system. This work is still in progress, and will be submitted soon.

Newtonian limit: weak convergence

The mathematical justification of the formal asymptotics requires a priori bounds uniform in We. Some bounds, like the $L^\infty\left((0,\infty);L^2\right)$ bound on τ for (6.1), are not uniform in We. They were usefull for the Cauchy theory, but are useless for the newtonian limit.

Notice that the initial data u_0, A_0 and τ_0 may depend on We. In order to get uniform bounds in We, we have to assume that initial data is well-prepared, in a sense to be made precise later on. We always start from data meeting the conditions of Result A or B leading to the existence of weak solutions.

Our first result is concerned with the weak convergence in the corotational system.

Proposition 6.1. *Let $d = 2,\ 3$. Let (u,τ) be a weak solution of* (6.1) *in the sense of Result A. Assume that*

$$\|u_0\|^2_{L^2(\Omega)} + \frac{\mathrm{We}}{2}\,\|\tau_0\|^2_{L^2(\Omega)} = O(1). \tag{6.7}$$

Then, there exist

$$u^0 \in L^\infty\Big((0,\infty);L^{2,\sigma}\Big) \cap L^2\Big((0,\infty);\dot{H}^1\Big) \quad \textit{and} \quad \tau^0 \in L^2\,\Big((0,\infty);L^2\Big)$$

such that u (resp. τ) converges to u^0 (resp. τ^0) at least in the sense of distribution, where u^0 is a weak solution of the Navier-Stokes system (6.6) *and* $\tau^0 = 2\omega D\left(u^0\right)$.

We turn to the weak convergence for the FENE-P model. It is formally clear that A converges to $A^0 := \frac{b}{b+d}\,\mathrm{I}$. The key to the convergence of u is a bound on τ in $L^2\left((0,\infty)\times\Omega\right)$ uniform in We, deduced from the decay of the free energy (6.5).

Proposition 6.2. *Let $d = 2,\ 3$. We consider a global weak solution (u,A,τ) of* (6.2) *in the sense of Result B. Assume that $\|u_0\|_{L^2(\Omega)} = O(1)$. Then we get several convergence results in the limit* $\mathrm{We} \to 0$.

- *Assume that initial data is* ill-prepared *in the sense that*

$$\int_\Omega \Big[-\ln\left(\det A_0\right) - b\ln\Big(1-\frac{\mathrm{Tr}\,A_0}{b}\Big) + (b+d)\ln\Big(\frac{b}{b+d}\Big)\Big] = O\left(1\right).$$

Then A tends to A^0 in $L^2\left((0,\infty);L^2\left(\Omega\right)\right)$ and

$$\left\|A - A^0\right\|_{L^2((0,\infty);L^2(\Omega))} = O\Big(\sqrt{\mathrm{We}}\Big). \tag{6.8}$$

• *Assume furthermore that initial data is* well-prepared *namely*

$$\int_\Omega \left[-\ln\left(\det A_0\right) - b\ln\left(1 - \frac{\operatorname{Tr} A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right] = O\left(\mathrm{We}\right). \tag{6.9}$$

Then we have the following improved convergences:

$$\left\| A - A^0 \right\|_{L^\infty((0,\infty);L^2(\Omega))} = O\left(\sqrt{\mathrm{We}}\right), \tag{6.10a}$$

$$\left\| A - A^0 \right\|_{L^2((0,\infty);L^2(\Omega))} = O\left(\mathrm{We}\right). \tag{6.10b}$$

Moreover, τ is bounded uniformly in $L^2\left((0,\infty)\times\Omega\right)$, and u (resp. τ) converges in the sense of distributions toward u^0 (resp. $2\omega D(u^0)$), where u^0 satisfies the Navier-Stokes system (6.6)*.*

Before coming to the strong convergence, let us state a slight generalization of Proposition 6.1 allowing to handle the case of oscillating initial data $(u_{0,n}, \tau_{0,n})$. For the sake of easiness, we temporarily consider initial data independent of We and treat only the case $d = 2$. We assume that $u_{0,n}$ strongly converges in $L^2(\Omega)$ toward u_0, and that $\tau_{0,n}$ is equicontinuous in $L^2(\Omega)$. In particular, we do not assume that $\tau_{0,n}$ converges strongly in $L^2(\Omega)$. We then call (u_n, τ_n) the associated weak solution of (6.1), which satisfies the energy inequality (6.3). We show that passing to the limit on n introduces defect measures in the limit system. These defect measures are due to the oscillations of the initial data $\tau_{0,n}$. We prove that they disappear in the limit $\mathrm{We} \to 0$.

Proposition 6.1 bis. Let $d = 2$ and $\Omega = \mathbb{R}^2$. The result is in two points:
• *Limit* $n \to \infty$. There exists

$$u \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\left((0,\infty); \dot H^1\right), \quad \text{and} \quad \tau \in L^\infty\left((0,\infty); L^2\right),$$

such that (u_n, τ_n) tends to (u, τ) at least in the sense of distributions and (u, τ) satisfies the system

$$\left\{ \begin{aligned} \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p &= \nabla\cdot\tau \\ \nabla\cdot u &= 0 \\ \mathrm{We}\left[\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau + \begin{pmatrix} -\varepsilon & \frac{1}{2}\delta \\ -\frac{1}{2}\delta & \varepsilon \end{pmatrix}\right] + \tau &= 2\omega D(u) \end{aligned} \right. \tag{6.11}$$

with defect measures $\delta,\ \varepsilon \in L^1_{loc}\left((0,\infty); L^1\right)$.
• *Limit* $\mathrm{We} \to 0$. There exists

$$u^0 \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\left((0,\infty); \dot H^1\right) \quad \text{and} \quad \tau^0 \in L^2\left((0,\infty); L^2\right),$$

solving the Navier-Stokes system (6.6) in the sense of distributions and such that (u, τ) converges weakly toward (u^0, τ^0).

This result is quite distant from the main focus of our paper. It is a natural continuation of some techniques involved in the article [LM00]. We therefore postpone its proof to the Appendix 6.A. The main difficulty is to get uniform in We a priori estimates on the defect measures δ and ε, so as to pass to the limit in (6.11).

Newtonian limit: strong convergence

Our strong convergence result follows from two steps: first we build an Ansatz for u and τ, and then we compare this approximation to u and τ in an appropriate norm derived from the energy associated to the system. This is the leitmotiv of relative entropy (or modulated energy) methods. Since the pioneering work of C. Dafermos [Daf79] and of H.-T. Yau [Yau91], relative entropy methods have become a crucial and widely used tool in the study of asymptotic limits to kinetic models in the contex of hydrodynamics [LM01, GSR04] and of the quasineutral limit for the Vlasov-Poisson system [Bre00, HK11]. They have also been successfully implemented in the approximation of incompressible fluids by hyperbolic systems [BNP04, NR06]; see also [Tza05, LT12] for an expository of the general method, which is close to ours, and the use of corrector terms. Let us also mention the use of relative entropies for the study of the long-time behavior of some micro-macro models for dilute solutions of polymers, and the convergence to equilibrium by B. Jourdain, C. Le Bris, T. Lelièvre and F. Otto [JLBLO06]. Relative entropy methods are also the key to the estimates of F. Otto and A. Tzavaras in [OT08].

The very rough idea is work with an energy $\mathfrak{e} = \mathfrak{e}(u, A, \tau)$ like the one in the left hand side of (6.3) (resp. (6.5)). Notice that $\mathfrak{e}(u, A, \tau) = \mathfrak{e}_1(u) + \mathfrak{e}_2(A) + \mathfrak{e}_3(\tau)$. The decisive point, is that the functions $\mathfrak{e}_i$, for $i = 1, \ldots\ 3$ are *globally* convex. Thus, one can make a Taylor expansion of $\mathfrak{e}$ around say $\left(\tilde{u}, \tilde{A}, \tilde{\tau}\right)$, and get that the quantity

$$\mathfrak{E}(u, A, \tau) := \mathfrak{e}(u, A, \tau) - \mathfrak{e}\left(\tilde{u}, \tilde{A}, \tilde{\tau}\right) - \nabla\mathfrak{e}_1\left(\tilde{u}\right)\cdot\left(u - \tilde{u}\right) - \nabla\mathfrak{e}_2\left(\tilde{A}\right)\cdot\left(A - \tilde{A}\right) - \nabla\mathfrak{e}_3\left(\tilde{\tau}\right)\cdot\left(\tau - \tilde{\tau}\right)$$

is positive and controls the norm of

$$|u - \tilde{u}|^2 + \alpha\left|A - \tilde{A}\right|^2 + \beta\,|\tau - \tilde{\tau}|^2\,,$$

with $\alpha,\ \beta \geq 0$, $\alpha = 0$ (resp. $\beta = 0$) for the corotational (resp. for the FENE-P) model. The quantity $\mathfrak{E}$ is called the relative entropy (or modulated energy). Its control grounds on an explicit computation of its total time derivative in order to establish a Gronwall type inequality. These computations may be quite tricky (especially in the case of the FENE-P system), as they involve the algebraic structure of the equations of (6.1) or (6.2).

This procedure yields an error estimate between u (resp. A, τ) and its approximation. The convergence result holds for solutions of (6.1) or (6.2) with very low regularity, typically weak solutions. However, in order to carry out the estimates of the relative entropies, we need quite a lot regularity on the profiles of our Ansatz.

Our theorem below handles the case of the corotational system.

Theorem 6.3. *Let $d = 2,\ 3$, $u_0 \in H^{4,\sigma}(\Omega)$ independent of* We *and $\tau_0 \in L^2(\Omega) \cap L^q(\Omega)$, with $2 < q \leq 3$. Notice that τ_0 may depend on* We *in the following sense: $\left\|\tau_0\right\|_{L^2(\Omega)} = O(1)$. Let also*

$$u \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\left((0,\infty); \dot{H}^1\right), \quad \textit{and} \quad \tau \in L^\infty\left((0,\infty); L^2\right) \cap L^\infty_{loc}\left((0,\infty); L^q\right)$$

be global weak solutions to (6.1) in the sense of Result A associated to the initial data u_0 and τ_0.
Then, there exists $0 < T^* < \infty$ independent of We *and*

$$u^0 \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\left((0,\infty); \dot{H}^1\right)$$

a global weak solution of (6.6) *associated to the initial data* u_0*, such that, in addition,* u^0 *belongs to* $L^\infty\left((0,T);H^4\right)$ *for all* $0<T<T^*$*. Moreover, for all* $0<T<T^*$*,*

$$\sup_{0<t<T}\left(\omega\left\|u(t,\cdot)-u^0(t,\cdot)\right\|_{L^2}^2+\omega(1-\omega)\int_0^t\left\|\nabla\left(u-u^0\right)\right\|_{L^2}^2\right.$$
$$\left.+\frac{\mathrm{We}}{2}\left\|\tau(t,\cdot)-\tau^0(t,\cdot)\right\|_{L^2}^2+\frac{1}{2}\int_0^t\left\|\tau-\tau^0\right\|_{L^2}^2\right)^{\frac{1}{2}}=O\big(\sqrt{\mathrm{We}}\big). \quad (6.12)$$

Let us comment on this theorem:

- The proof is done in the case when $\Omega=\mathbb{R}^3$. It is not hard to adapt our arguments to the easier case $\Omega=\mathbb{R}^2$. Furthermore, as no boundary condition is prescribed on τ, and as $u^0=0$ can be imposed on the boundary for the limit velocity field, there is no boundary layer in the limit $\mathrm{We}\to 0$. Thus, our analysis extends straightforwardly to the case when Ω is a bounded domain.
- We assume that u_0 does not depend on We only in order to alleviate the proof. Of course, one can start from an initial data for u depending on We and get the convergence (6.12) on condition that one assumes
$$\left\|u(0,\cdot)-u^0(0,\cdot)\right\|_{L^2(\Omega)}=O\big(\sqrt{\mathrm{We}}\big).$$
- Our result does not require further regularity for (u,τ) than the natural regularity yielded by (6.3). However, it is quite demanding on the limit profile u^0. It might be possible to weaken the regularity requirements on the initial velocity field. In particular, we do not take advantage of the regularizing effect of the Navier-Stokes equation, which yields $u^0\in L^2\left((0,T);H^5\right)$, because we need the L^∞ bound in time.
- The regularity needed on u^0 in order to carry out the computations of the proof is the reason why the convergence result only holds as long as u^0 is remains sufficiently regular.
- Note that nothing is prescribed on the initial stress tensor τ_0, except the requirements to get a global weak solution to (6.1). In other words, Theorem 6.3 is a convergence result for ill-prepared data τ_0.
- Our theorem complements the study of L. Molinet and R. Talhouk [MT08]. They manage to get a convergence result of strong solutions for ill-prepared data, in the case when the equation on τ has the additional term $a\left(D(u)\tau+\tau D(u)\right)$, with $-1\leq a\leq 1$. Our system (6.1) corresponds to $a=0$. In the general case, $a\neq 0$, there is no known energy associated to the system. Hence their proof, unlike ours, relies on a cutting up of u and τ in low and high frequencies.
- This convergence result can be used, in the two-dimensional case, to prove the existence of global in time strong solutions to (6.1) for We small enough. Of course, we would rely on the global existence of strong solutions to the Navier-Stokes system when $d=2$.

The proof of Theorem 6.3 serves as a guideline for our result on the FENE-P system, which is a work in progress.

6.1.3 Organization of the paper

For the reader's convenience we devote the first section of this paper (Section 6.2) to the proof of the results concerning the corotational system. The proofs are easier in this case, and sheds some light on some features, which help to understand our analysis of the FENE-P system. We show the weak convergence of Proposition 6.1 in the Section 6.2.1, and the strong convergence of Theorem 6.3 in Section 6.2.2. In Section 6.3 we address the weak convergence result related to the FENE-P system, i.e. Proposition 6.2. The proof of Proposition 6.1 bis is postponed to the Appendix 6.A. In this appendix, we also show refined $L^p\left((0,T);L^q\right)$ a priori estimates on the corotational system. These estimates are also the key for the existence of weak solutions to the corotational system. We recall in the Appendix 6.B some elements of the proof of Result A due to P.-L. Lions and N. Masmoudi, in the case $d=2$. Finally, Appendix 6.C is concerned with auxiliary properties of the entropy (6.5).

6.2 Low Weissenberg limit for the corotational system

We concentrate on the low Weissenberg asymptotic analysis of the corotational system (6.1). The first part of this section is devoted the weak convergence. In the second subsection, we show the strong convergence result of Theorem 6.3, relying on a relative entropy method.

6.2.1 Weak convergence

We carry out the proof of Proposition 6.1 in the case when $\Omega \subset \mathbb{R}^d$ is a bounded domain and $d = 2,\ 3$. Our analysis extends straightforwardly to the case $\Omega = \mathbb{T}^d$, and $\Omega = \mathbb{R}^d$, as we only work with local in space bounds. Let (u,τ) be a sequence of weak solutions to (6.1) satisfying the a priori bound (6.3). According to the assumption (6.7) on the initial data, we deduce that

$$\begin{array}{lll} u & \text{is uniformly bounded in We in} & L^\infty\left((0,\infty);L^2\right)\cap L^2\left((0,\infty);\dot{H}^1\right), \\ \tau & \text{u.b. in We in} & L^2\left((0,\infty);L^2\right). \end{array}$$

Let us notice that the bound on τ in $L^\infty\left((0,\infty);L^2\right)$ is not uniform in We.

Compactness As is usual, the former bounds imply the existence of

$$u^0 \in L^\infty\left((0,\infty);L^{2,\sigma}\right)\cap L^2\left((0,\infty);\dot{H}^1\right) \quad \text{and} \quad \tau^0 \in L^2\left((0,\infty);L^2\right),$$

such that the following convergences hold (extracting subsequences if necessary), for all $0<T<\infty$,

$$\begin{aligned} u &\rightharpoonup u^0 && L^2\left((0,T);H^1\right), && (6.13a)\\ u(t,\cdot) &\rightharpoonup u^0(t,\cdot) && L^2(\Omega), && (6.13b)\\ u &\overset{*}{\rightharpoonup} u^0 && L^\infty\left((0,\infty);L^2\right), && (6.13c)\\ \nabla u &\rightharpoonup \nabla u^0 && L^2\left((0,\infty);L^2\right), && (6.13d)\\ \tau &\rightharpoonup \tau^0 && L^2\left((0,\infty);L^2\right), && (6.13e)\\ \tau(t,\cdot) &\rightharpoonup \tau^0(t,\cdot) && L^2(\Omega), && (6.13f) \end{aligned}$$

and

$$\partial_t u \qquad \text{is uniformly bounded in } L^{4/d}\left((0,T);V'\right),$$

where V' is the dual of $V := \{v \in H_0^1,\ \nabla\cdot v = 0\}$. For the latter bound we note that $\nabla\cdot\tau$ is bounded in $L^2\left((0,T);H^{-1}\right)$. The non-linear term is the only tricky one to estimate: for all $\varphi_\sigma \in C_c^\infty\left((0,\infty);C_c^{\infty,\sigma}\right)$, for all $0<t<T$,

$$\left|\langle -u\cdot\nabla u(t,\cdot),\varphi_\sigma(t,\cdot)\rangle_{\mathcal{D}',\mathcal{D}}\right| \le \|u(t,\cdot)\|_{L^2}^{2-d/2}\,\|u(t,\cdot)\|_{H^1}^{d/2}\,\|\varphi_\sigma(t,\cdot)\|_V\,.$$

Therefore, using the Aubin-Lions lemma [CF88], we get the strong convergence

$$u \longrightarrow u^0 \qquad L^2\left((0,T);L^2\right). \tag{6.13g}$$

Weak convergence of τ and u The simple observation leading to the convergence of τ is that

$$2\omega D(u) - \tau = \mathrm{We}\left(\partial_t\tau + u\cdot\nabla\tau + \tau W(u) - W(u)\tau\right).$$

The Weissenberg number in front of the right hand side makes the convergence follow directly from the bounds on u and τ above. The newtonian limit is therefore much more simple than passing to the limit on a sequence of approximated solutions, when proving the existence of weak solutions. We do not need refined convergence results for τ. For all $\psi \in C_c^\infty\left((0,\infty)\times\Omega\right)$,

$$\begin{aligned}
&\langle 2\omega D\left(u\right) - \tau, \psi\rangle_{\mathcal{D}',\mathcal{D}}\\
&= \langle \partial_t\tau + u\cdot\nabla\tau + \tau W\left(u\right) - W\left(u\right)\tau, \psi\rangle_{\mathcal{D}',\mathcal{D}}\\
&= \mathrm{We}\left[-\langle \tau, \partial_t\psi + u\cdot\nabla\psi\rangle_{L^2,L^2} + \langle \tau W\left(u\right) - W\left(u\right)\tau, \psi\rangle_{\mathcal{D}',\mathcal{D}}\right] \overset{\mathrm{We}\to 0}{\longrightarrow} 0.
\end{aligned}$$

Moreover,

$$\langle 2\omega D\left(u\right) - \tau, \psi\rangle_{\mathcal{D}',\mathcal{D}} \overset{\mathrm{We}\to 0}{\longrightarrow} \left\langle 2\omega D\left(u^0\right) - \tau^0, \psi\right\rangle_{\mathcal{D}',\mathcal{D}},$$

which yields at the limit, $\tau^0 = 2\omega D\left(u^0\right)$.

Let $\varphi_\sigma \in C_c^\infty\left((0,\infty); C_c^{\infty,\sigma}\right)$. The strong convergence of the velocity field in $L^2\left((0,T);L^2\right)$ allows to pass to the weak limit in the nonlinear term $u\cdot\nabla u$. Hence, we can pass to the limit in the momentum equation and get that u^0 satisfies

$$\left\langle \partial_t u^0 + u^0\cdot\nabla u^0 - (1-\omega)\Delta u^0, \varphi_\sigma\right\rangle_{\mathcal{D}',\mathcal{D}} = \left\langle \tau^0, \varphi_\sigma\right\rangle_{\mathcal{D}',\mathcal{D}} = \omega\left\langle \Delta u^0, \varphi_\sigma\right\rangle_{\mathcal{D}',\mathcal{D}}.$$

It remains to apply De Rham's theorem (see for example [Sim03] for a rigorous statement) to ensure that u^0 is a weak solution to the Navier-Stokes system (6.6).

Remark 6.4. We can pass to the limit in the inequality (6.3) assuming furthermore that $\sqrt{\mathrm{We}}\,\tau_0$ tends to zero in L^2. Then using that $\tau^0 = 2\omega D(u^0)$ and the obtained weak convergences, we get the standard energy inequality related to the Navier-Stokes equations.

6.2.2 Strong convergence

Our goal is to give a proof of Theorem 6.3 stating the strong convergence of the velocity field u and of the symmetric stress tensor τ of the fluid flow solving (6.1). Our modus operandi emphasizes in a simple case some features of the relative entropy method, which we also use for the strong convergence in the FENE-P system.

To get the strong convergence, we expand u, p and τ in powers of We:

$$u \simeq u^0 + \mathrm{We}\,u^1, \quad p \simeq p^0 + \mathrm{We}\,p^1, \quad \tau \simeq \tau^0 + \mathrm{We}\,\tau^1.$$

Very formal computations yield, as expected, that the lower order term $u^0 = u^0(t,x) \in \mathbb{R}^3$ should solve the three-dimensional Navier-Stokes equation

$$\left\{\begin{aligned} \partial_t u^0 + u^0\cdot\nabla u^0 - \Delta u^0 + \nabla p^0 &= 0\\ \nabla\cdot u^0 &= 0\end{aligned}\right., \tag{6.14}$$

and that the stress tensor

$$\tau^0 = 2\omega D(u^0). \tag{6.15}$$

At first order in We, we expect $u^1 = u^1(t,x) \in \mathbb{R}^3$ and the symmetric tensor $\tau^1 = \tau^1(t,x) \in \mathbb{R}^3$ to solve

$$\begin{cases} \partial_t u^1 + u^0 \cdot \nabla u^1 + u^1 \cdot \nabla u^0 - (1-\omega)\Delta u^1 + \nabla p^1 & = \nabla \cdot \tau^1 \\ \nabla \cdot u^1 & = 0 \\ \partial_t \tau^0 + u^0 \cdot \nabla \tau^0 + \tau^0 W(u^0) - W(u^0)\tau^0 + \tau^1 & = 2\omega D(u^1) \end{cases} . \tag{6.16}$$

We aim at showing, in the case of $\Omega = \mathbb{R}^3$ that these expansions are in fact correct, as well for well-prepared as for ill-prepared data τ_0. The idea of the proof is classical and consists in using the relative entropy of the viscoelastic system. We proceed in two steps for the proof: first we show the well-posedness of system (6.16), then we prove a Gronwall type inequality on the relative entropy.

On the necessity of the first-order correctors

The first-order correctors u^1 and τ^1 are needed to achieve our estimates, although there are transparent in the final convergence result (6.12). Indeed, if one stops the expansion of u and τ at order 0, we get the estimate:

$$\begin{aligned} \omega \left\| u(t,\cdot) - u^0(t,\cdot) \right\|_{L^2}^2 + 2\omega(1-\omega) \int_0^t \left\| \nabla \left(u - u^0\right) \right\|_{L^2}^2 + \frac{\mathrm{We}}{2} \left\| \tau(t,\cdot) \right\|_{L^2}^2 \\ + \int_0^t \left\| \tau - 2\omega D(u^0) \right\|_{L^2}^2 \le \frac{\mathrm{We}}{2} \left\| \tau_0 \right\|_{L^2}^2 - 2\omega \int_0^t \int_{\mathbb{R}^3} \left(\left(u - u^0\right) \cdot \nabla u^0 \right) \cdot \left(u - u^0\right) \\ - 2\omega \int_0^t \int_{\mathbb{R}^3} (\tau - 2\omega D(u)) : D(u^0), \end{aligned} \tag{6.17}$$

which does not seem to allow to conclude. In fact, the natural idea would be to bound

$$\left| -2\omega \int_0^t \int_{\mathbb{R}^3} \left(\left(u - u^0\right) \cdot \nabla u^0 \right) \cdot \left(u - u^0\right) \right| \le 2\omega \int_0^t \left\| \nabla u^0 \right\|_{L^\infty} \left\| u - u^0 \right\|_{L^2}^2 ,$$

and to split the term

$$\begin{aligned} - 2\omega \int_0^t \int_{\mathbb{R}^3} (\tau - 2\omega D(u)) \cdot D(u^0) = -2\omega \int_0^t \int_{\mathbb{R}^3} \left(\tau - 2\omega D(u^0)\right) \cdot D(u^0) \\ - 4\omega^2 \int_0^t \int_{\mathbb{R}^3} \left(D(u^0) - D(u)\right) : D(u^0). \end{aligned}$$

We can bound the latter by

$$\begin{aligned} 2\omega \int_0^t \left\| \tau - 2\omega D(u^0) \right\|_{L^2} \left\| D(u^0) \right\|_{L^2} + 4C\omega^2 \int_0^t \left\| \nabla \left(u - u^0\right) \right\|_{L^2} \left\| D(u^0) \right\|_{L^2} \\ \le C \left[\nu \int_0^t \left\| \tau - 2\omega D(u^0) \right\|_{L^2}^2 + \nu \int_0^t \left\| \nabla \left(u - u^0\right) \right\|_{L^2}^2 + \frac{1}{\nu} \int_0^t \left\| D(u^0) \right\|_{L^2}^2 \right] \end{aligned}$$

and absorb some terms for ν small in the left hand side of (6.17). Yet, the term

$$\frac{1}{\nu} \int_0^t \left\| D(u^0) \right\|_{L^2}^2$$

remaining in the right hand side need not be small in the limit $\mathrm{We} \to 0$.

Well-posedness of the profiles

We carry out an H^m a priori estimate on (6.14) in the same fashion as was done in [Tem75] for the Euler system: there exists $C>0$ such that, for $m \geq 3$, for all t sufficiently small,

$$\frac{1}{2}\left\|u^0(t,\cdot)\right\|_{H^m}^2 + \int_0^t \left\|\nabla u^0(s,\cdot)\right\|_{H^m}^2 ds \leq \frac{1}{2}\left\|u_0\right\|_{H^m}^2 + C_0\left\|u^0(t,\cdot)\right\|_{H^m}^3.$$

Hence, for all $0<T<\frac{1}{C_0\|u_0\|_{H^m}} =: T^*$,

$$\left\|u^0\right\|_{L^\infty((0,T);H^m)} \leq \frac{1}{\frac{1}{\|u_0\|_{H^m}} - C_0 T} < \infty. \tag{6.18}$$

Therefore, there exists a global weak solution $u^0 \in C^0\left([0,\infty);H^{-1}\right) \cap L^\infty\left((0,\infty);L^{2,\sigma}\right) \cap L^2\big((0,\infty);\dot{H}^1\big)$ of (6.14), such that, for all $0<T<T^*$, $u^0 \in L^\infty\left((0,T);H^m\right)$. This regularity for $m=4$ is sufficient for the rest of the computations. Note that an $L^2\left((0,T);H^4\right)$ bound on u^0 is not enough to us, so that we do not take advantage of the regularizing effect of (6.14) to weaken the assumption on the initial data u_0.

From (6.16), one retrieves

$$\tau^1 = 2\omega D(u^1) - \partial_t\tau^0 - u^0\cdot\nabla\tau^0 - \tau^0 W(u^0) + W(u^0)\tau^0, \tag{6.19}$$

so that we can introduce it in the momentum equation at first order:

$$\begin{cases} \partial_t u^1 + u^0\cdot\nabla u^1 + u^1\cdot\nabla u^0 - \Delta u^1 + \nabla p^1 &= f^1 \\ \nabla\cdot u^1 &= 0 \end{cases}. \tag{6.20}$$

We complement (6.20) with the initial data $u^1(0,\cdot)=0$. Using the equation satisfied by u^0, the source term may be written under the form

$$\begin{aligned} f^1 &:= \partial_t\left(\nabla\cdot\tau^0\right) - \nabla\cdot\left(u^0\cdot\nabla\tau^0\right) - \nabla\cdot\left(\tau^0 W(u^0)\right) + \nabla\cdot\left(W(u^0)\tau^0\right) \\ &= -\omega\Delta\left(u^0\cdot\nabla u^0\right) + \omega\Delta^2 u^0 - \nabla\Delta p^0 - 2\omega\nabla\cdot\left(u^0\cdot\nabla D(u^0)\right) - 2\omega\nabla\cdot\left(D(u^0)W(u^0)\right) \\ &\quad + 2\omega\nabla\cdot\left(W(u^0)D(u^0)\right) \\ &= -\omega\Delta\left(u^0\cdot\nabla u^0\right) + \omega\Delta^2 u^0 + \nabla\nabla\cdot\left(u^0\cdot\nabla u^0\right) - 2\omega\nabla\cdot\left(u^0\cdot\nabla D(u^0)\right) \\ &\quad - 2\omega\nabla\cdot\left(D(u^0)W(u^0)\right) + 2\omega\nabla\cdot\left(W(u^0)D(u^0)\right) \end{aligned}$$

which is in $L^\infty\left((0,T);L^2\right)$. Straightforward energy estimates on (6.20) show that a sequence of approximated solutions is bounded in $L^\infty\left((0,T);L^2\right) \cap L^2\left((0,T);H^1\right)$, which yields the existence of a weak solution

$$u^1 \in C^0\left([0,T);H^{-1}\right) \cap L^\infty\left((0,T);L^{2,\sigma}\right) \cap L^2\big((0,T);H^1\big).$$

Using the regularity of u^0, we get the extra estimate $u^1 \in L^\infty((0,T);H^1)$. Hence, one deduces from the latter, (6.19) and the equation satisfied by u^0, that $\tau^1 \in L^2\left((0,T);L^2\right)$.

Weak strong estimate

We now turn to the estimation of the remainders

$$U^{(r)} := u - u^0 - \mathrm{We}\, u^1 \tag{6.21}$$

and $\tau-\tau^0$. In order to carry out the computations below, we need the regularity on the profiles u^0, u^1, τ^0 and τ^1 we have assumed above. On the one hand

$$\begin{aligned}\partial_t U^{(r)}+U^{(r)}\cdot\nabla u+u^0\cdot\nabla U^{(r)}-(1-\omega)\Delta U^{(r)}+\nabla\left(p-p^0-\mathrm{We}\,p^1\right)&\\=-\mathrm{We}\left(u^1\cdot\nabla(u-u^0)\right)+\nabla\cdot\left(\tau-\tau^0-\mathrm{We}\,\tau^1\right),&\end{aligned}$$

which yields

$$\begin{aligned}&\frac{1}{2}\left\|U^{(r)}(t,\cdot)\right\|_{L^2}^2+(1-\omega)\int_0^t\left\|\nabla U^{(r)}\right\|_{L^2}^2+\int_0^t\int_{\mathbb{R}^3}\left(U^{(r)}\cdot\nabla u\right)\cdot U^{(r)}\\&\qquad\le-\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\left(u^1\cdot\nabla(u-u^0)\right)\cdot U^{(r)}-\int_0^t\int_{\mathbb{R}^3}\left(\tau-\tau^0-\mathrm{We}\,\tau^1\right):\nabla U^{(r)}.\end{aligned}\tag{6.22}$$

On the other hand

$$\begin{aligned}\partial_t(\tau-\tau^0)+u\cdot\nabla(\tau-\tau^0)+\frac{\tau-\tau^0}{\mathrm{We}}&=-(u-u^0)\cdot\nabla\tau^0-(\tau-\tau^0)W(u)-\tau^0\left(W(u)-W(u^0)\right)\\&\quad+W(u)(\tau-\tau^0)+\left(W(u)-W(u^0)\right)\tau^0+\tau^1+2\omega\frac{D\left(U^{(r)}\right)}{\mathrm{We}},\end{aligned}$$

which gives

$$\begin{aligned}&\frac{1}{2}\left\|\tau(t,\cdot)-\tau^0(t,\cdot)\right\|_{L^2}^2+\frac{1}{\mathrm{We}}\int_0^t\left\|\tau-\tau^0\right\|_{L^2}^2=\frac{1}{2}\left\|\tau_0-2\omega D\left(u_0\right)\right\|_{L^2}^2\\&\quad+\int_0^t\int_{\mathbb{R}^3}\tau^1:\left(\tau-\tau^0\right)-\int_0^t\int_{\mathbb{R}^3}\left(\left(u-u^0\right)\cdot\nabla\tau^0\right):\left(\tau-\tau^0\right)\\&\quad+\int_0^t\int_{\mathbb{R}^3}\left(\tau^0W(u-u^0)\right):(\tau-\tau^0)+\int_0^t\int_{\mathbb{R}^3}\left(W(u^0-u)\tau^0\right):\left(\tau-\tau^0\right)\\&\quad+\frac{2\omega}{\mathrm{We}}\int_0^t\int_{\mathbb{R}^3}\nabla U^{(r)}:\left(\tau-\tau^0\right).\end{aligned}\tag{6.23}$$

The linear combination 2ω(6.22)+We(6.23) gives the energy equality

$$\omega\left\|U^{(r)}(t,\cdot)\right\|_{L^2}^2+2\omega(1-\omega)\int_0^t\left\|\nabla U^{(r)}\right\|_{L^2}^2+\frac{\mathrm{We}}{2}\left\|\tau(t,\cdot)-\tau^0(t,\cdot)\right\|_{L^2}^2+\int_0^t\left\|\tau-\tau^0\right\|_{L^2}^2\tag{6.24}$$

$$\begin{aligned}&\le\frac{\mathrm{We}}{2}\left\|\tau_0-2\omega D\left(u_0\right)\right\|_{L^2}^2-2\omega\int_0^t\int_{\mathbb{R}^3}\left(U^{(r)}\cdot\nabla u\right)\cdot U^{(r)}\\&\quad-2\omega\,\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\left(u^1\cdot\nabla(u-u^0)\right)\cdot U^{(r)}+\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\tau^1:\left(\tau-\tau^0\right)\\&\quad-\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\left(\left(u-u^0\right)\cdot\nabla\tau^0\right):\left(\tau-\tau^0\right)+\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\left(\tau^0W(u-u^0)\right):(\tau-\tau^0)\\&\quad+\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\left(W(u^0-u)\tau^0\right):\left(\tau-\tau^0\right)+4\omega^2\,\mathrm{We}\int_0^t\int_{\mathbb{R}^3}\tau^1:\nabla U^{(r)}\\&=\frac{\mathrm{We}}{2}\left\|\tau_0-2\omega D\left(u_0\right)\right\|_{L^2}^2+\int_0^t(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}+\mathrm{F}+\mathrm{G}).\end{aligned}\tag{6.25}$$

We estimate each term of the right hand side of (6.25) separately. The goal is to split each term into a part which is sufficiently small to be absorbed by the left hand side of

(6.25), a part which is controlled through a Gronwall type inequality and remainder terms of order $O(\mathrm{We})$. Let $\nu > 0$. This parameter is going to be taken small independently of $0 < \mathrm{We} < 1$ in the sequel. We have

$$\begin{aligned}
|\mathrm{A}| &\leq 2\omega \left| \int_{\mathbb{R}^3} \left(U^{(r)} \cdot \nabla u^0\right) \cdot U^{(r)} \right| + 2\omega\,\mathrm{We} \left| \int_{\mathbb{R}^3} \left(U^{(r)} \cdot \nabla u^1\right) \cdot U^{(r)} \right| \\
&\leq 2\omega \left\| U^{(r)} \right\|_{L^3} \left\| u^0 \right\|_{L^6} \left\| \nabla U^{(r)} \right\|_{L^2} + 2\omega\,\mathrm{We} \left\| U^{(r)} \right\|_{L^3} \left\| u^1 \right\|_{L^6} \left\| \nabla U^{(r)} \right\|_{L^2} \\
&\leq 2\omega\nu^{-\frac{3}{4}} \left\| U^{(r)} \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla u^0 \right\|_{L^2} \nu^{\frac{3}{4}} \left\| \nabla U^{(r)} \right\|_{L^2}^{\frac{3}{2}} + 2\omega\,\mathrm{We} \left\| U^{(r)} \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla u^1 \right\|_{L^2} \left\| \nabla U^{(r)} \right\|_{L^2}^{\frac{3}{2}} \\
&\leq \frac{\omega}{2\nu^3} \left\| U^{(r)} \right\|_{L^2}^2 \left\| \nabla u^0 \right\|_{L^2}^4 + \frac{3\omega\nu}{2} \left\| \nabla U^{(r)} \right\|_{L^2}^2 \\
&\quad + \frac{\omega}{2\nu^3}\,\mathrm{We} \left\| U^{(r)} \right\|_{L^2}^2 \left\| \nabla u^1 \right\|_{L^2}^4 + \frac{3\omega\nu}{2}\,\mathrm{We} \left\| \nabla U^{(r)} \right\|_{L^2}^2 .
\end{aligned}$$

The second term is of order $O\left(\mathrm{We}^2\right)$. Indeed,

$$\begin{aligned}
|\mathrm{B}| &\leq 2\omega\,\mathrm{We} \left| \int_{\mathbb{R}^3} \left(u^1 \cdot \nabla U^{(r)}\right) \cdot U^{(r)} \right| + 2\omega\,\mathrm{We}^2 \left| \int_{\mathbb{R}^3} \left(u^1 \cdot \nabla u^1\right) \cdot U^{(r)} \right| \\
&= 2\omega\,\mathrm{We}^2 \left| \int_{\mathbb{R}^3} \left(u^1 \cdot \nabla u^1\right) \cdot U^{(r)} \right| \\
&\leq 2\omega\,\mathrm{We}^2 \left\| u^1 \right\|_{L^3} \left\| u^1 \right\|_{L^6} \left\| \nabla U^{(r)} \right\|_{L^2} \leq 2\omega\,\mathrm{We}^2 \left\| u^1 \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla u^1 \right\|_{L^2}^{\frac{3}{2}} \left\| \nabla U^{(r)} \right\|_{L^2} \\
&\leq \frac{\omega}{\nu}\,\mathrm{We}^2 \left\| u^1 \right\|_{L^2} \left\| \nabla u^1 \right\|_{L^2}^3 + \omega\nu\,\mathrm{We}^2 \left\| \nabla U^{(r)} \right\|_{L^2}^2 .
\end{aligned}$$

The third term is estimated in a simple way

$$|\mathrm{C}| \leq \mathrm{We} \left\| \tau^1 \right\|_{L^2} \left\| \tau - \tau^0 \right\|_{L^2} \leq \frac{\mathrm{We}}{2} \left\| \tau^1 \right\|_{L^2}^2 + \frac{\mathrm{We}}{2} \left\| \tau - \tau^0 \right\|_{L^2}^2 ;$$

so is the last term

$$|\mathrm{G}| \leq \frac{2\omega^2}{\nu}\,\mathrm{We} \left\| \tau^1 \right\|_{L^2}^2 + 2\omega^2\nu\,\mathrm{We} \left\| \nabla U^{(r)} \right\|_{L^2}^2 .$$

For the fourth term, we rely again on a convexity inequality

$$\begin{aligned}
|\mathrm{D}| &\leq \mathrm{We} \left| \int_{\mathbb{R}^3} \left(U^{(r)} \cdot \nabla \tau^0\right) : \left(\tau - \tau^0\right) \right| + \mathrm{We}^2 \left| \int_{\mathbb{R}^3} \left(u^1 \cdot \nabla \tau^0\right) : \left(\tau - \tau^0\right) \right| \\
&\leq \mathrm{We} \left\| U^{(r)} \right\|_{L^2} \left\| \nabla \tau^0 \right\|_{L^\infty} \left\| \tau - \tau^0 \right\|_{L^2} + \mathrm{We}^2 \left\| u^1 \cdot \nabla \tau^0 \right\|_{L^2} \left\| \tau - \tau^0 \right\|_{L^2} \\
&\leq \frac{\mathrm{We}}{2} \left\| U^{(r)} \right\|_{L^2}^2 \left\| \nabla \tau^0 \right\|_{L^\infty}^2 + \frac{\mathrm{We}}{2} \left\| \tau - \tau^0 \right\|_{L^2}^2 \\
&\quad + \frac{\mathrm{We}^2}{2} \left\| u^1 \right\|_{L^2}^2 \left\| \nabla \tau^0 \right\|_{L^\infty}^2 + \frac{\mathrm{We}^2}{2} \left\| \tau - \tau^0 \right\|_{L^2}^2 .
\end{aligned}$$

The next two terms are treated analogously:

$$\begin{aligned}
|\mathrm{E}| &\leq \mathrm{We} \left| \int_{\mathbb{R}^3} \left(\tau^0 W\left(U^{(r)}\right)\right) : \left(\tau - \tau^0\right) \right| + \mathrm{We}^2 \left| \int_{\mathbb{R}^3} \left(\tau^0 W(u^1)\right) : \left(\tau - \tau^0\right) \right| \\
&\leq \mathrm{We} \left\| \tau^0 \right\|_{L^\infty} \left\| \nabla U^{(r)} \right\|_{L^2} \left\| \tau - \tau^0 \right\|_{L^2} + \mathrm{We}^2 \left\| \tau^0 \right\|_{L^\infty} \left\| \nabla u^1 \right\|_{L^2} \left\| \tau - \tau^0 \right\|_{L^2} \\
&\leq \frac{\mathrm{We}\,\nu}{2} \left\| \tau^0 \right\|_{L^\infty}^2 \left\| \nabla U^{(r)} \right\|_{L^2}^2 + \frac{\mathrm{We}}{2\nu} \left\| \tau - \tau^0 \right\|_{L^2}^2 \\
&\quad + \frac{\mathrm{We}^2}{2} \left\| \tau^0 \right\|_{L^\infty}^2 \left\| \nabla u^1 \right\|_{L^2}^2 + \frac{\mathrm{We}^2}{2} \left\| \tau - \tau^0 \right\|_{L^2} ,
\end{aligned}$$

and the same type of estimate holds for F. We deduce from these estimates that there exists a value of $\nu > 0$, depending (among others) on ω and $\|u_0\|_{H^4(\Omega)}$, but not on We, such that for all $0 < \mathrm{We} < 1$,

$$\begin{aligned}
&\omega\left\|U^{(r)}(t,\cdot)\right\|_{L^2}^2 + \omega(1-\omega)\int_0^t \left\|\nabla U^{(r)}\right\|_{L^2}^2 + \frac{\mathrm{We}}{2}\left\|\tau(t,\cdot)-\tau^0(t,\cdot)\right\|_{L^2}^2 + \frac{1}{2}\int_0^t \left\|\tau-\tau^0\right\|_{L^2}^2 \\
&\leq \frac{\mathrm{We}}{2}\left\|\tau_0 - 2\omega D\left(u_0\right)\right\|_{L^2}^2 \\
&\quad + C_\nu \int_0^t \left(\left\|\nabla u^0\right\|_{L^2}^4 + \mathrm{We}\left\|\nabla\tau^0\right\|_{L^\infty}^2 + \mathrm{We}\left\|\nabla u^1\right\|_{L^2}^4 + 1\right)\left(\omega\left\|U^{(r)}\right\|_{L^2}^2 + \frac{\mathrm{We}}{2}\left\|\tau-\tau^0\right\|_{L^2}\right) \\
&\quad + \frac{\omega}{\nu}\mathrm{We}^2\int_0^t \left\|u^1\right\|_{L^2}\left\|\nabla u^1\right\|_{L^2}^3 + \frac{\mathrm{We}}{2}\int_0^t \left\|\tau^1\right\|_{L^2}^2 \\
&\quad + \frac{\mathrm{We}^2}{2}\int_0^t \left\|u^1\right\|_{L^2}^2\left\|\nabla\tau^0\right\|_{L^\infty}^2 + \frac{2\omega^2}{\nu}\mathrm{We}\int_0^t \left\|\tau^1\right\|_{L^2}^2 + \frac{\mathrm{We}^2}{2}\int_0^t \left\|\tau^0\right\|_{L^\infty}^2\left\|\nabla u^1\right\|_{L^2}^2 .
\end{aligned} \tag{6.26}$$

The constant C_ν depends on the choice of ν, on ω and again on $\|u_0\|_{H^4(\Omega)}$, but is independent of We. Via Gronwall's lemma, we finally manage to control a relative entropy associated to the viscoelastic system (6.1): for all $0 \leq t \leq T < T^*$,

$$\begin{aligned}
&\omega\left\|U^{(r)}(t,\cdot)\right\|_{L^2}^2 + \omega(1-\omega)\int_0^t \left\|\nabla U^{(r)}\right\|_{L^2}^2 + \frac{\mathrm{We}}{2}\left\|\tau(t,\cdot)-\tau^0(t,\cdot)\right\|_{L^2}^2 + \frac{1}{2}\int_0^t \left\|\tau-\tau^0\right\|_{L^2}^2 \\
&\leq \mathrm{We}\Bigg[\frac{1}{2}\left\|\tau_0 - 2\omega D\left(u_0\right)\right\|_{L^2}^2 + \int_0^t \Big(\frac{\omega}{\nu}\mathrm{We}\left\|u^1\right\|_{L^2}\left\|\nabla u^1\right\|_{L^2}^3 \\
&\quad + \frac{1}{2}\left\|\tau^1\right\|_{L^2}^2 + \frac{\mathrm{We}}{2}\left\|u^1\right\|_{L^2}^2\left\|\nabla\tau^0\right\|_{L^\infty}^2 + \frac{\mathrm{We}^2}{2}\left\|\tau^0\right\|_{L^\infty}^2\left\|\nabla u^1\right\|_{L^2}^2 \\
&\quad + \frac{2\omega^2}{\nu}\left\|\tau^1\right\|_{L^2}^2\Big)\Bigg]\exp\left(t + C_\nu\int_0^t\left(\left\|\nabla u^0\right\|_{L^2}^4 + \mathrm{We}\left\|\nabla\tau^0\right\|_{L^\infty}^2 + \mathrm{We}\left\|\nabla u^1\right\|_{L^2}^4 + 1\right)\right).
\end{aligned} \tag{6.27}$$

This estimate shows the convergence statement (6.12) of Theorem 6.3 in the modulated energy norm.

6.3 Low Weissenberg limit for a FENE-P type fluid: weak convergence

In this section, we focus on the low Weissenberg limit for global weak solutions of (6.2) posed in a bounded domain $\Omega \subset \mathbb{R}^d$, in the torus $\Omega = \mathbb{T}^d$ or in $\mathbb{R}^d$.

The free energy (6.5) is the fundamental tool for the mathematical analysis of the FENE-P system. It plays a role analogous to the energy (6.3) of the corotational system. However, due to its non-trivial form, it leads to intricate computations.

Because of its importance, we recall the result of D. Hu and T. Lelièvre [HL07]

$$\begin{aligned}
&\frac{1}{2}\left\|u(t,\cdot)\right\|_{L^2(\Omega)}^2 + (1-\omega)\int_0^t \left\|\nabla u\right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega \left[-\ln\left(\det A\right) - b\ln\left(1-\frac{\mathrm{Tr}(A)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](t) \\
&\quad + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}^2}\int_0^t\int_\Omega \left[\frac{\mathrm{Tr}\,A}{\left(1-\frac{\mathrm{Tr}\,A}{b}\right)^2} - \frac{2d}{1-\frac{\mathrm{Tr}\,A}{b}} + \mathrm{Tr}\left(A^{-1}\right)\right] \\
&\leq \frac{1}{2}\left\|u_0\right\|_{L^2(\Omega)}^2 + \frac{\omega(b+d)}{2b}\frac{1}{\mathrm{We}}\int_\Omega \left[-\ln\left(\det A_0\right) - b\ln\left(1-\frac{\mathrm{Tr}(A_0)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right]
\end{aligned} \tag{6.28}$$

and give an outline of how this a priori estimate is derived. We work with regular solutions of (6.2). The free energy estimate relies on the computation of the total time derivative of the left hand side of (6.28) using the formula: for any invertible matrix $M = M(t)$ depending smoothly on t

$$\left(\partial_t + u\cdot\nabla\right)\left(\ln \det M\right) = \mathrm{Tr}\left(M^{-1}(\partial_t + u\cdot\nabla)M\right). \tag{6.29}$$

A simple energy estimate yields on the one hand,

$$\begin{aligned}
\frac{1}{2}\left\|u(t,\cdot)\right\|_{L^2} + (1-\omega)\int_0^t \left\|\nabla u(t,\cdot)\right\|_{L^2} &= -\int_0^t\int_\Omega \tau : \nabla u + \frac{1}{2}\left\|u_0\right\|_{L^2} \\
&= -\frac{1}{\mathrm{We}}\frac{(b+d)\omega}{b}\int_0^t\int_\Omega \frac{A:\nabla u}{1-\frac{\mathrm{Tr}\,A}{b}} + \frac{1}{2}\left\|u_0\right\|_{L^2}
\end{aligned} \tag{6.30}$$

On the other hand, using (6.29) and the equation on the structure tensor A, we have

$$\begin{aligned}
-(\partial_t + u\cdot\nabla)\ln\left(\det A\right) &= \frac{1}{\mathrm{We}}\frac{d}{1-\frac{\mathrm{Tr}\,A}{b}} - \frac{1}{\mathrm{We}}\mathrm{Tr}\left(A^{-1}\right), \\
-b(\partial_t + u\cdot\nabla)\ln\left(1-\frac{\mathrm{Tr}\,A}{b}\right) &= \frac{2D(u):A}{1-\frac{\mathrm{Tr}\,A}{b}} - \frac{1}{\mathrm{We}}\frac{\mathrm{Tr}\,A}{\left(1-\frac{\mathrm{Tr}\,A}{b}\right)^2} + \frac{1}{\mathrm{We}}\frac{d}{1-\frac{\mathrm{Tr}\,A}{b}},
\end{aligned}$$

which boils down to

$$\begin{aligned}
\int_\Omega \left[-\ln\left(\det A\right) - b\ln\left(1-\frac{\mathrm{Tr}\,A}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](T) &= \int_0^t\int_\Omega \frac{2D(u):A}{1-\frac{\mathrm{Tr}\,A}{b}} \\
&\quad - \frac{1}{\mathrm{We}}\int_0^t\int_\Omega \left[\frac{\mathrm{Tr}\,A}{\left(1-\frac{\mathrm{Tr}\,A}{b}\right)^2} - \frac{2d}{1-\frac{\mathrm{Tr}\,A}{b}} + \mathrm{Tr}\left(A^{-1}\right)\right] \\
&\quad + \int_\Omega \left[-\ln\left(\det A_0\right) - b\ln\left(1-\frac{\mathrm{Tr}\,A_0}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](t).
\end{aligned} \tag{6.31}$$

The estimate (6.28) is seen to hold thanks to the linear combination (6.30)$+\frac{1}{\mathrm{We}}\frac{(b+d)\omega}{2b}$(6.31).

As for the asymptotic analysis of the corotational system, we first investigate the weak convergence of u. The strong convergence is the truly tricky point, and is still in progress.

Letting We go to 0 in (6.2) yields formally that A converges toward

$$A^0 := \frac{b}{b+d}\,\mathrm{I}\,.$$

In the rest of this section we handle the proof of Proposition 6.2.

The convergence of A toward A^0 comes directly from the entropy inequality (6.28). The main observations, which lead to such a convergence result, are that the functionals

$$\mathcal{F}:\ A \longmapsto -\ln(\det A) - b\ln\left(1-\frac{\operatorname{Tr} A}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right) \tag{6.32}$$

$$\mathcal{H}:\ A \longmapsto \frac{\operatorname{Tr} A}{\left(1-\frac{\operatorname{Tr} A}{b}\right)^2} - \frac{2d}{1-\frac{\operatorname{Tr} A}{b}} + \operatorname{Tr}\left(A^{-1}\right) \tag{6.33}$$

defined for appropriate symmetric positive definite matrices A with $0 < \operatorname{Tr} A < b$,

- have a global minimum at $A = A^0 = \frac{b}{b+d}\mathrm{I}$ with value 0,
- are globally strictly convex,
- and thus yield a bound on $|A - A^0|^2$.

Hence, one can use the decay of the free energy to prove the estimates (6.8), (6.10a) and (6.10b). Notice that we also have the inequality $0 \le \mathcal{F} \le \mathcal{G}$. For the reader's convenience, we give the proof of such properties in the two-dimensional case in the Appendix 6.C.

We conclude, using these properties and the decay of the free energy that for all $t > 0$,

$$\begin{aligned}
0 &\le \int_\Omega \left|A - A^0\right|^2(t) \le C\int_\Omega \left[-\ln(\det A) - b\ln\left(1-\frac{\operatorname{Tr}(A)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right](t)\\
&\le C\,\mathrm{We}\,\|u_0\|^2_{L^2(\Omega)}\\
&\qquad + \frac{\omega(b+d)}{2b}\int_\Omega \left[-\ln(\det A_0) - b\ln\left(1-\frac{\operatorname{Tr}(A_0)}{b}\right) + (b+d)\ln\left(\frac{b}{b+d}\right)\right].
\end{aligned} \tag{6.34}$$

and that

$$\begin{aligned}
0 &\le \int_0^t\int_\Omega \left|A - A^0\right|^2\\
&\le C\int_0^t\int_\Omega \left[\frac{\operatorname{Tr} A}{\left(1-\frac{\operatorname{Tr} A}{b}\right)^2} - \frac{2d}{1-\frac{\operatorname{Tr} A}{b}} + \operatorname{Tr}\left(A^{-1}\right)\right]\\
&\le \frac{1}{2}\,\mathrm{We}^2\,\|u_0\|^2_{L^2(\Omega)}\\
&\qquad + \frac{\omega(b+2)}{2b}\,\mathrm{We}\int_\Omega \left[-\ln(\det A_0) - b\ln\left(1-\frac{\operatorname{Tr}(A_0)}{b}\right) + (b+2)\ln\left(\frac{b}{b+2}\right)\right].
\end{aligned} \tag{6.35}$$

The bounds (6.34) and (6.35) imply the estimates of Proposition 6.2.

It remains to establish the convergence of u and τ. Assume now that initial data is well-prepared, i.e. that (6.9) is satisfied. We deduce a uniform $L^2\left((0,\infty);L^2\right)$ bound on τ from the convergence of A. Indeed, let us rewrite τ in the following way:

$$\tau = \frac{(b+d)\omega}{b}\frac{1}{\mathrm{We}}\left[\frac{A}{1-\frac{\operatorname{Tr} A}{b}} - \mathrm{I}\right] = \frac{(b+d)\omega}{b}\frac{1}{\mathrm{We}}\left[\frac{A - A^0}{1-\frac{\operatorname{Tr} A^0}{b}} + A\frac{\operatorname{Tr} A - \operatorname{Tr} A^0}{b\left(1-\frac{\operatorname{Tr} A^0}{b}\right)\left(1-\frac{\operatorname{Tr} A}{b}\right)}\right]. \tag{6.36}$$

The first term in the right hand side of (6.36)

$$\frac{1}{\mathrm{We}}\frac{A - A^0}{1-\frac{\operatorname{Tr} A^0}{b}}$$

is bounded in $L^2\left((0,\infty);L^2\right)$ uniformly in We, thanks to the convergence result (6.10b) for well-prepared data. For the second term, we notice that A is bounded in $L^\infty\left((0,\infty);L^\infty\right)$

by b and that

$$\frac{(b+d)\omega}{b}\frac{1}{\mathrm{We}}\frac{\operatorname{Tr}A-\operatorname{Tr}A^0}{\left(1-\frac{\operatorname{Tr}A^0}{b}\right)\left(1-\frac{\operatorname{Tr}A}{b}\right)}=\frac{(b+d)\omega}{b}\frac{1}{\mathrm{We}}\left[\frac{\operatorname{Tr}A}{1-\frac{\operatorname{Tr}A}{b}}-\frac{\operatorname{Tr}A^0}{1-\frac{\operatorname{Tr}A^0}{b}}\right]=\operatorname{Tr}\tau.$$

This part is bounded thanks to the decay of the free energy. Indeed, using the inequality

$$\operatorname{Tr}A\ \operatorname{Tr}\left(A^{-1}\right)\geq d^2$$

valid for the positive definite matrices A, we find that the fourth term in the right hand side of (6.28) bounds the $L^2\left((0,\infty);L^2\right)$ norm of $\operatorname{Tr}\tau$:

$$\begin{aligned}\frac{\operatorname{Tr}A}{\left(1-\frac{\operatorname{Tr}A}{b}\right)^2}-\frac{2d}{1-\frac{\operatorname{Tr}A}{b}}+\operatorname{Tr}A^{-1}&\geq\frac{\operatorname{Tr}A}{\left(1-\frac{\operatorname{Tr}A}{b}\right)^2}-\frac{2d}{1-\frac{\operatorname{Tr}A}{b}}+\frac{d^2}{\operatorname{Tr}A}\\&\geq\frac{\left[\operatorname{Tr}A-d\left(1-\frac{\operatorname{Tr}A}{b}\right)\right]^2}{\operatorname{Tr}A\left(1-\frac{\operatorname{Tr}A}{b}\right)^2}\\&\geq\frac{\left[\operatorname{Tr}A-d\left(1-\frac{\operatorname{Tr}A}{b}\right)\right]^2}{b\left(1-\frac{\operatorname{Tr}A}{b}\right)^2}\\&=\frac{1}{b}\left[\frac{\operatorname{Tr}A}{1-\frac{\operatorname{Tr}A}{b}}-d\right]^2\geq\frac{b}{(b+d)^2\omega^2}\mathrm{We}^2\left(\operatorname{Tr}\tau\right)^2.\end{aligned}\tag{6.37}$$

It remains to see the convergence of τ and u. Assume that $d=2,\ 3$. From (6.28), it comes that u is uniformly bounded in We in $L^\infty\left((0,\infty);L^2\right)$ and $L^2\big((0,\infty);\dot{H}^1\big)$. Reasoning in the same manner as in Section 6.2.1, we deduce from the uniform bound on τ in $L^2\left((0,\infty)\times\Omega\right)$ the existence of

$$u^0\in L^\infty\Big((0,\infty);L^{2,\sigma}\Big)\cap L^2\Big((0,\infty);\dot{H}^1\Big),$$

such that u converges to u^0 in a fashion similar to (6.13). To see the weak convergence of τ, the idea is to use the equation on A in the system (6.2):

$$\tau=-\frac{(b+d)\omega}{b}\left[\partial_tA+\nabla\cdot(uA)-\nabla uA-A\left(\nabla u\right)^T\right].\tag{6.38}$$

The right hand side of (6.38) converges in the sense of distributions toward

$$-\frac{(b+d)\omega}{b}\left[\partial_tA^0+u^0\cdot\nabla A^0-2\frac{b}{b+d}D\left(u^0\right)\right]=2\omega D\left(u^0\right).$$

Finally, we can pass to the weak limit in the equation for u^0, and get that u^0 solves the Navier-Stokes system (6.6).

Remark 6.5. Notice that if one further assumes (6.9) with $o\,(\mathrm{We})$ instead of $O\,(\mathrm{We})$, we can prove that u^0 satisfies the energy estimate associated to the Navier-Stokes system.

6.A A remark on the newtonian limit for oscillating initial data

This appendix is devoted to the proof of Proposition 6.1 bis. Recall that we consider here only the domain $\Omega=\mathbb{R}^2$. We assume that the initial data $\tau_{0,n}$ is strongly oscillating in n (not in We). In our setting (see below), these oscillations introduce defect measures at the limit $n\to\infty$. In order to handle these defect measures, we need refined a priori bounds on the solutions of (6.1). Our analysis is close to the paper [LM00] by P.-L. Lions and N. Masmoudi, where the existence of weak solutions is proved.

6.A.1 Further a priori bounds for the corotational system

We complement the bound (6.3) by carrying out $L^p\left((0,T),L^q\right)$ estimates on the system (6.1). In particular, these bounds are crucial for the Cauchy theory of weak solutions as developed in [LM00].

The proof of such estimates is divided into two parts. First, assuming an $L^p\left((0,T),L^q\right)$ estimate on τ, we prove a control on the velocity using the momentum equation. Then, using the equation on τ, we use a Gronwall lemma to show an $L^p\left((0,T),L^q\right)$ estimate on τ taking advantage of the estimate of the velocity in terms of τ.

An $L^p\left((0,T);L^q\right)$ control of the velocity field u through an $L^p\left((0,T);L^q\right)$ estimate on τ

In order to estimate ∇u in $L^p\left((0,T);L^q\right)$, we decompose u in a sum $u=u^1+u^2+u^3$, where u^1 and u^2 are the unique solutions to the Stokes system

$$\left\{\begin{aligned} \partial_t u^i - \nu\Delta u^i + \nabla p^i &= f^i \\ \nabla\cdot u^i &= 0 \\ u^i(0,\cdot) &= 0 \end{aligned}\right. , \tag{6.39}$$

with $f^1 := -u\cdot\nabla u$, $f^2 := \nabla\cdot\tau$, and where u^3 is the unique solution to

$$\left\{\begin{aligned} \partial_t u^3 - \nu\Delta u^3 + \nabla p^3 &= 0 \\ \nabla\cdot u^3 &= 0 \\ u^3(0,\cdot) &= u_0 \end{aligned}\right. . \tag{6.40}$$

As u_0 belongs to $I_{p,q}$ for $4\le q<\infty$ and $1<p\le\frac{q}{q-1}$,

$$\left\|\nabla u^3\right\|_{L^p((0,T);L^q)} \le C\left\|u_0\right\|_{I_{p,q}}. \tag{6.41}$$

The second term u^2 can be estimated by applying the nonstationary version of Cattabriga's estimate given in [GGS93] (see corollary 4.2 and estimate (1.6)):

$$\left\|\nabla u^2\right\|_{L^p((0,T);L^q)} \le C\left\|\tau\right\|_{L^p((0,T);L^q)}. \tag{6.42}$$

From (6.3), we conclude that $u\cdot\nabla u$ is bounded in $L^r\left((0,T);L^s\right)$ for all $1\le r\le 2$ and $s=\frac{2r}{3r-2}$. Hence, applying Theorem 2.8 from [GS91] and Sobolev's injection theorem, we get

$$\left\|\nabla u^1\right\|_{L^r\left((0,T);L^{s^\star}\right)} \le C\left\|\nabla^2 u^1\right\|_{L^r((0,T);L^s)} \le C\left\|u\cdot\nabla u\right\|_{L^r((0,T);L^s)} \le C\left[r,s,\left\|u_0\right\|_{L^2},\left\|\tau_0\right\|_{L^2}\right]$$

with $1<r\le 2$ et $s^\star=\frac{2s}{2-s}=\frac{r}{r-1}$, so that

$$\left\|\nabla u^1\right\|_{L^{\frac{q}{q-1}}((0,T);L^q)} \le C\left[\left\|u_0\right\|_{L^2},\left\|\tau_0\right\|_{L^2}\right]. \tag{6.43}$$

From (6.41), (6.42) and (6.43) we conclude that

$$\left\|\nabla u\right\|_{L^{\frac{q}{q-1}}((0,T);L^q)} \le C\left\|\tau\right\|_{L^{\frac{q}{q-1}}((0,T);L^q)} + C\left[\left\|u_0\right\|_{L^2},\left\|u_0\right\|_{I_{p,q}},\left\|\tau_0\right\|_{L^2}\right], \tag{6.44}$$

with constants uniform in T and We.

Extra $L^p((0,T);L^q)$ estimates on τ through a Gronwall estimate

Let us now estimate $\|\tau\|_{L^p((0,T);L^q)}$, for any $0 \leq T$,

$$1 \leq p \leq \frac{q}{q-1} \qquad \text{and} \qquad 3 < q < \infty. \tag{6.45}$$

Testing the equation on τ against $\tau|\tau|^{q-2}$ gives

$$\frac{1}{q}\partial_t \|\tau\|_{L^q}^q + \int_{\mathbb{R}^2} (u\cdot\nabla\tau) : \tau|\tau|^{q-2} + \int_{\mathbb{R}^2} \tau W(u)\cdot\tau|\tau|^{q-2} - \int_{\mathbb{R}^2} W(u)\tau : \tau|\tau|^{q-2} + \frac{1}{\mathrm{We}}\|\tau\|^q = \frac{2\omega}{\mathrm{We}}\int_{\mathbb{R}^2} D(u) : \tau|\tau|^{q-2}.$$

Yet, it follows from the incompressibility of the velocity field

$$\begin{aligned}\int_{\mathbb{R}^2} (u\cdot\nabla\tau) : \tau|\tau|^{q-2} &= \int_{\mathbb{R}^2} u_\alpha\left(\partial_\alpha\tau_{\beta\gamma}\right)\tau_{\beta\gamma}|\tau|^{q-2} \\ &= \int_{\mathbb{R}^2} u_\alpha \frac{1}{q}\partial_\alpha\left(|\tau|^q\right) = \int_{\mathbb{R}^2} \partial_\alpha\left(u_\alpha\frac{|\tau|^q}{q}\right) = 0,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} \tau W(u) : \tau|\tau|^{q-2} &= \int_{\mathbb{R}^2} \tau_{\alpha\gamma}W(u)_{\gamma\beta}\tau_{\alpha\beta}|\tau|^{q-2} \\ &= \frac{1}{2}\left[\int_{\mathbb{R}^2} \tau_{\alpha\gamma}\partial_\gamma u_\beta\tau_{\alpha\beta}|\tau|^{q-2} - \int_{\mathbb{R}^2} \tau_{\alpha\gamma}\partial_\gamma u_\beta\tau_{\alpha\beta}|\tau|^{q-2}\right] = 0.\end{aligned}$$

Therefore,

$$\frac{1}{q}\partial_t\|\tau\|_{L^q}^q + \frac{1}{\mathrm{We}}\|\tau\|_{L^q}^q = \frac{2\omega}{\mathrm{We}}\int_{\mathbb{R}^2} D(u) : \tau|\tau|^{q-2} \leq \frac{2\omega}{\mathrm{We}}\|\nabla u\|_{L^q}\|\tau\|_{L^q}^{q-1}. \tag{6.46}$$

Gronwall Lemma related to τ

Letting $Z := \|\tau\|_{L^q}^q$, it follows from (6.46),

$$\frac{1}{q}\frac{\partial_t Z}{Z^{\frac{q-1}{q}}} + \frac{1}{\mathrm{We}}Z^{\frac{1}{q}} \leq \frac{2\omega}{\mathrm{We}}\|\nabla u\|_{L^q},$$

which rewrites

$$\partial_t\left(Z^{\frac{1}{q}}\right) + \frac{1}{\mathrm{We}}Z^{\frac{1}{q}} \leq \frac{2\omega}{\mathrm{We}}\|\nabla u\|_{L^q}.$$

A Gronwall-type argument yields for all $t > 0$,

$$\|\tau\|_{L^q}(t) = Z^{\frac{1}{q}}(t) \leq \|\tau\|_{L^q}(0)\exp\left(-\frac{t}{\mathrm{We}}\right) + \frac{2\omega}{\mathrm{We}}\int_0^t \|\nabla u\|_{L^q}(s)\exp\left(\frac{s-t}{\mathrm{We}}\right)ds. \tag{6.47}$$

On the one hand, for all $0 < t \leq T$,

$$\begin{aligned}&\left\|\int_0^t \|\nabla u\|_{L^q}(s)\exp\left(\frac{s-t}{\mathrm{We}}\right)ds\right\|_{L^\infty(0,T)} \leq C\,\mathrm{We}^{\frac{1}{q}}\|\nabla u\|_{L^{\frac{q}{q-1}}((0,T);L^q)} \\ &\leq C\,\mathrm{We}^{\frac{1}{q}}\|\tau\|_{L^{\frac{q}{q-1}}((0,T);L^q)} + \mathrm{We}^{\frac{1}{q}}C\left[\|u_0\|_{L^2},\|u_0\|_{I_{p,q}},\|\tau_0\|_{L^2}\right] \\ &\leq C\,\mathrm{We}^{\frac{1}{q}}T^{\frac{q-1}{q}}\|\tau\|_{L^\infty((0,T);L^q)} + \mathrm{We}^{\frac{1}{q}}C\left[\|u_0\|_{L^2},\|u_0\|_{I_{p,q}},\|\tau_0\|_{L^2}\right]\end{aligned}$$

which implies for T_1 sufficiently small

$$\|\tau\|_{L^\infty((0,T_1);L^q)} \le \frac{1}{1-2\omega\left(\frac{T_1}{\mathrm{We}}\right)^{\frac{q}{q-1}}}\|\tau\|_{L^q}(0) + \frac{C\left[\|u_0\|_{L^2}, \|u_0\|_{I_{p,q}}, \|\tau_0\|_{L^2}\right]}{\mathrm{We}^{\frac{q-1}{q}}\left(1-2\omega\left(\frac{T_1}{\mathrm{We}}\right)^{\frac{q}{q-1}}\right)}. \tag{6.48}$$

The latter yields a local in time a priori estimate on τ and then on ∇u using the first part: for all $T>0$,

$$\|\tau\|_{L^\infty((0,T);L^q(\mathbb{R}^2))} \le C\left[T, \mathrm{We}, \|u_0\|_{L^2}, \|u_0\|_{I_{p,q}}, \|\tau_0\|_{L^2}, \|\tau_0\|_{L^q}\right] \tag{6.49}$$

and

$$\|\nabla u\|_{L^{\frac{q}{q-1}}((0,T);L^q(\mathbb{R}^2))} \le C\left[T, \mathrm{We}, \|u_0\|_{L^2}, \|u_0\|_{I_{p,q}}, \|\tau_0\|_{L^2}, \|\tau_0\|_{L^q}\right]. \tag{6.50}$$

The explicit computation of the constants appearing in (6.49) and (6.50) is tedious. Note that they blow up exponentially fast when $T\to\infty$ or $\mathrm{We}\to 0$.

On the other hand,

$$\begin{aligned}\left\|\int_0^t \|\nabla u\|_{L^q}(s)\exp\left(\frac{s-t}{\mathrm{We}}\right)ds\right\|_{L^1(0,T)} &\le \int_0^T \|\nabla u\|_{L^q}(s)\int_s^T \exp\left(\frac{s-T}{\mathrm{We}}\right)dtds\\ &\le \mathrm{We}\,\|\nabla u\|_{L^1((0,T);L^q)}\end{aligned}$$

and

$$\left\|\int_0^t \|\nabla u\|_{L^q}(s)\exp\left(\frac{s-t}{\mathrm{We}}\right)ds\right\|_{L^\infty(0,T)} \le \mathrm{We}\,\|\nabla u\|_{L^\infty((0,T);L^q)},$$

from which we get by interpolation

$$\left\|\int_0^t \|\nabla u\|_{L^q}(s)\exp\left(\frac{s-t}{\mathrm{We}}\right)ds\right\|_{L^{\frac{q}{q-1}}(0,T)} \le \mathrm{We}\,\|\nabla u\|_{L^{\frac{q}{q-1}}((0,T);L^q)}.$$

Combining this last inequality with (6.47) yields

$$\|\tau\|_{L^{\frac{q}{q-1}}((0,T);L^q)} \le \mathrm{We}^{\frac{q-1}{q}}\|\tau\|_{L^q}(0) + 2\omega\|\nabla u\|_{L^{\frac{q}{q-1}}((0,T);L^q)},$$

and for ω sufficiently small, thanks to (6.44), a bound

$$\|\tau\|_{L^{\frac{q}{q-1}}((0,\infty);L^q)} \le C\left[\|u_0\|_{L^2}, \|u_0\|_{I_{p,q}}, \|\tau_0\|_{L^2}, \|\tau_0\|_{L^q}\right] \tag{6.51}$$

uniform in time and in We.

Remark 6.6. The singularity in $1/\mathrm{We}^{\frac{q-1}{q}}$ in the estimate (6.48) prevents us from obtaining a bound uniform in We in $L^\infty_{loc}((0,\infty);L^q)$. However, for fixed We, we can use this bound. This explains why this bound is usefull for the proof of weak solutions, and not for the low Weissenberg asymptotic analysis.

6.A.2 Newtonian limit with defect measures in the initial data

Let (u_n,τ_n) be a sequence of weak solutions to (6.1) associated to the initial conditions

$$u_n(0,\cdot) := u_{0,n}(\cdot), \qquad \tau_n(0,\cdot) := \tau_{0,n}(\cdot).$$

We assume that:

- The sequence (u_n, τ_n) satisfies (6.3) and the a priori bounds (6.49) and (6.50). Note that the energy bound (6.3) bounds u_n (resp. τ_n) uniformly in n and We in the spaces $L^2\big((0,\infty);\dot{H}^1\big)$, $L^\infty\left((0,\infty);L^2\right)$ (resp. $L^2\left((0,\infty);L^2\right)$).
- $u_{0,n}$ converges strongly in L^2 toward u_0.
- $\tau_{0,n}$ is equicontinuous in L^2, i.e. that

$$\sup_n \int_{|\tau_{0,n}|\geq M} |\tau_{0,n}|^2 \overset{M\to\infty}{\longrightarrow} 0. \tag{6.52}$$

In particular, we do not assume that $\tau_{0,n}$ converges strongly in L^2, which allows the presence of defect measures initially.

Defect measures

We begin our analysis by making a change of unknown, underlining some special features of (6.1) when $d = 2$. Following [CS12, equation (251)], let us introduce the new unknowns

$$a_n := \tau_{n,11} - \tau_{n,22}, \qquad b_n := \tau_{n,12}, \qquad c_n := \tau_{n,11} + \tau_{n,22}.$$

We compute (dropping for the moment the subscripts n)

$$\tau W(u) - W(u)\tau = \begin{pmatrix} \tau_{12}\left(\partial_2 u_1 - \partial_1 u_2\right) & \frac{1}{2}\left(\tau_{11}-\tau_{22}\right)\left(\partial_1 u_2 - \partial_2 u_1\right) \\ \frac{1}{2}\left(\tau_{11}-\tau_{22}\right)\left(\partial_1 u_2 - \partial_2 u_1\right) & \tau_{12}\left(\partial_1 u_2 - \partial_2 u_1\right) \end{pmatrix}.$$

Hence, the transport equation on τ_n becomes

$$\begin{cases} \partial_t a_n + u_n\cdot\nabla a_n - 2b_n \operatorname{curl} u_n + \frac{a_n}{\mathrm{We}} &= \frac{2\omega}{\mathrm{We}}\left(\partial_1 u_{n,1} - \partial_2 u_{n,2}\right) \\ \partial_t b_n + u_n\cdot\nabla b_n + \frac{1}{2}a_n \operatorname{curl} u_n + \frac{b_n}{\mathrm{We}} &= \frac{\omega}{\mathrm{We}}\left(\partial_1 u_{n,2} + \partial_2 u_{n,1}\right) \\ \partial_t c_n + u_n\cdot\nabla c_n + \frac{c_n}{\mathrm{We}} &= 0 \end{cases} \tag{6.53}$$

where $\operatorname{curl} u_n := \partial_1 u_{n,2} - \partial_2 u_{n,1}$. In particular c_n is decoupled from a_n and b_n. Of course, the sequences (a_n), (b_n) and (c_n) inherit from the properties of (τ_n), and (a_n) (resp. (b_n), (c_n)) satisfy the a priori bounds (6.3) and (6.49).

The presence of defect measures in the initial data means that

$$|a_{0,n}|^2 \overset{*}{\rightharpoonup} |a_0|^2 + \alpha_0 \quad L^\infty\big((0,T);L^{\frac{q}{2}}\big), \tag{6.54a}$$

$$|b_{0,n}|^2 \overset{*}{\rightharpoonup} |b_0|^2 + \beta_0 \quad L^\infty\big((0,T);L^{\frac{q}{2}}\big). \tag{6.54b}$$

A way to quantify the possible loss of convergence in products of weakly converging sequences is to introduce defect measures. As

$$\begin{array}{rcl} |a_n|^2 & \text{is uniformly bounded in } n \text{ in (u.b.)} & L^\infty\big((0,T);L^{\frac{q}{2}}\big), \\ |b_n|^2 & \text{u.b. in} & L^\infty\big((0,T);L^{\frac{q}{2}}\big), \\ a_n \operatorname{curl} u_n & \text{u.b. in} & L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big), \\ b_n \operatorname{curl} u_n & \text{u.b. in} & L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big), \\ \left(\partial_1 u_{1,n} - \partial_2 u_{2,n}\right) a_n & \text{u.b. in} & L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big), \\ \left(\partial_1 u_{2,n} + \partial_2 u_{1,n}\right) b_n & \text{u.b. in} & L^2\big((0,T);L^{\frac{q}{2}}\big), \\ |\nabla u_n|^2 & \text{u.b. in} & L^1\left((0,\infty);L^1\right), \end{array}$$

there exists $\alpha \in L^\infty_{loc}\big((0,\infty);L^{\frac{q}{2}}\big)$ (resp. $\beta \in L^\infty_{loc}\big((0,\infty);L^{\frac{q}{2}}\big)$, $\delta, \varepsilon, \eta, \lambda \in L^{\frac{q}{q-1}}_{loc}\big((0,\infty);L^{\frac{q}{2}}\big)$, $\mu \in [L^\infty((0,\infty);L^\infty)]'$) such that for all $T>0$,

$$\begin{aligned}
|a_n|^2 &\overset{*}{\rightharpoonup} |a|^2+\alpha && L^\infty\big((0,T);L^{\frac{q}{2}}\big),\\
|b_n|^2 &\overset{*}{\rightharpoonup} |b|^2+\beta && L^\infty\big((0,T);L^{\frac{q}{2}}\big),\\
a_n \operatorname{curl} u_n &\rightharpoonup a \operatorname{curl} u+\delta && L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big),\\
b_n \operatorname{curl} u_n &\rightharpoonup b \operatorname{curl} u+\varepsilon && L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big),\\
(\partial_1 u_{1,n}-\partial_2 u_{2,n})\, a_n &\rightharpoonup (\partial_1 u^1-\partial_2 u^2)\, a+\eta && L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big),\\
(\partial_1 u_{2,n}+\partial_2 u_{1,n})\, b_n &\rightharpoonup (\partial_1 u^2+\partial_2 u^1)\, b+\lambda && L^{\frac{q}{q-1}}\big((0,T);L^{\frac{q}{2}}\big),\\
|\nabla u_n|^2 &\overset{*}{\rightharpoonup} |\nabla u|^2+\mu && [L^\infty((0,\infty);L^\infty)]'.
\end{aligned}$$

Let us state a couple of straightforward properties on the defect measures:
- The measures $\alpha,\ \beta,\ \mu$ are positive.
- For every bounded measurable set $E\subset(0,\infty)\times\mathbb{R}^2$,

$$\begin{aligned}
|\eta(E)| &\le \sqrt{\mu(E)}\sqrt{\alpha(E)}, \quad |\lambda(E)| \le \sqrt{\mu(E)}\sqrt{\beta(E)},\\
|\delta(E)| &\le \sqrt{\mu(E)}\sqrt{\alpha(E)}, \quad |\varepsilon(E)| \le \sqrt{\mu(E)}\sqrt{\beta(E)}.
\end{aligned} \tag{6.55}$$

Moreover, testing the momentum equation against u_n then passing to the limit, and passing to the limit in the momentum equation then testing against u yields an equality between the terms appearing in the averaging process:

$$2(1-\omega)\mu+\eta+2\lambda=0. \tag{6.56}$$

Such an inequality implies in particular $\mu \in L^2_{loc}\big((0,\infty);L^{\frac{2q}{q+2}}\big)$.

Weak convergence analysis

Our purpose is to pass to the limit on n, then on We. There are three steps:

1. We show the convergence of u_n (resp. τ_n) toward u (resp. τ), and pass to the weak limit $n\to\infty$ in the system (6.1).
2. We pass to the limit $n\to\infty$ in (6.3), in order to get a priori bounds on u and τ uniform in We.
3. We study the limit $\mathrm{We}\to 0$, as in the Section 6.2.1.

We will have recourse to the equicontinuity of τ_n in $L^\infty_{loc}((0,\infty);L^2)$ showed by P.-L. Lions and N. Masmoudi in [LM00].

First step Classical arguments yield the existence of

$$u\in L^\infty\big((0,\infty);L^{2,\sigma}\big)\cap L^2\big((0,\infty);\dot H^1\big) \quad\text{and}\quad \tau\in L^\infty\big((0,\infty);L^2\big)\cap L^\infty_{loc}((0,\infty);L^q),$$

satisfying the energy inequality (6.3) with a right hand side slightly modified to account for the fact that $\tau_{0,n}$ does not converge to τ_0 in L^2. Passing to the limit $n\to\infty$ in the system (6.1) for (u_n,τ_n), we get that (u,τ) solves

$$\begin{cases}
\partial_t u+u\cdot\nabla u-(1-\omega)\Delta u+\nabla p &= \nabla\cdot\tau\\
\nabla\cdot u &= 0\\
\partial_t\tau+u\cdot\nabla\tau+\tau W(u)-W(u)\tau+\begin{pmatrix} -\varepsilon & \frac{1}{2}\delta\\ -\frac{1}{2}\delta & \varepsilon\end{pmatrix}+\frac{1}{\mathrm{We}}\tau &= \frac{2\omega}{\mathrm{We}}D(u)
\end{cases} \tag{6.57}$$

in the sense of distributions. We focus now on the low Weissenberg limit in (6.57).

Second step We intend to let $n \to \infty$ in the energy estimate

$$\omega \|u_n\|_{L^2}^2 + 2\omega(1-\omega)\int_0^T \|\nabla u_n\|_{L^2}^2 + \frac{\mathrm{We}}{2}\|\tau_n\|_{L^2}^2 + \int_0^T \|\tau_n\|_{L^2}^2 = \omega \|u_{0,n}\|_{L^2}^2 + \frac{\mathrm{We}}{2}\|\tau_{0,n}\|_{L^2}^2 \tag{6.58}$$

so as to retrieve bounds on the defect measures δ and ε uniform in We. Assuming no defect measures initially (i.e. $\alpha_0 = \beta_0 = 0$), we already know that (6.3) is satisfied at the limit. Nevertheless, in the presence of defect measures, using the inequalities

$$\begin{aligned}
\|u\|_{L^\infty((0,\infty);L^2)} &\le \liminf_n \|u_n\|_{L^\infty((0,\infty);L^2)}\,,\\
\|\nabla u\|_{L^2((0,\infty);L^2)} &\le \liminf_n \|\nabla u_n\|_{L^2((0,\infty);L^2)}\,,\\
\|\tau\|_{L^\infty((0,\infty);L^2)} &\le \liminf_n \|\tau_n\|_{L^\infty((0,\infty);L^2)}\,,\\
\|\tau\|_{L^2((0,\infty);L^2)} &\le \liminf_n \|\tau_n\|_{L^2((0,\infty);L^2)}\,,
\end{aligned}$$

is responsible for the loss of a lot of information. In particular, we lose all information on the defect measures propagation.

The heart of the matter is to justify the following formal limit $n \to \infty$ in (6.58)

$$\begin{aligned}
\omega \|u\|_{L^2}^2 &+ 2\omega(1-\omega)\int_0^T \|\nabla u\|_{L^2}^2 + 2\omega(1-\omega)\int_0^T\int_{\mathbb{R}^2}\mu\\
&+ \frac{\mathrm{We}}{2}\|\tau\|_{L^2}^2 + \frac{\mathrm{We}}{4}\int_{\mathbb{R}^2}(\alpha+4\beta) + \int_0^T\|\tau\|_{L^2}^2 + \frac{1}{2}\int_0^T\int_{\mathbb{R}^2}(\alpha+4\beta)\\
&= \omega\|u_0\|_{L^2}^2 + \frac{\mathrm{We}}{2}\|\tau_0\|_{L^2}^2 + \frac{\mathrm{We}}{4}\int_{\mathbb{R}^2}(\alpha_0+4\beta_0)\,.
\end{aligned}\tag{6.59}$$

If the latter holds uniformly with respect to We, then we get uniform bounds on μ, α and β in $L^1\left((0,\infty);L^1\right)$, and by (6.55)

$$|\eta| \le \frac{1}{2}(\mu+\alpha)\,, \qquad |\delta| \le \frac{1}{2}(\mu+\alpha)\,,$$

we bound η and δ uniformly in We in $L^1\left((0,\infty);L^1\right)$. The same holds of course for λ and ε as well. In order to show (6.59), we need to prove that μ, α and β belong to $L^1_{loc}\left((0,\infty);L^1\right)$. According to the results above,

$$\alpha,\ \beta \in L^\infty\left((0,\infty), L^{\frac{q}{2}}\right) \quad \text{and} \quad \mu \in L^2_{loc}\left((0,\infty), L^{\frac{q}{2}}\right) \quad \text{(see (6.56))}.$$

However, these bounds are not uniform in We, and do not imply a $L^1_{loc}\left((0,\infty);L^1\right)$ bound. Here a stronger result is needed on the sequence τ_n:

Result C (P.-L. Lions and N. Masmoudi). *We assume (6.52), i.e. the equicontinuity of $\tau_{0,n}$ in L^2. Then, τ_n is equicontinuous in $L^\infty_{loc}\left((0,\infty);L^2\right)$, i.e. for all $T>0$,*

$$\sup_{t\in(0,T)}\sup_n \int_{|\tau_n|\ge M} |\tau_n|^2 \overset{M\to\infty}{\longrightarrow} 0. \tag{6.60}$$

We refer to [LM00] Section III.3 for details concerning the proof. Let us point out the main idea. It is to consider τ_n as a solution of the linear system

$$\begin{cases} \partial_t\tau_n + u\cdot\nabla\tau_n + \tau_n W(u) - W(u)\tau_n + \frac{1}{\mathrm{We}}\tau_n &= \frac{2\omega D(u)}{\mathrm{We}}\\ \tau_n(0,\cdot) &= \tau_{0,n}\end{cases}$$

with u fixed, which yields an affine mapping $K_u : \tau_{0,n} \longmapsto \tau_n$ depending on u. Yet, K_u satisfies estimates independent of u. For R an auxilliary parameter, one then decomposes the initial data into

$$\tau_{0,n} = \tau_{0,n} 1_{|\tau_{0,n}|<R} + \tau_{0,n} 1_{|\tau_{0,n}|\geq R}$$

and bounds

$$\int_{|\tau_n|\geq M} |\tau_n|^2 \leq \int_{|\tau_n|\geq M} \left| K_u \left(\tau_{0,n} 1_{|\tau_{0,n}|<R}\right)\right|^2 + \int_{\mathbb{R}^2} \left| K_u \left(\tau_{0,n} 1_{|\tau_{0,n}|\geq R}\right)\right|^2 .$$

The second integral is made small for R large thanks to (6.52). The first is small in the limit $M \to \infty$. Note that up to this point, we do not take care on the dependence on We.

We deduce from (6.60), that

$$\begin{gathered} a_n \operatorname{curl} u_n - a \operatorname{curl} u, \quad b_n \operatorname{curl} u_n - b \operatorname{curl} u, \\ \left(\partial_1 u_{1,n} - \partial_2 u_{2,n}\right) a_n - \left(\partial_1 u_1 - \partial_2 u_2\right) a, \quad \left(\partial_1 u_{2,n} + \partial_2 u_{1,n}\right) b_n - \left(\partial_1 u_2 + \partial_2 u_1\right) b \end{gathered}$$

are equiintegrable in $L^1_{loc}\left((0,\infty); L^1\right)$. Therefore, they converge weakly in $L^1_{loc}\left((0,\infty); L^1\right)$, and their weak limits δ, ε, η and λ belong to $L^1_{loc}\left((0,\infty); L^1\right)$. The defect measure μ is in $L^1_{loc}\left((0,\infty); L^1\right)$ because of the equality (6.56).

Final step We proceed exactly as in the Section 6.2.1: from (6.58) we get the existence of

$$u^0 \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\big((0,\infty); \dot{H}^1\big) \quad \text{and} \quad \tau^0 \in L^2\left((0,\infty); L^2\right),$$

such that convergences analogous to (6.13) hold. Passing to the limit We $\to 0$ in (6.57) leads to the fact that u^0 satisfies the Navier-Stokes system (6.6) in the weak sense.

Remark 6.7. In order to pass to the limit in the energy equality (6.59), we assume moreover that $\sqrt{\mathrm{We}}\,\tau_0$ tends to zero in L^2 and that $\mathrm{We}(\alpha_0 + 4\beta_0)$ tends to zero in L^1. Thus passing to the limit, using the sign of the defect measures and the obtained weak convergences, we recover the usual energy estimate for the Navier-Stokes system.

6.B Existence of weak solutions to the corotational system

In this section we give some details about the proof, originally developped by P.-L. Lions and N. Masmoudi in [LM00], of the existence of global weak solutions to the corotational model (6.1). We hope this section could be of interest for the reader who is interested in understanding the tools introduced by P.-L. Lions and N. Masmoudi. The proof of global existence to the FENE-P system is in some sense in the same spirit even if it is a little bit more tricky due to a more complex entropy a priori estimate, extra nonlinearities and a lack of equiintegrability.

Througout this appendix, we restrict ourselves to the two dimensional setting $d = 2$. Our purpose is:

- To write the proof of the global existence result in a simpler situation, avoiding the technicalities of the dimension 3.
- To underline some special features of the two-dimensional situation.
- Furthermore, as our main concern in this paper is to study the asymptotics of the model when We $\to 0$, we take special care of keeping track of the dependence on We in the estimates.

We address the case of $\Omega = \mathbb{R}^2$. The analysis extends to $\Omega = \mathbb{T}^2$, and with small changes, when it comes to estimate the pressure, to a bounded domain Ω. We consider initial velocity fields in the space $I_{p,q} \subset W^{-1,q}$ defined by (6.4). We emphasize two points in the proof:

1. the construction of approximated solutions (u_n, p_n, τ_n) to (6.1),
2. the compactness of weak solutions to (6.1) satisfying the a priori bounds.

The key point is the second. We skip the step which consists in passing to the limit in the equations for (u_n, p_n, τ_n). Indeed the truly tricky point in the latter, which is to pass to the limit in the product $\tau_n W(u_n)$, rests on the compactness analysis.

6.B.1 Approximate solutions

Our construction of approximate solutions by the mean of a truncation in Fourier space is very classical. We explain it for the sake of completeness. The second part in this section deals with the convergence properties of the sequence of approximate solutions.

Construction

For all $n \in \mathbb{N}$, we define the multipliers in Fourier space $\mathbb{P}$ (Leray projector) and J_n (truncation in low frequencies):

$$\mathbb{P} := \mathcal{F}^{-1}\left[\left(\mathrm{I}_2 - \left(\frac{\xi_i\xi_j}{|\xi|^2}\right)\right)\cdot\right], \quad J_n := \mathcal{F}^{-1}\left[1_{B(0,n)}\cdot\right],$$

i.e. for all $u \in L^2(\mathbb{R}^2)$,

$$\widehat{\mathbb{P}(u)} = \left(\mathrm{I}_d - \left(\frac{\xi_i\xi_j}{|\xi|^2}\right)\right)\hat{u} \quad \text{and} \quad \widehat{J_n(u)} = 1_{B(0,n)}\hat{u}.$$

Let $n \in \mathbb{N}$ be fixed, $n \geq 1$. The spaces $L_n^{2,\sigma}(\mathbb{R}^2)$ and $L_n^2(\mathbb{R}^2)$ are defined by

$$\begin{aligned} L_n^{2,\sigma}\big(\mathbb{R}^2\big) &:= \left\{u \in L^2\big(\mathbb{R}^2;\mathbb{R}^2\big),\ u = J_n u, \nabla\cdot u = 0\right\} \subset C^\infty\big(\mathbb{R}^2\big), \\ L_n^2\big(\mathbb{R}^2\big) &:= \left\{\tau \in L^2\big(\mathbb{R}^2;\mathbb{R}^{2\times 2}_{sym}\big),\ \tau = J_n\tau\right\} \subset C^\infty\big(\mathbb{R}^2\big). \end{aligned}$$

We consider $u_n = u_n(t,x) \in \mathbb{R}^2$ and $\tau_n = \tau_n(t,x) \in M_2(\mathbb{R})$ solving the approximated system

$$\left\{\begin{aligned} \partial_t u_n + J_n\mathbb{P}\left(u_n\cdot\nabla u_n\right) - (1-\omega)\Delta u_n &= \mathbb{P}\left(\nabla\cdot\tau_n\right) \\ J_n\mathbb{P}u_n &= u_n \\ \mathrm{We}\left(\partial_t\tau_n + J_n\left(u_n\cdot\nabla\tau_n\right) + J_n\left(\tau_n W(u_n)\right) - J_n\left(W(u_n)\tau_n\right)\right) + \tau_n &= 2\omega D(u_n) \\ J_n\tau_n &= \tau_n \end{aligned}\right. . \tag{6.61}$$

The Cauchy problem

$$\partial_t\begin{pmatrix} u_n \\ \tau_n \end{pmatrix} = F_n\begin{pmatrix} u_n \\ \tau_n \end{pmatrix}, \qquad \begin{pmatrix} u_n \\ \tau_n \end{pmatrix}(0,\cdot) = \begin{pmatrix} u_{0,n} \\ \tau_{0,n} \end{pmatrix} = \begin{pmatrix} J_n u_0 \\ J_n\tau_0 \end{pmatrix} \in L_n^{2,\sigma}\times L_n^2,$$

where

$$\begin{aligned} F_n \ :\ L_n^{2,\sigma}\times L_n^2 &\longrightarrow L_n^{2,\sigma}\times L_n^2 \\ (u,\tau) &\longmapsto \begin{pmatrix} -J_n\mathbb{P}\left(u\cdot\nabla u\right) + (1-\omega)\Delta u + \mathbb{P}\left(\nabla\cdot\tau\right) \\ -J_n\left(u\cdot\nabla\tau\right) - J_n\left(\tau W(u)\right) + J_n\left(W(u)\tau\right) - \frac{\tau}{\mathcal{W}} + \frac{2\omega D(u)}{\mathcal{W}} \end{pmatrix} \end{aligned}$$

has a unique solution $(u_n, \tau_n) \in C^1\Big([0,T], L^{2,\sigma}_n\Big) \times C^1\left([0,T], L^2_n\right)$ locally in time. Indeed, the Cauchy-Lipschitz theorem applies as F is C^1, thus locally Lipschitz: Bernstein's inequality (see [BCD11] Lemma 2.1) yields, for all $u,\ v \in L^{2,\sigma}_n$,

$$\|J_n\mathbb{P}\left(u\cdot\nabla v\right)\|_{L^2(\mathbb{R}^2)} \le \|u\cdot\nabla v\|_{L^2(\mathbb{R}^2)} \le \|u\|_{L^4(\mathbb{R}^2)}\|\nabla v\|_{L^4(\mathbb{R}^2)} \le Cn^{\frac{3}{2}}\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}.$$

Therefore $(u,v)\mapsto J_n\mathbb{P}\left(u\cdot\nabla v\right)$ is bilinear and continuous on $L^{2,\sigma}_n$; the same holds for the other terms.

A priori bounds and hints for the convergence

It is easy to carry out the same computations as in Section 6.A.1 to see that (u_n,τ_n) satisfies (6.3), (6.49) and (6.50). These bounds uniform in n are crucial at least for two reasons:

- The energy bound (6.3) and the blow-up condition for ordinary differential equations imply that the approximate solutions (u_n,τ_n) are globally defined.
- They yield some compactness properties on the solutions.

It follows from the a priori bounds on (u_n,τ_n) that there exists $u\in L^\infty\left((0,\infty);L^{2,\sigma}\right)\cap L^2\Big((0,\infty);\dot{H}^1\Big)$ such that for all $T,t>0$, extracting subsequences if necessary,

$$u_n \rightharpoonup u \qquad L^2\Big((0,T);H^1\Big), \tag{6.62a}$$
$$u_n(t,\cdot) \rightharpoonup u(t,\cdot) \quad L^2, \tag{6.62b}$$
$$u_n \overset{*}{\rightharpoonup} u \qquad L^\infty\Big((0,\infty);L^2\Big), \tag{6.62c}$$
$$\nabla u_n \rightharpoonup \nabla u \qquad L^2\Big((0,\infty);L^2\Big), \tag{6.62d}$$
$$\nabla u_n \rightharpoonup \nabla u \qquad L^{\frac{q}{q-1}}\left((0,T);L^q\right), \tag{6.62e}$$

and for all $\varphi\in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$,

$$(\partial_t u_n) \qquad \text{is uniformly bounded in } L^2\Big((0,T);H^{-1}\Big), \tag{6.62f}$$
$$\varphi u_n \longrightarrow \varphi u \qquad L^\infty\Big((0,T);H^{-1}\Big), \tag{6.62g}$$
$$\varphi u_n \longrightarrow \varphi u \qquad L^2\Big((0,T);L^2\Big). \tag{6.62h}$$

There exists $\tau\in L^\infty\left((0,\infty);L^2\right)\cap L^\infty_{loc}\left((0,\infty);L^q\right)$ such that for all $T,\ t>0$, extracting subsequences if necessary,

$$\tau_n \rightharpoonup \tau \qquad L^2\Big((0,\infty);L^2\Big), \tag{6.63a}$$
$$\tau_n(t,\cdot) \rightharpoonup \tau(t,\cdot) \quad L^2, \tag{6.63b}$$
$$\tau_n \overset{*}{\rightharpoonup} \tau \qquad L^\infty\Big((0,\infty);L^2\Big), \tag{6.63c}$$
$$\tau_n \overset{*}{\rightharpoonup} \tau \qquad L^\infty\left((0,T);L^q\right) \tag{6.63d}$$

and for all $\psi\in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$,

$$(\partial_t \tau_n) \qquad \text{is uniformly bounded in } L^2\Big((0,T);H^{-1}\Big), \tag{6.63e}$$
$$\psi\tau_n \longrightarrow \psi\tau \qquad L^\infty\Big((0,T);H^{-1}\Big). \tag{6.63f}$$

The strong convergence of the velocity field u_n makes it possible to pass to the limit in the nonlinear term $u_n\cdot\nabla u_n$. To see that (6.62h) holds, one can either apply directly the Aubin-Lions lemma, or resort to the Ascoli theorem. In the latter case, the convergence follows from:

– the bound on u_n in $L^\infty\left((0,\infty);L^2\right)$, which yields some strong compactness thanks to Rellich's theorem,
– together with the bound (6.62f), which allows to appeal to Ascoli's theorem.

Let us expound these two points. Let $\varphi\in C_c^\infty\left(\mathbb{R}_+\times\mathbb{R}^2\right)$. Then for all $t\in(0,\infty)$, Rellich's compactness theorem [BCD11] implies the relative compactness of the sequence $\varphi(t,\cdot)u_n(t,\cdot)$ in the space $H^{-1}\left(\mathbb{R}^2\right)$. This is for the first point. In order to apply Ascoli's theorem, we have to see that the sequence $\varphi(t,\cdot)u_n(t,\cdot)$ is equicontinuous in t. Let us prove (6.62f), i.e. that $\varphi\partial_t u_n$ is uniformly bounded in n in the space $L^2\left((0,\infty);H^{-1}\right)$. Indeed, we have that

$$\varphi\partial_t u_n=-\varphi J_n\mathbb{P}\left(u_n\cdot\nabla u_n\right)-(1-\omega)\varphi\Delta u_n+\varphi\mathbb{P}\left(\nabla\cdot\tau_n\right), \tag{6.64}$$

and for all $v\in H^1\left(\mathbb{R}^2\right)$,

$$\begin{aligned}\left\langle J_n\mathbb{P}\left(u_n\cdot\nabla u_n\right),v\right\rangle_{H^{-1},H^1}&=\left\langle\varphi J_n\mathbb{P}\left(u_n\otimes u_n\right),\nabla\left(\varphi v\right)\right\rangle_{H^{-1},H^1}\\&\le C\left\|u_n\right\|_{L^4}^2\left\|\nabla\left(\varphi v\right)\right\|_{L^2}\le C\left\|u_n\right\|_{L^2}\left\|\nabla u_n\right\|_{L^2}\left\|v\right\|_{H^1},\end{aligned}$$

from which we infer the bound

$$\left\|J_n\mathbb{P}\left(u_n\cdot\nabla u_n\right)\right\|_{L^2((0,\infty);H^{-1})}\le\left\|u_n\right\|_{L^\infty((0,\infty);L^2)}\left\|\nabla u_n\right\|_{L^2((0,\infty);L^2)};$$

the other terms in the right hand side of (6.64) are easier to handle; notice that $\nabla\cdot\tau_n$ is uniformly bounded in n in $L^2\left((0,\infty);H^{-1}\right)$. Finally,

$$\left\|\varphi(t,\cdot)u_n(t,\cdot)-\varphi(t',\cdot)u_n(t',\cdot)\right\|\le\int_t^{t'}\left\|\varphi\partial_t u_n\right\|_{H^{-1}}\le|t-t'|^{\frac12}\left\|\varphi\partial_t u_n\right\|_{L^2((0,\infty);L^2)},$$

which is the equicontinuity we are looking for. Ascoli's theorem and a diagonal extraction yields the existence of $u\in C^0\left((0,\infty);H^{-1}\right)$ and (6.62g).

It is worth noticing that these steps work for τ_n, up to minor changes in the estimates: we use the local in time bound (6.49) on τ_n to get for example

$$\begin{aligned}\left\langle\varphi J_n\tau_n W(u_n),v\right\rangle_{H^{-1},H^1}&\le\int_{\mathbb{R}^2}J_n\tau_n W(u_n)\varphi v\\&\le C\left\|\tau_n\right\|_{L^q}\left\|\nabla u_n\right\|_{L^2}\left\|\varphi v\right\|_{L^{\frac{2q}{q-2}}}\le C\left\|\tau_n\right\|_{L^q}\left\|\nabla u_n\right\|_{L^2}\left\|\varphi v\right\|_{H^1}.\end{aligned}$$

This leads to (6.63e), the existence of $\tau\in C^0\left((0,\infty);H^{-1}\right)$ and (6.63f).

Thanks to the control of the derivatives of u_n, we conclude for u_n to the strong convergence in $L^2\left((0,\infty);L^2\right)$:

$$\left\|\varphi\left(u_n-u\right)\right\|_{L^2((0,\infty);L^2)}^2\le\left\|\varphi\left(u_n-u\right)\right\|_{L^\infty((0,\infty);H^{-1})}\left\|\varphi\left(u_n-u\right)\right\|_{L^2((0,\infty);H^1)}\overset{n\to\infty}{\longrightarrow}0.$$

The trouble with τ_n is that we cannot proceed using this approach to show the strong convergence in $L^2\left((0,\infty);L^2\right)$. Indeed, we do not control any derivative of τ_n. Hence, the best we can get using a naive approach, is the weak compactness of the sequence τ_n. As the terms $\tau_n W(u_n)$ and $W(u_n)\tau_n$ do not seem to have an evident div-curl structure, this weak compactness is not enough to pass to the limit in the product. The goal of the next section is to recover some strong convergence of τ_n, but using far more elaborate methods.

6.B.2 Compactness

As emphasized in [LM00], the heart of the matter when proving the existence of weak solutions (u,τ) to (6.1) is to pass to the limit in the product $\tau_n W(u_n)$ (resp. $W(u_n)\tau_n$) of approximated weak solutions (u_n,τ_n). Indeed, for such a sequence, $W(u_n)$ (resp. τ_n) is only known to converge weakly toward $W(u)$ (resp. τ). The convergence of the product requires therefore further investigations.

Assume that (u_n,τ_n) is a sequence of weak solutions to (6.1) with initial data

$$u_{0,n} \in L^2 \cap I_{p,q} \qquad \text{and} \qquad \tau_{0,n} \in L^2 \cap L^q,$$

satisfying the a priori bounds (6.3), (6.49) and (6.50). Following [CS12] (see equation (251)), let us introduce the new unknowns

$$a_n := \tau_{n,11} - \tau_{n,22}, \qquad b_n := \tau_{n,12}, \qquad c_n := \tau_{n,11} + \tau_{n,22}.$$

We compute (dropping for the moment the subscripts n)

$$\begin{aligned}
\tau W(u) - W(u)\tau &= \frac{1}{2}\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}\begin{pmatrix} 0 & \partial_1 u_2 - \partial_2 u_1 \\ \partial_2 u_1 - \partial_1 u_2 & 0 \end{pmatrix} \\
&\quad - \frac{1}{2}\begin{pmatrix} 0 & \partial_1 u_2 - \partial_2 u_1 \\ \partial_2 u_1 - \partial_1 u_2 & 0 \end{pmatrix}\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \tau_{21}\left(\partial_2 u_1 - \partial_1 u_2\right) & \tau_{11}\left(\partial_1 u_2 - \partial_2 u_1\right) \\ \tau_{22}\left(\partial_2 u_1 - \partial_1 u_2\right) & \tau_{21}\left(\partial_1 u_2 - \partial_2 u_1\right) \end{pmatrix} \\
&\quad - \frac{1}{2}\begin{pmatrix} \tau_{21}\left(\partial_1 u_2 - \partial_2 u_1\right) & \tau_{22}\left(\partial_1 u_2 - \partial_2 u_1\right) \\ \tau_{11}\left(\partial_2 u_1 - \partial_1 u_2\right) & \tau_{21}\left(\partial_2 u_1 - \partial_1 u_2\right) \end{pmatrix} \\
&= \begin{pmatrix} \tau_{12}\left(\partial_2 u_1 - \partial_1 u_2\right) & \frac{1}{2}\left(\tau_{11} - \tau_{22}\right)\left(\partial_1 u_2 - \partial_2 u_1\right) \\ \frac{1}{2}\left(\tau_{11} - \tau_{22}\right)\left(\partial_1 u_2 - \partial_2 u_1\right) & \tau_{12}\left(\partial_1 u_2 - \partial_2 u_1\right) \end{pmatrix}.
\end{aligned}$$

Hence, the transport equation on τ_n becomes

$$\left\{\begin{array}{ll} \partial_t a_n + u_n \cdot \nabla a_n - 2b_n \operatorname{curl} u_n + \frac{a_n}{\mathrm{We}} & = \frac{2\omega}{\mathrm{We}}\left(\partial_1 u_{n,1} - \partial_2 u_{n,2}\right) \\ \partial_t b_n + u_n \cdot \nabla b_n + \frac{1}{2} a_n \operatorname{curl} u_n + \frac{b_n}{\mathrm{We}} & = \frac{\omega}{\mathrm{We}}\left(\partial_1 u_{n,2} + \partial_2 u_{n,1}\right) \\ \partial_t c_n + u_n \cdot \nabla c_n + \frac{c_n}{\mathrm{We}} & = 0 \end{array}\right. \tag{6.65}$$

where $\operatorname{curl} u_n := \partial_1 u_{n,2} - \partial_2 u_{n,1}$. In particular c_n is decoupled from a_n and b_n. Of course, the sequences (a_n), (b_n) and (c_n) inherit from the properties of (τ_n), and (a_n) (resp. (b_n), (c_n)) satisfy the a priori bounds (6.3) and (6.49).

Existence of a weak pressure and regularity

We address the existence of the pressure p_n associated to (u_n,τ_n). Let $T>0$. From the variational formulation of system (6.1), we get that for all $\varphi_\sigma \in C_c^\infty\left((0,\infty); C_c^{\infty,\sigma}\right)$,

$$\langle \partial_t u_n + u_n \cdot \nabla u_n - (1-\omega)\Delta u_n - \nabla \cdot \tau_n, \varphi_\sigma \rangle_{\mathcal{D}',\mathcal{D}} = 0. \tag{6.66}$$

Yet, (u_n) is bounded in $L^\infty\left((0,\infty); L^2\right)$, so that $(\partial_t u_n)$ is bounded in $W^{-1,\infty}\left((0,\infty); L^2\right)$. Furthermore, let us show that

$$u_n \cdot \nabla u_n - (1-\omega)\Delta u_n - \nabla \cdot \tau_n \in L^2\left((0,T); H^{-1}\right).$$

One immediately sees that Δu_n (resp. $\nabla \cdot \tau_n$) is bounded in $L^2\left((0,T);H^{-1}\right)$. Hence the only tricky term is the nonlinear one: for all $\varphi \in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$,

$$\begin{aligned}\left|\langle -u_n\cdot\nabla u_n,\varphi\rangle_{\mathcal{D}',\mathcal{D}}\right| = \left|\langle u_n\otimes u_n,\nabla\varphi\rangle_{\mathcal{D}',\mathcal{D}}\right| &\leq \int_0^T \|u_n\otimes u_n\|_{L^2(\mathbb{R}^2)}\|\varphi\|_{H^1(\mathbb{R}^2)}\\ \leq \int_0^T \|u_n\|^2_{L^4(\mathbb{R}^2)}\|\varphi\|_{H^1(\mathbb{R}^2)} &\leq \int_0^T \|u_n\|_{L^2(\mathbb{R}^2)}\|u_n\|_{H^1(\mathbb{R}^2)}\|\varphi\|_{H^1(\mathbb{R}^2)}\\ &\leq \|u_n\|_{L^\infty(L^2)}\|u_n\|_{L^2(H^1)}\|\varphi\|_{L^2(H^1)},\end{aligned}$$

which yields the conclusion. As a consequence

$$\partial_t u_n + u_n\cdot\nabla u_n - (1-\omega)\Delta u_n - \nabla\cdot\tau_n$$

is uniformly bounded in $W^{-1,\infty}\left((0,T);H^{-1}\right)$. This fact, (6.66) and De Rham's theorem (cf. Theorem 8.3 in [Sim03]) imply the existence of a sequence of pressures (p_n), uniformly bounded in $W^{-1,\infty}\left((0,T);L^2_{loc}\right)$, satisfying

$$\nabla p_n = -\partial_t u_n - u_n\cdot\nabla u_n + (1-\omega)\Delta u_n + \nabla\cdot\tau_n \in W^{-1,\infty}\left((0,T);H^{-1}\right), \tag{6.67a}$$

$$\|p_n\|_{W^{-1,\infty}((0,T);L^2(K))} \leq C(K)\left\|-\partial_t u_n - u_n\cdot\nabla u_n + (1-\omega)\Delta u_n + \nabla\cdot\tau_n\right\|_{W^{-1,\infty}((0,T);H^{-1})} \tag{6.67b}$$

for all $K \Subset \mathbb{R}^2$. For all $\varphi,\ \psi$ in $C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$,

$$\langle \partial_t u_n + u_n\cdot\nabla u_n - (1-\omega)\Delta u_n + \nabla p_n - \nabla\cdot\tau_n,\varphi\rangle_{\mathcal{D}',\mathcal{D}} = 0, \tag{6.68a}$$

$$\left\langle \partial_t\tau_n + u_n\cdot\nabla\tau_n + \tau_n W(u_n) - W(u_n)\tau_n + \frac{\tau_n - 2\omega D(u_n)}{\mathrm{We}},\psi\right\rangle_{\mathcal{D}',\mathcal{D}} = 0. \tag{6.68b}$$

Take $\varphi = \nabla\phi$, with $\phi \in C_c^\infty\left((0,\infty),\mathbb{R}^2\right)$, as a test function in (6.68a): using the incompressibility condition,

$$\begin{aligned}0 = \langle \partial_t u_n + u_n\cdot\nabla u_n - (1-\omega)\Delta u_n + \nabla p_n - \nabla\cdot\tau_n,\nabla\phi\rangle_{\mathcal{D}',\mathcal{D}}&\\ = \langle \nabla\cdot(u_n\cdot\nabla u_n) + \Delta p_n - \nabla\cdot(\nabla\cdot\tau_n),\varphi\rangle_{\mathcal{D}',\mathcal{D}}&,\end{aligned}$$

so that p_n is a solution to

$$-\Delta p_n = \nabla\cdot(u_n\cdot\nabla u_n) - \nabla\cdot(\nabla\cdot\tau_n) \tag{6.69}$$

in the sense of distributions. The right hand side belongs to $L^2\left((0,T);H^{-2}\right)$. Indeed,

$$-\nabla\cdot(\nabla\cdot\tau_n) \in L^2\left((0,T);H^{-2}\right)$$

and

$$\nabla\cdot(u_n\cdot\nabla u_n) \in L^2\left((0,T);W^{-1,1}\right)$$

which injects in in $L^1\left((0,T);H^{-2}\right)$. To see this injection into a Sobolev space of negative exponent, use the fact that $H^{-2} = \left(H_0^2\right)'$ and the classical Sobolev injection $H^2 \hookrightarrow W^{1,1}$: for all $f \in W^{-1,1}$, for all $\varphi \in H_0^2 \subset W_0^{1,1}$,

$$\langle f,\varphi\rangle_{W^{-1,1},W_0^{1,1}} \leq \|f\|_{W^{-1,1}}\|\varphi\|_{W_0^{1,1}} \leq C\|f\|_{W^{-1,1}}\|\varphi\|_{H_0^2},$$

so $f \in H^{-2}$ and $\|f\|_{H^{-2}} \leq C\|f\|_{W^{-1,1}}$. Applying the regularity estimates of [Neč67], we finally get that

$$p_n \in L^2\left((0,T);L^2\right). \tag{6.70}$$

We conclude from these estimates that for all $T>0$,

$$\partial_t u_n = -u_n\cdot\nabla u_n + (1-\omega)\Delta u_n + \nabla\cdot\tau_n - \nabla p_n \in L^2\left((0,T);H^{-1}\right). \tag{6.71}$$

Defect measures

It follows from the a priori bounds on (u_n, τ_n) and the bound (6.70), that there exists $u \in L^\infty\left((0,\infty); L^{2,\sigma}\right) \cap L^2\big((0,\infty); \dot{H}^1\big)$, $p \in L^2_{loc}\left((0,\infty); L^2\right)$ such that the convergences (6.62) hold, and for all $T > 0$,

$$p_n \rightharpoonup p \quad L^2\left((0,T); L^2\right).$$

Moreover, there exists $a \in L^\infty\left((0,\infty); L^2\right) \cap L^\infty_{loc}\left((0,\infty); L^q\right)$ such that for all $T,\ t > 0$, the convergences (6.63), with τ_n (resp. τ) formally replaced by a_n (resp. a), take place. The same type of convergences hold of course for (b_n) and (c_n).

A way to quantify the possible loss of convergence in products of weakly converging sequences is to introduce defect measures. As

$$\begin{aligned}
|a_n|^2 &\quad \text{is uniformly bounded in (u.b.)} && L^\infty\big((0,T); L^{\frac{q}{2}}\big),\\
|b_n|^2 &\quad \text{u.b. in} && L^\infty\big((0,T); L^{\frac{q}{2}}\big),\\
a_n \operatorname{curl} u_n &\quad \text{u.b. in} && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big),\\
b_n \operatorname{curl} u_n &\quad \text{u.b. in} && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big),\\
(\partial_1 u_{1,n} - \partial_2 u_{2,n})\, a_n &\quad \text{u.b. in} && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big),\\
(\partial_1 u_{2,n} + \partial_2 u_{1,n})\, b_n &\quad \text{u.b. in} && L^2\big((0,T); L^{\frac{q}{2}}\big),\\
|\nabla u_n|^2 &\quad \text{u.b. in} && L^1\left((0,\infty); L^1\right),
\end{aligned}$$

there exists $\alpha \in L^\infty_{loc}\big((0,\infty); L^{\frac{q}{2}}\big)$ (resp. $\beta \in L^\infty_{loc}\big((0,\infty); L^{\frac{q}{2}}\big)$, $\delta, \varepsilon, \eta, \lambda \in L^{\frac{q}{q-1}}_{loc}\big((0,\infty); L^{\frac{q}{2}}\big)$, $\mu \in [L^\infty\left((0,\infty); L^\infty\right)]'$) such that for all $T > 0$,

$$\begin{aligned}
|a_n|^2 &\overset{*}{\rightharpoonup} |a|^2 + \alpha && L^\infty\big((0,T); L^{\frac{q}{2}}\big), && (6.72a)\\
|b_n|^2 &\overset{*}{\rightharpoonup} |b|^2 + \beta && L^\infty\big((0,T); L^{\frac{q}{2}}\big), && (6.72b)\\
a_n \operatorname{curl} u_n &\rightharpoonup a \operatorname{curl} u + \delta && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big), && (6.72c)\\
b_n \operatorname{curl} u_n &\rightharpoonup b \operatorname{curl} u + \varepsilon && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big), && (6.72d)\\
(\partial_1 u_{1,n} - \partial_2 u_{2,n})\, a_n &\rightharpoonup (\partial_1 u^1 - \partial_2 u^2)\, a + \eta && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big), && (6.72e)\\
(\partial_1 u_{2,n} + \partial_2 u_{1,n})\, b_n &\rightharpoonup (\partial_1 u^2 + \partial_2 u^1)\, b + \lambda && L^{\frac{q}{q-1}}\big((0,T); L^{\frac{q}{2}}\big), && (6.72f)\\
|\nabla u_n|^2 &\overset{*}{\rightharpoonup} |\nabla u|^2 + \mu && [L^\infty\left((0,\infty); L^\infty\right)]'. && (6.72g)
\end{aligned}$$

Let us state a couple of properties.

Lemma 6.8. *The defect measures satisfy:*

1. *The measures α, β, μ are positive.*
2. *For every bounded measurable set $E \subset (0,\infty) \times \mathbb{R}^2$,*

$$\begin{aligned}
|\eta(E)| \le \sqrt{\mu(E)}\sqrt{\alpha(E)}, \quad |\lambda(E)| \le \sqrt{\mu(E)}\sqrt{\beta(E)},\\
|\delta(E)| \le \sqrt{\mu(E)}\sqrt{\alpha(E)}, \quad |\varepsilon(E)| \le \sqrt{\mu(E)}\sqrt{\beta(E)}.
\end{aligned} \tag{6.73}$$

Proof. We have

$$|a_n - a|^2 = |a_n|^2 + |a|^2 - 2a_n a \overset{*}{\rightharpoonup} \alpha \qquad L^\infty\big((0,T); L^{\frac{q}{2}}\big).$$

Indeed $L^\infty\big((0,T);L^{\frac{q}{2}}\big) = \Big[L^1\big((0,T);L^{\frac{q}{q-2}}\big)\Big]'$ and for all $v \in L^1\big((0,T);L^{\frac{q}{q-2}}\big)$,

$$\begin{aligned}\langle a_n a, v\rangle_{L^\infty\left(L^{\frac{q}{2}}\right),L^1\left(L^{\frac{q}{q-2}}\right)} &= \langle a_n, av\rangle_{L^\infty(L^q),L^1\left(L^{\frac{q}{q-1}}\right)}\\ &\overset{n\to\infty}{\longrightarrow} \langle a, av\rangle_{L^\infty(L^q),L^1\left(L^{\frac{q}{q-1}}\right)} = \langle |a|^2, v\rangle_{L^\infty\left(L^{\frac{q}{2}}\right),L^1\left(L^{\frac{q}{q-2}}\right)}.\end{aligned}$$

In the same way we get

$$\begin{aligned}|b_n - b|^2 &\overset{*}{\rightharpoonup} \beta \quad && L^\infty\big((0,T);L^{\frac{q}{2}}\big),\\ |\nabla u_n - \nabla u|^2 &\overset{*}{\rightharpoonup} \mu \quad && [L^\infty((0,T);L^\infty)]',\end{aligned}$$

as for all $v \in L^\infty((0,T);L^\infty)$

$$\begin{aligned}\langle \nabla u_n \nabla u, v\rangle_{[L^\infty(L^\infty)]',L^\infty(L^\infty)} &= \langle \nabla u_n, \nabla u v\rangle_{L^2(L^2),L^2(L^2)}\\ &\overset{n\to\infty}{\longrightarrow} \langle \nabla u, \nabla u v\rangle_{L^2(L^2),L^2(L^2)} = \langle |\nabla u|^2, v\rangle_{[L^\infty(L^\infty)]',L^\infty(L^\infty)}.\end{aligned}$$

This concludes the proof of the first point.

Let $E \subset (0,\infty)\times\mathbb{R}^2$ be a bounded measurable set. Then, by inequality of Cauchy-Schwarz

$$\int (\partial_1 u_{1,n} - \partial_2 u_{2,n} - \partial_1 u_1 + \partial_2 u_2)(a_n - a)1_E \le C\Big(\int |a_n - a|^2 1_E\Big)^{\frac{1}{2}}\Big(\int |\nabla u_n - \nabla u|^2 1_E\Big)^{\frac{1}{2}}$$

which yields, using the weak convergences (6.72a),

$$\int_E d\eta \le C\Big(\int_E d\alpha\Big)^{\frac{1}{2}}\Big(\int_E d\mu\Big)^{\frac{1}{2}}$$

in the limit $n\to\infty$. The same proof holds for the other inequalities. □

We proceed with the estimates for the defect measures.

First estimate

The approach consists in

– multiplying the equation on the velocity by u_n and passing to the limit,

– passing to the limit in the equation on the velocity, and then multiplying by u.

This way of doing yields an equality between the terms appearing in the averaging process. In this section, we aim at showing:

$$2(1-\omega)\mu + \eta + 2\lambda = 0. \tag{6.74}$$

Such an inequality implies in particular $\mu \in L^2_{loc}\big((0,\infty);L^{\frac{2q}{q+2}}\big)$.

Let $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R}^2)$. Testing (6.68a) against $u_n\varphi$ yields

$$\begin{aligned}0 &= \langle \partial_t u_n + u_n\cdot\nabla u_n - (1-\omega)\Delta u_n - \nabla\cdot\tau_n + \nabla p_n, u_n\varphi\rangle_{L^2(H^{-1}),L^2(H^1)}\\ &= \Big\langle \frac{1}{2}\partial_t|u_n|^2 + \frac{1}{2}u_n\cdot\nabla|u_n|^2 - \frac{1-\omega}{2}\Delta|u_n|^2 + (1-\omega)|\nabla u_n|^2\\ &\qquad -\nabla\cdot(\tau_n u_n) + \tau_n : \nabla u_n + \nabla\cdot(p_n u_n), \varphi\Big\rangle_{\mathcal{D}',\mathcal{D}}\end{aligned} \tag{6.75}$$

and passing to the weak limit in (6.75) we get

$$\left\langle \frac{1}{2}\partial_t|u|^2 + \frac{1}{2}u\cdot\nabla|u|^2 - \frac{1-\omega}{2}\Delta|u|^2 + (1-\omega)|\nabla u|^2 + (1-\omega)\mu \right.$$
$$\left. -\nabla\cdot(\tau u) + \tau : \nabla u + \frac{\eta}{2} + \lambda + \nabla\cdot(pu)\,, \varphi \right\rangle_{\mathcal{D}',\mathcal{D}} = 0. \quad (6.76)$$

Indeed,

$$\begin{aligned}
\tau_n : \nabla u_n &= \tau_{n,\alpha\beta}\partial_\alpha u_{n,\beta} \\
&= \tau_{n,11}\partial_1 u_{n,1} + \tau_{n,21}\left(\partial_2 u_{n,1} + \partial_1 u_{n,2}\right) + \tau_{n,22}\partial_2 u_{n,2} \\
&= \frac{1}{2}(a_n + c_n)\partial_1 u_{n,1} + b_n\left(\partial_2 u_{n,1} + \partial_1 u_{n,2}\right) + \frac{1}{2}(-a_n + c_n)\partial_2 u_{n,2} \\
&= \frac{1}{2}a_n\left(\partial_1 u_{n,1} - \partial_2 u_{n,2}\right) + b_n\left(\partial_2 u_{n,1} + \partial_1 u_{n,2}\right).
\end{aligned}$$

The convergence of the nonlinear term $\frac{1}{2}u_n\cdot\nabla|u_n|^2$ is faced using the strong convergence (6.62h): we have

$$\begin{aligned}
&\left|\left\langle u_n\cdot\nabla|u_n|^2 - u\cdot\nabla|u|^2, \varphi\right\rangle_{\mathcal{D}',\mathcal{D}}\right| \\
&\le \left|\left\langle u_n\cdot\nabla\left(|u_n|^2 - |u|^2\right), \varphi\right\rangle_{\mathcal{D}',\mathcal{D}}\right| + \left|\left\langle (u_n - u)\cdot\nabla|u|^2, \varphi\right\rangle_{\mathcal{D}',\mathcal{D}}\right| \\
&\le \left|\left\langle \nabla\varphi\cdot u_n, |u_n|^2 - |u|^2\right\rangle_{L^\infty(L^2),L^1(L^2)}\right| + \left|\left\langle \nabla\varphi|u|^2, u_n - u\right\rangle_{L^2(L^2),L^2(L^2)}\right|,
\end{aligned}$$

which tends to 0 when $n\to\infty$. We turn to the second step. For all $v\in L^2\left((0,T);H^1\right)$, passing to the limit in (6.68a)

$$\begin{aligned}
0 &= \left\langle \partial_t u_n + u_n\cdot\nabla u_n - (1-\omega)\Delta u_n + \nabla p_n - \nabla\cdot\tau_n, v\right\rangle_{L^2(H^{-1}),L^2(H^1)} \\
&\xrightarrow{n\to\infty} \left\langle \partial_t u + u\cdot\nabla u - (1-\omega)\Delta u + \nabla p - \nabla\cdot\tau, v\right\rangle_{L^2(H^{-1}),L^2(H^1)}. \quad (6.77)
\end{aligned}$$

Taking $v = \varphi u \in L^2\left((0,T);H^1\right)$, we get

$$\left\langle \partial_t\frac{|u|^2}{2} + \frac{1}{2}u\cdot\nabla|u|^2 - \frac{1-\omega}{2}\Delta|u|^2 + (1-\omega)|\nabla u|^2 \right.$$
$$\left. -\nabla\cdot(\tau u) + \tau : \nabla u + \nabla\cdot(pu)\,, \varphi \right\rangle_{\mathcal{D}',\mathcal{D}} = 0. \quad (6.78)$$

Comparing (6.76) and (6.78) leads to

$$\left\langle 2(1-\omega)\mu + \eta + 2\lambda, \varphi\right\rangle_{\mathcal{D}',\mathcal{D}} = 0,$$

which is nothing but (6.74).

Second estimate

Following the same scheme, we want to estimate

$$\gamma := \alpha + 4\beta \in L^\infty_{loc}\left((0,\infty); L^{\frac{q}{2}}\right). \quad (6.79)$$

Let $\psi \in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$. The main source of difficulties comes from the low regularity of a_n (resp. b_n), which is not better than $L^\infty_{loc}\left((0,\infty);L^q\right)$. Hence it is not possible to proceed exactly as in the previous section: as such, it does not make sense to integrate $u_n\cdot\nabla a_n$ against $a_n\psi$. We need to have recourse to other types of arguments.

Let $n\in\mathbb{N}$. When the fields $b_n\in L^\infty_{loc}\left((0,\infty);L^q\right)$ and $u_n\in L^1_{loc}\left((0,\infty);W^{1,q}\right)\cap L^\infty\left((0,\infty);L^2\right)$ are fixed, we notice that $a_n\in L^\infty_{loc}\left((0,\infty);L^q\right)$ is the unique renormalized solution to

$$\partial_t a_n + u_n\cdot\nabla a_n + \frac{a_n}{\mathrm{We}} = \frac{2\omega}{\mathrm{We}}\left(\partial_1 u_{n,1}-\partial_2 u_{n,2}\right) + 2b_n \operatorname{curl} u_n =: f_n$$

with source term f_n in $L^1_{loc}\left((0,\infty);L_{loc}^{\frac{2q}{q+2}}\right)$. This is a consequence of Theorem II.2 in [DL89]. Indeed $u\in L^1_{loc}\left((0,\infty);W^{1,q}\right)\subset L^1_{loc}\left((0,\infty);L^1\right)$, so that $\frac{u}{1+|x|}\in L^1_{loc}\left((0,\infty);L^1\right)$. Furthermore, for $q\geq 4$, $\frac{q+2}{2q}+\frac{1}{q}=\frac{q+4}{q}\leq 1$ and the product of f_n with a_n is well defined. We have from corollary II.2 in [DL89], that the equality

$$\frac{1}{2}\partial_t|a_n|^2 + \frac{1}{2}u_n\cdot\nabla|a_n|^2 + \frac{|a_n|^2}{\mathrm{We}} = \frac{2\omega}{\mathrm{We}}\left(\partial_1 u_{n,1}-\partial_2 u_{n,2}\right)a_n + 2b_n a_n \operatorname{curl} u_n \tag{6.80a}$$

is true in the sense of distributions. The same holds if instead of a_n, one looks at b_n. We have

$$\frac{1}{2}\partial_t|b_n|^2 + \frac{1}{2}u_n\cdot\nabla|b_n|^2 + \frac{|b_n|^2}{\mathrm{We}} = \frac{\omega}{\mathrm{We}}\left(\partial_1 u_{n,2}+\partial_2 u_{n,1}\right)b_n - \frac{1}{2}b_n a_n \operatorname{curl} u_n. \tag{6.80b}$$

Passing to the limit in (6.80a)+4(6.80b) leads to

$$\begin{aligned}\frac{1}{2}\partial_t\left(|a|^2+4|b|^2\right) + \frac{1}{2}\partial_t\left(\alpha+4\beta\right) + \frac{1}{2}\nabla\cdot\left(u\left(|a|^2+4|b|^2\right)\right) + \frac{1}{2}\nabla\cdot\left(u\left(\alpha+4\beta\right)\right)\\ + \frac{|a|^2+4|b|^2}{\mathrm{We}} + \frac{\alpha+4\beta}{\mathrm{We}} = \frac{2\omega}{\mathrm{We}}\left[\left(\partial_1 u_1-\partial_2 u_2\right)a + \eta + 2\left(\partial_1 u_2+\partial_2 u_1\right)b + 2\lambda\right]\end{aligned} \tag{6.81}$$

in the sense of $\mathcal{D}'\left((0,\infty)\times\mathbb{R}^2\right)$.

Passing to the limit in the system (6.65), we obtain that $a\in L^\infty_{loc}\left((0,\infty);L^q\right)$ is a weak solution to

$$\partial_t a + u\cdot\nabla a + \frac{a}{\mathrm{We}} = \frac{2\omega}{\mathrm{We}}\left(\partial_1 u_1-\partial_2 u_2\right) + 2b\operatorname{curl} u + 4\varepsilon,$$

and that $b\in L^\infty_{loc}\left((0,\infty);L^q\right)$ is a weak solution to

$$\partial_t b + u\cdot\nabla b + \frac{b}{\mathrm{We}} = \frac{\omega}{\mathrm{We}}\left(\partial_1 u_2+\partial_2 u_1\right) - \frac{1}{2}a\operatorname{curl} u - \delta.$$

The same arguments of [DL89] than those invoked for a_n (resp. b_n) yield uniqueness for a and b, as well as the equalities in $\mathcal{D}'\left((0,\infty)\times\mathbb{R}^2\right)$:

$$\frac{1}{2}\partial_t|a|^2 + \frac{1}{2}u\cdot\nabla|a|^2 + \frac{|a|^2}{\mathrm{We}} = \frac{2\omega}{\mathrm{We}}\left(\partial_1 u_1-\partial_2 u_2\right)a + 2ba\operatorname{curl} u + 4\varepsilon a, \tag{6.82a}$$

$$\frac{1}{2}\partial_t|b|^2 + \frac{1}{2}u\cdot\nabla|b|^2 + \frac{|b|^2}{\mathrm{We}} = \frac{\omega}{\mathrm{We}}\left(\partial_1 u_2+\partial_2 u_1\right)b - \frac{1}{2}ba\operatorname{curl} u - \delta b. \tag{6.82b}$$

Hence, $|a|^2+4|b|^2$ is a weak solution to

$$\begin{aligned}\frac{1}{2}\partial_t\left(|a|^2+4|b|^2\right) + \frac{1}{2}u\cdot\nabla\left(|a|^2+4|b|^2\right) + \frac{|a|^2+4|b|^2}{\mathrm{We}}\\ = \frac{2\omega}{\mathrm{We}}\left[\left(\partial_1 u_1-\partial_2 u_2\right)a + 2\left(\partial_1 u_2+\partial_2 u_1\right)b\right] + 2\varepsilon a - 2\delta b.\end{aligned} \tag{6.83}$$

Subsequently, substracting (6.83) to (6.81), one finds that $\gamma \in L^\infty_{loc}\big((0,\infty); L^{\frac{q}{2}}\big)$ defined by (6.79) is a weak solution to the damped transport equation

$$\frac{1}{2}\partial_t \gamma + \frac{1}{2} u \cdot \nabla \gamma + \frac{\gamma}{\mathrm{We}} = 2\omega \frac{\eta + 2\lambda}{\mathrm{We}} - 2\varepsilon a + 2\delta b \tag{6.84}$$

with initial data

$$\gamma_0 := \gamma(0,\cdot) = \alpha(0) + 4\beta(0) \in L^{\frac{q}{2}},$$

a_0 (resp. b_0) being a weak limit in L^q to $a_{0,n}$ (resp. $b_{0,n}$), extracting subsequences if necessary.

Compactness: end of the argument

The goal is to show that if the defect measure γ is initially zero, then it vanishes for all time. Using (6.73) in combination with (6.74), yields

$$\eta + 2\lambda \leq 0, \qquad |\varepsilon| \leq C\gamma, \qquad |\delta| \leq C\gamma,$$

and for all $\psi \in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$, such that $\psi \geq 0$,

$$\langle \varepsilon a, \psi\rangle_{\mathcal{D}',\mathcal{D}} \leq C \int_{(0,\infty)\times\mathbb{R}^2} |a|\gamma\psi = C\,\langle \gamma|a|, \psi\rangle_{\mathcal{D}',\mathcal{D}}\,.$$

This allows to control the right hand side of (6.84), and finally gives the inequality

$$\frac{1}{2}\partial_t \gamma + \frac{1}{2} u \cdot \nabla \gamma \leq C\,[|a| + |b|]\,\gamma, \tag{6.85}$$

with C independent of the Weissenberg number We, and which, of course, is to be understood in the distribution sense: for all $\psi \in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$, $\psi \geq 0$,

$$\left\langle \frac{1}{2}\partial_t \gamma + \frac{1}{2} u \cdot \nabla \gamma, \psi \right\rangle_{\mathcal{D}',\mathcal{D}} \leq C\,\langle \gamma\left(|a| + |b|\right), \psi\rangle_{\mathcal{D}',\mathcal{D}}\,. \tag{6.86}$$

Adapting the arguments of P.-L. Lions and N. Masmoudi in [LM00] and taking advantage of the sign of $\eta + 2\lambda$, we introduce the unique renormalized solution $\Upsilon \in C^0\left((0,\infty); L^2\right)$ of

$$\frac{1}{2}\partial_t \Upsilon + \frac{1}{2} u \cdot \nabla \Upsilon = C\left(|a| + |b|\right) \tag{6.87}$$

with initial value $\Upsilon(0,\cdot) = 0$. Note that $\Upsilon \geq 0$ a.e.. Therefore, $\exp(-\Upsilon) \in L^\infty_{loc}\left((0,\infty); L^\infty\right)$ and

$$\widehat{\gamma} := \gamma \exp(-\Upsilon) \in L^\infty_{loc}\left((0,\infty); L^{\frac{q}{2}}\right)$$

is non negative a.e.. We aim at showing that $\widehat{\gamma}$ is a renormalized solution to

$$\partial_t \widehat{\gamma} + u \cdot \nabla \widehat{\gamma} + \frac{1}{\mathrm{We}}\widehat{\gamma} \leq 0, \tag{6.88}$$

i.e. that for all admissible function $\vartheta \in C^1(\mathbb{R})$ such that $\vartheta' \geq 0$, ϑ, $\vartheta'\left(1 + |t|\right)^{-1}$ are bounded on $\mathbb{R}$ and ϑ vanishes around 0, the following holds

$$\partial_t \vartheta\left(\widehat{\gamma}\right) + u \cdot \nabla \vartheta\left(\widehat{\gamma}\right) + \frac{1}{\mathrm{We}}\widehat{\gamma}\,\vartheta'\left(\widehat{\gamma}\right) \leq 0.$$

In order to justify the application of the chain rule, we regularize γ, Υ, u:

$$\gamma_\varepsilon := \rho_\varepsilon * \gamma, \quad \Upsilon_\varepsilon := \rho_\varepsilon * \Upsilon, \quad \text{and} \quad u_\varepsilon := \rho_\varepsilon * u,$$

with ρ_ε being an approximate identity. Then, on the one hand Υ_ε satisfies

$$\frac{1}{2}\partial_t \Upsilon_\varepsilon + \frac{1}{2} u_\varepsilon \cdot \nabla \Upsilon_\varepsilon = C\left(|a| + |b|\right) + r_\varepsilon,$$

with

$$r_\varepsilon = \frac{1}{2} u_\varepsilon \cdot \nabla \Upsilon_\varepsilon - \frac{1}{2}\rho_\varepsilon * \left(u \cdot \nabla \Upsilon\right) + C\rho_\varepsilon * \left(|a| + |b|\right) - C\left(|a| + |b|\right).$$

On the other hand, γ_ε solves

$$\frac{1}{2}\partial_t \gamma_\varepsilon + \frac{1}{2} u_\varepsilon \cdot \nabla \gamma_\varepsilon + \frac{1}{\mathrm{We}}\gamma_\varepsilon \leq C\left(|a| + |b|\right)\gamma_\varepsilon + \widetilde{r}_\varepsilon,$$

with

$$\widetilde{r}_\varepsilon = \frac{1}{2} u_\varepsilon \cdot \nabla \gamma_\varepsilon - \frac{1}{2}\rho_\varepsilon * \left(u \cdot \nabla \gamma\right) + C\rho_\varepsilon * \left(\left(|a| + |b|\right)\gamma\right) - C\left(|a| + |b|\right)\gamma_\varepsilon.$$

Hence, for ϑ as above

$$\begin{aligned}\partial_t\left(\vartheta\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\right) + u \cdot \nabla\left(\vartheta\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\right) + \frac{1}{\mathrm{We}}\left(\vartheta'\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\\ \leq \vartheta'\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\exp\left(-\Upsilon_\varepsilon\right)\left[\widetilde{r}_\varepsilon - \gamma_\varepsilon r_\varepsilon\right]. \quad (6.89)\end{aligned}$$

It remains to pass to the limit $\varepsilon \to 0$ in (6.89). The following convergences hold:

$$\gamma_\varepsilon \quad (\text{resp. } \Upsilon_\varepsilon,\ u_\varepsilon) \longrightarrow u \quad (\text{resp. } \Upsilon,\ u) \quad \text{a.e.}.$$

Moreover, the rest terms r_ε and $\widetilde{r}_\varepsilon$ tend to 0 in $L^1_{loc}\left((0,\infty)\times\mathbb{R}^2\right)$ and a.e.. Therefore, for all $\psi \in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$,

$$\begin{aligned}\langle \partial_t\left(\vartheta\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\right), \psi\rangle_{\mathcal{D}',\mathcal{D}} &= -\langle \vartheta\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right), \partial_t \psi\rangle_{\mathcal{D}',\mathcal{D}}\\ &= -\int_{(0,\infty)\times\mathbb{R}^2} \vartheta\left(\gamma_\varepsilon \exp\left(-\Upsilon_\varepsilon\right)\right)\partial_t \psi \xrightarrow{\varepsilon\to 0} \langle \partial_t\left(\vartheta\left(\gamma \exp\left(-\Upsilon\right)\right)\right), \psi\rangle_{\mathcal{D}',\mathcal{D}}\end{aligned}$$

by the dominated convergence theorem. Treating the other terms in a similar way yields finally (6.88).

To conclude that $\widehat{\gamma}(0,\cdot) = 0$ implies $\widehat{\gamma} = 0$, we mimic the proof of Theorem II.2 in [DL89]: for every $M > 0$,

$$0 \leq \int_{\mathbb{R}^2}\left(\inf\left(\widehat{\gamma}, M\right)\right)^{\frac{q}{2}} \leq 0$$

Hence, $\widehat{\gamma} = 0$ a.e., from which we deduce $\gamma = 0$ a.e..

As a consequence, for $\psi \in C_c^\infty\left((0,\infty)\times\mathbb{R}^2\right)$ such that $\psi = 1$ in $[0,1]\times B\left(0,1\right)$ and $\psi = 0$ outside $[0,2]\times B\left(0,2\right)$, for all $R > 0$,

$$\begin{aligned}\left|\int_{(0,\infty)\times\mathbb{R}^2}|a_n|^2 - \int_{(0,\infty)\times\mathbb{R}^2}|a|^2\right| &\leq \left|\int_{(0,\infty)\times\mathbb{R}^2}\left[|a_n|^2 - |a|^2\right]\psi\left(\frac{\cdot}{R}\right)\right|\\ &\quad + \left|\int_{(0,\infty)\times\mathbb{R}^2}\left[|a_n|^2 - |a|^2\right]\left(1 - \psi\left(\frac{\cdot}{R}\right)\right)\right|. \quad (6.90)\end{aligned}$$

The second term in the right hand side of (6.90) can be made arbitrarily small when $R \to \infty$ as a_n is bounded uniformly in $L^2\left((0,\infty);L^2\right)$. Then fix $R > 0$. For this R, the first integral tends to 0 when $n \to \infty$ because of (6.72a) and $\alpha = 0$. We conclude that

$$\|a_n\|_{L^2((0,\infty);L^2)} \longrightarrow \|a\|_{L^2((0,\infty);L^2)}$$

and therefore

$$a_n \overset{n\to\infty}{\longrightarrow} a \qquad L^2\left((0,\infty);L^2\right).$$

The same holds for b_n, c_n and thus for τ_n.

6.C Properties of $\mathcal{F}$ and $\mathcal{H}$

We sketch the calculations in the case when $d = 2$. The computations for $d > 2$ are completely analogous, albeit heavier.

Instead of working on A directly, we rather diagonalize A:

$$A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

with P an orthogonal matrix. Notice that $|A|^2 = \lambda_1^2 + \lambda_2^2$, because A is symmetric. It is thus sufficient to control the eigenvalues of $A - A_0$ to get a control of $|A - A^0|$. So, $\mathcal{F}$ becomes a function

$$\mathcal{F}:\ (x, y) \in \mathbb{R}_+^2 \cap B(0, b) \longmapsto\ -\ln x - \ln y - b \ln\left(1 - \frac{x+y}{b}\right) + (b+2)\ln\left(\frac{b}{b+2}\right),$$

where $B(0, b)$ is the ball of radius b for the norm $|(x, y)| = |x| + |y|$. The first-order derivatives

$$\begin{aligned} \partial_x \mathcal{F}(x, y) &= -\frac{1}{x} + \frac{b}{b-(x+y)} = \frac{(b+1)x + y - b}{x(b-(x+y))}, \\ \partial_y \mathcal{F}(x, y) &= -\frac{1}{y} + \frac{b}{b-(x+y)} = \frac{(b+1)y + x - b}{x(b-(x+y))} \end{aligned}$$

are zero for $x^0,\ y^0$ such that

$$\begin{cases} (b+1)x^0 + y^0 &= b \\ x^0 + (b+1)y^0 &= b \end{cases}$$

i.e. $x^0 = y^0 = \frac{b}{b+2}$. For every $(x, y) \in \mathbb{R}_+^2 \cap B(0, b)$, the Hessian matrix of $\mathcal{F}$ containing the second-order derivatives at the point (x, y) is

$$\mathrm{Hess}(\mathcal{F})(x, y) = \begin{pmatrix} \frac{1}{x^2} + \frac{b}{(b-(x+y))^2} & \frac{b}{(b-(x+y))^2} \\ \frac{b}{(b-(x+y))^2} & \frac{1}{y^2} + \frac{b}{(b-(x+y))^2} \end{pmatrix},$$

with caracteristic polynomial

$$\begin{aligned} &\left(\frac{1}{x^2} + \frac{b}{(b-(x+y))^2} - \lambda\right)\left(\frac{1}{y^2} + \frac{b}{(b-(x+y))^2} - \lambda\right) - \frac{b^2}{(b-(x+y))^4} \\ &\quad = \lambda^2 - \left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{2b}{(b-(x+y))^2}\right]\lambda + \frac{1}{x^2 y^2} + \frac{b}{(b-(x+y))^2}\left(\frac{1}{x^2} + \frac{1}{y^2}\right). \end{aligned} \tag{6.91}$$

Yet the discriminant of (6.91) is equal to

$$\frac{1}{x^4} + \frac{1}{y^4} - \frac{2}{x^2 y^2} + \frac{4b^2}{(b-(x+y))^4} = \left(\frac{1}{x^2} - \frac{1}{y^2}\right)^2 + \frac{4b^2}{(b-(x+y))^4} > 4b > 0,$$

and the two eigenvalues of the Hessian matrix are:

$$\begin{aligned} \lambda_+ &= \frac{1}{2}\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{2b}{(b-(x+y))^2} + \sqrt{\left(\frac{1}{x^2} - \frac{1}{y^2}\right)^2 + \frac{4b^2}{(b-(x+y))^4}}\right] > \frac{1}{b^2} > 0, \\ \lambda_- &= \frac{1}{2}\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{2b}{(b-(x+y))^2} - \sqrt{\left(\frac{1}{x^2} - \frac{1}{y^2}\right)^2 + \frac{4b^2}{(b-(x+y))^4}}\right]. \end{aligned}$$

Trace and determinant of the Hessian matrix being positive, we infer that its eigenvalues are positive. However, we need a lower bound on λ_- uniform in $(x,y) \in \mathbb{R}^2_+$, $x+y<b$: if $y \geq x$,

$$\frac{1}{2}\left[\frac{1}{x^2}+\frac{1}{y^2}+\frac{2b}{(b-(x+y))^2}-\sqrt{\left(\frac{1}{x^2}-\frac{1}{y^2}\right)^2+\frac{4b^2}{(b-(x+y))^4}}\right]$$
$$\geq \frac{1}{2}\left[\frac{1}{x^2}-\frac{1}{y^2}+\frac{2b}{(b-(x+y))^2}-\sqrt{\left(\frac{1}{x^2}-\frac{1}{y^2}\right)^2+\frac{4b^2}{(b-(x+y))^4}}\right]+\frac{1}{y^2}\geq\frac{1}{b^2}.$$

The same trick for $y<x$ shows that $\lambda_- \geq \frac{1}{b^2}$. Thus the Hessian matrix is positive definite and

$$\left|\mathcal{F}(x,y)-\mathcal{F}\left(x^0,y^0\right)\right| \geq C\left(\left|x-x^0\right|^2+\left|y-y^0\right|^2\right). \tag{6.92}$$

The latter (6.92) together with (6.28) implies for all $t>0$,

$$\begin{aligned}
0 &\leq \int_\Omega \left|A-A^0\right|^2(t) \leq C\int_\Omega\left[-\ln(\det A)-b\ln\left(1-\frac{\mathrm{Tr}(A)}{b}\right)+(b+d)\ln\left(\frac{b}{b+d}\right)\right](t)\\
&\leq C\,\mathrm{We}\,\|u_0\|^2_{L^2(\Omega)}\\
&\quad+\frac{\omega(b+d)}{2b}\int_\Omega\left[-\ln(\det A_0)-b\ln\left(1-\frac{\mathrm{Tr}(A_0)}{b}\right)+(b+d)\ln\left(\frac{b}{b+d}\right)\right].
\end{aligned} \tag{6.93}$$

Doing the same type of calculations for

$$\mathcal{H}:\ (x,y)\in\mathbb{R}^2_+\cap B(0,b)\longmapsto \frac{x+y}{\left(1-\frac{x+y}{b}\right)^2}-\frac{4}{1-\frac{x+y}{b}}+\frac{1}{x}+\frac{1}{y}.$$

We have, for all $(x,y)\in\mathbb{R}^2_+\cap B(0,b)$,

$$\partial_x\mathcal{H}(x,y)=\frac{1-\frac{4}{b}}{\left(1-\frac{x+y}{b}\right)^2}+\frac{2}{b}\frac{x+y}{\left(1-\frac{x+y}{b}\right)^3}-\frac{1}{x^2},$$
$$\partial_y\mathcal{H}(x,y)=\frac{1-\frac{4}{b}}{\left(1-\frac{x+y}{b}\right)^2}+\frac{2}{b}\frac{x+y}{\left(1-\frac{x+y}{b}\right)^3}-\frac{1}{y^2}$$

and these derivatives are zero at the point $(x^0,y^0)=\left(\frac{b}{b+2},\frac{b}{b+2}\right)$. Differentiating once more, we get

$$\mathrm{Hess}(\mathcal{H})(x,y)=\begin{pmatrix}\frac{X}{Y}+\frac{2}{x^3} & \frac{X}{Y}\\ \frac{X}{Y} & \frac{X}{Y}+\frac{2}{y^3}\end{pmatrix},$$

where

$$X:=\frac{4}{b}\left(1-\frac{2}{b}\right)+\frac{2}{b^2}\left(1+\frac{4}{b}\right)(x+y)\quad\text{and}\quad Y:=\left(1-\frac{x+y}{b}\right)^4.$$

with caracteristic polynomial

$$\left(\frac{X}{Y}+\frac{2}{x^3}-\lambda\right)\left(\frac{X}{Y}+\frac{2}{y^3}-\lambda\right)-\frac{X^2}{Y^2}$$
$$=\lambda^2-2\left[\frac{1}{x^3}+\frac{1}{y^3}+\frac{X}{Y}\right]\lambda+\frac{2X}{Y}\left(\frac{1}{x^3}+\frac{1}{y^3}\right)+\frac{4}{x^3y^3}. \tag{6.94}$$

The discriminant of (6.94) is equal to

$$4\left(\frac{1}{x^3}-\frac{1}{y^3}\right)^2+\frac{4X^2}{Y^2}\geq 0$$

and the two eigenvalues of the Hessian matrix are:

$$\lambda_+=\frac{1}{x^3}+\frac{1}{y^3}+\frac{X}{Y}+\sqrt{\left(\frac{1}{x^3}-\frac{1}{y^3}\right)^2+\frac{X^2}{Y^2}}>\frac{2}{b^2},$$
$$\lambda_-=\frac{1}{x^3}+\frac{1}{y^3}+\frac{X}{Y}-\sqrt{\left(\frac{1}{x^3}-\frac{1}{y^3}\right)^2+\frac{X^2}{Y^2}}.$$

Again, trace and determinant of the Hessian matrix being positive, we infer that its eigenvalues are positive. However, we need a lower bound on λ_- uniform in $(x,y)\in\mathbb{R}_+^2$, $x+y<b$: if $y\geq x$,

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{X}{Y}-\sqrt{\left(\frac{1}{x^3}-\frac{1}{y^3}\right)^2+\frac{X^2}{Y^2}}$$
$$\geq\frac{1}{x^3}-\frac{1}{y^3}+\frac{X}{Y}-\sqrt{\left(\frac{1}{x^3}-\frac{1}{y^3}\right)^2+\frac{X^2}{Y^2}}+\frac{2}{y^3}\geq\frac{2}{b^2}>0.$$

The same trick for $y<x$ shows that $\lambda_-\geq\frac{2}{b^2}$. Thus the Hessian matrix is positive definite and

$$\left|\mathcal{H}(x,y)-\mathcal{H}\left(x^0,y^0\right)\right|\geq C\left(\left|x-x^0\right|^2+\left|y-y^0\right|^2\right). \tag{6.95}$$

We conclude, using the decay of the free energy that for all $t>0$,

$$\begin{aligned}0&\leq\int_0^t\int_\Omega\left|A-A^0\right|^2\\&\leq C\int_0^t\int_\Omega\left[\frac{\operatorname{Tr}A}{\left(1-\frac{\operatorname{Tr}A}{b}\right)^2}-\frac{2d}{1-\frac{\operatorname{Tr}A}{b}}+\operatorname{Tr}\left(A^{-1}\right)\right]\\&\leq\frac{1}{2}\,\mathrm{We}^2\left\|u_0\right\|_{L^2(\Omega)}^2\\&\quad+\frac{\omega(b+2)}{2b}\,\mathrm{We}\int_\Omega\left[-\ln\left(\det A_0\right)-b\ln\left(1-\frac{\operatorname{Tr}(A_0)}{b}\right)+(b+2)\ln\left(\frac{b}{b+2}\right)\right].\end{aligned} \tag{6.96}$$

The bounds (6.93) and (6.96) imply now the estimates of Proposition 6.2.

Remark 6.9 (On the limit $b\to\infty$). This is the limit of infinitely extensible dumbbels. Unfortunately, we note that the constants above are not uniform in the parameter b. For instance, the constant C in (6.95) behaves like $O\left(\frac{1}{b^2}\right)$. In (6.37), the bound also blows up when $b\to\infty$. Therefore, we do not get, by these means, any information at the limit $b\to\infty$, neither on A, nor on τ. This limit happens to be very important from a mathematical viewpoint. Indeed, it is straightforward to see that the formal limit is the Oldroyd-B system, for which no existence result of weak solutions is known. We plan to carry out further investigations on this question.

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