IMPROVED REGULARITY IN BUMPY LIPSCHITZ DOMAINS

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ABSTRACT. This paper is devoted to the proof of Lipschitz regularity, down to the microscopic scale, for solutions of an elliptic system with highly oscillating coefficients, over a highly oscillating Lipschitz boundary. The originality of this result is that it does not assume more than Lipschitz regularity on the boundary. In particular, we bypass the use of the classical regularity theory. Our Theorem, which is a significant improvement of our previous work on Lipschitz estimates in bumpy domains, should be read as an improved regularity result for an elliptic system over a Lipschitz boundary. Our progress in this direction is made possible by an estimate for a boundary layer corrector. We believe that this estimate in the Sobolev-Kato class is of independent interest.

1. INTRODUCTION

This paper is devoted to the proof of Lipschitz regularity, down to the microscopic scale, for weak solutions $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^N$ of the elliptic system

$$\begin{cases} -\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon &= 0, \quad x \in D^\varepsilon_\psi(0,1), \\ u^\varepsilon &= 0, \quad x \in \Delta^\varepsilon_\psi(0,1), \end{cases} \tag{1.1}$$

over a highly oscillating Lipschitz boundary.

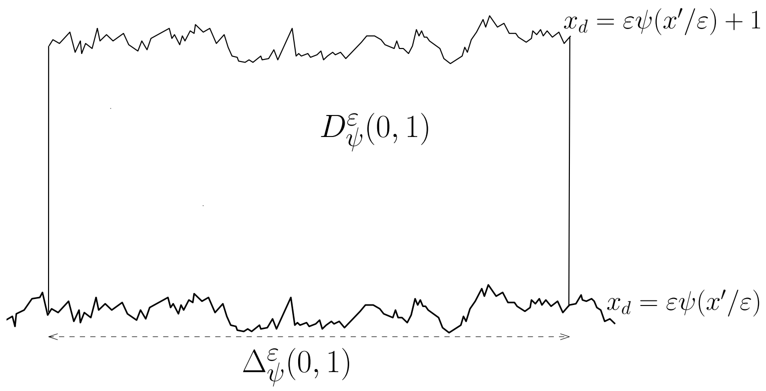

FIGURE 1. The domain $D^\varepsilon(0,1)$ and the portion of the boundary $\Delta^\varepsilon_\psi(0,1)$

Throughout this work, ψ is a Lipschitz graph,

$$D^\varepsilon_\psi(0,1) := \{x' \in (-1,1)^{d-1},\ \varepsilon\psi(x'/\varepsilon) < x_d < \varepsilon\psi(x'/\varepsilon) + 1\} \subset \mathbb{R}^d$$

and

$$\Delta^\varepsilon_\psi(0,1) := \{x' \in (-1,1)^{d-1},\ x_d = \varepsilon\psi(x'/\varepsilon)\}$$

is the lower highly oscillating boundary on which homogeneous Dirichlet boundary conditions are imposed (see Figure 1).

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1.1. **Statement of our results.** Our main theorem is the following.

Theorem 1. *There exists $C > 0$ such that for all $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$, for all matrix $A = A(y) = (A^{\alpha\beta}_{ij}(y)) \in \mathbb{R}^{d^2\times N^2}$, elliptic with constant λ, 1-periodic and Hölder continuous with exponant $\nu > 0$, for all $0 < \varepsilon < 1/2$, for all weak solutions u^ε to* (1.1)*, for all $r \in [\varepsilon, 1/2]$*

$$\int_{(-r,r)^{d-1}} \int_{\varepsilon\psi(x'/\varepsilon)}^{\varepsilon\psi(x'/\varepsilon)+r} |\nabla u^\varepsilon|^2 dx_d dx' \leq Cr^d \int_{(-1,1)^{d-1}} \int_{\varepsilon\psi(x'/\varepsilon)}^{\varepsilon\psi(x'/\varepsilon)+1} |\nabla u^\varepsilon|^2 dx_d dx', \tag{1.2}$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}})$.

The uniform estimate of Theorem 1 should be read as an improved regularity result. Indeed, estimate (1.2) can be seen as a Lipschitz estimate down to the microscopic scale $O(\varepsilon)$. For an elliptic system over a slowly oscillating boundary $x_d = \psi(x')$ for ψ merely Lipschitz, it is of course not possible to get an improved regularity estimate. What we manage to prove here is that the highly oscillating Lipschitz boundary $x_d = \varepsilon\psi(x'/\varepsilon)$ being close in the limit to the flat boundary, system (1.1) inherits some regularity properties up to the scale $O(\varepsilon)$ of the limit system when $\varepsilon \to 0$.

Theorem 1 represents a considerable improvement of a recent result obtained by the two authors, namely Result B and Theorem 16 in [KP15]. This first work dealt with uniform Lipschitz regularity over highly oscillating $C^{1,\nu}$ boundaries. As is classical in the proof of uniform estimates by compactness methods, we need to build (interior and boundary) correctors. Our breakthrough is made possible by estimating a boundary layer corrector $v = v(y)$ solution to the system

$$\left\{ \begin{aligned} -\nabla \cdot A(y)\nabla v &= 0, \quad y_d > \psi(y'), \\ v &= v_0, \quad y_d = \psi(y'), \end{aligned} \right. \tag{1.3}$$

in the Lipschitz half-space $y_d > \psi(y')$ with non localized Dirichlet boundary data v_0, without resorting to regularity theory.

Theorem 2. *Assume $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$ and $v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1})$ i.e.*

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \|v_0\|_{H^{1/2}(\xi+(0,1)^{d-1})} < \infty.$$

Then, there exists a unique weak solution v of (1.3) *such that*

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \int_{\xi+(0,1)^{d-1}} \int_{\psi(y')}^{\infty} |\nabla v|^2 dy_d dy' \leq C\|v_0\|^2_{H^{1/2}_{uloc}} < \infty, \tag{1.4}$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}})$.

This estimate enables to bypass the classical regularity theory for elliptic systems over $C^{1,\nu}$ boundaries, which is heavily relied on in the paper [KP15].

Originality of our results. Let us describe two main aspects of our work:

(1) the lack of regularity of the boundary, which is merely Lipschitz,
(2) the lack of structure of the oscillations of the bumpy boundary.

The originality of Theorem 1 lies in the fact that no smoothness of the boundary, which is just assumed to be Lipschitz, is needed for it to hold. Previous results in this direction, in particular our previous work [KP15], always relied on some smoothness of the boundary, typically $\psi \in C^{1,\nu}$ with $\nu > 0$, or $\psi \in C^1_\omega$ with ω a modulus of continuity satisfying a Dini type condition, i.e. $\int_0^1 \omega(t)/tdt < \infty$. The difficulty of dealing with Lipschitz boundaries lies in the fact that zooming in close to the boundary does not yield any improvement of flatness. Therefore, classical regularity theory (for instance Schauder theory) is not effective. The theory for boundary value problems in Lipschitz domains, not based on regularity, is very dissimilar from the theory in $C^{1,\nu}$ domains. Work on boundary value problems in Lipschitz domains, in particular the development of potential theory, has started in the late 70's and the 80's with seminal works by Dahlberg [Dah77, Dah79], Dahlberg and Kenig [DK87] and Jerison and Kenig [JK81]. Recent progress toward uniform estimates for systems with oscillating coefficients has been achieved by Kenig and Shen [KS11], and Shen [She15a]. Notice in particular that estimate (1.2) of Theorem 1 implies

$$\int_{(-1/2,1/2)^{d-1}} \int_{\varepsilon\psi(x'/\varepsilon)}^{\varepsilon\psi(x'/\varepsilon)+\varepsilon} |\nabla u^\varepsilon|^2 dx_d dx' \leq C\varepsilon \int_{(-1,1)^{d-1}} \int_{\varepsilon\psi(x'/\varepsilon)}^{\varepsilon\psi(x'/\varepsilon)+1} |\nabla u^\varepsilon|^2 dx_d dx',$$

with a constant C uniform in ε. This is the so-called Rellich estimate (over a bumpy Lipschitz boundary), which is the keystone of the potential theory in Lipschitz domains.

Pioneering work on uniform estimates in homogenization has been achieved by Avellaneda and Lin in the late 80's [AL87a, AL87b, AL89a, AL89b, AL91]. The regularity theory for operators with highly oscillating coefficients has recently attracted a lot of attention, and important contributions have been made to relax the structure assumptions on the oscillations [AS14a, AS14b, GNO14]. Our work is in a different vein. It is focused on the boundary behavior of solutions. Of course, one can flatten the boundary, and put the oscillations of the boundary into the coefficients. Here, nothing is prescribed on the boundary (except that it is Lipschitz and bounded). Notably, we do not prescribe any structure assumption on the oscillations of the boundary: ψ is neither periodic, nor quasiperiodic, nor stationary ergodic. It oscillates in a completely unprescribed way. This is the main difference with the recent developments on interior estimates in homogenization, which always assume some structure on the oscillations of the coefficients. To conclude, it is remarkable that the existence of an appropriate boundary layer corrector can be proved in such generality, when the existence of bounded interior correctors in homogenization relies on some structure: see for instance [Koz78, She15b, AGK15] for almost periodic structures and [GNO15] for random structures.

Overview of the paper. In section 2 we recall several results related to Sobolev-Kato spaces, homogenization and uniform Lipschitz estimates. These results are of constant use in our work. Then the paper has two main parts. The first aim is to prove Theorem 2 about the well-posedness of the boundary layer system in a space of non localized energy over a Lipschitz boundary. The key idea is to carry out a domain decomposition. Subsequently, there are three steps. Firstly, we prove the well-posedness of the boundary layer system over a flat boundary, namely in the domain $\mathbb{R}^d_+$. This is done in section 3. Secondly, we define and estimate a Dirichlet to Neumann operator over $H^{1/2}_{uloc}$. This key tool is introduced in section 4. Thirdly, we show that proving the well-posedness of the boundary layer system over a Lipschitz boundary boils down to analyzing a problem in a layer $\{\psi(y') < y_d < 0\}$ close to the boundary. The energy estimates for this problem are carried out in section 5. Eventually in section 6, and this is the last part of this work, we are able to prove Theorem 1 using a compactness scheme.

Framework and notations. Let $\lambda > 0$ and $0 < \nu < 1$ be fixed in what follows. We assume that the coefficients matrix $A = A(y) = (A^{\alpha\beta}_{ij}(y))$, with $1 \leq \alpha,\ \beta \leq d$ and $1 \leq i,\ j \leq N$ is real, that

$$A \in C^{0,\nu}(\mathbb{R}^d), \tag{1.5}$$

that A is uniformly elliptic i.e.

$$\lambda|\xi|^2 \leq A^{\alpha\beta}_{ij}(y)\xi^\alpha_i\xi^\beta_j \leq \frac{1}{\lambda}|\xi|^2, \quad \text{for all } \xi = (\xi^\alpha_i) \in \mathbb{R}^{dN},\ y \in \mathbb{R}^d \tag{1.6}$$

and periodic i.e.

$$A(y+z) = A(y), \quad \text{for all } y \in \mathbb{R}^d,\ z \in \mathbb{Z}^d. \tag{1.7}$$

We say that A belongs to the class $\mathcal{A}^\nu$ if A satisfies (1.5), (1.6) and (1.7).

For easy reference, we summarize here the standard notations used throughout the text. For $x \in \mathbb{R}^d$, $x = (x', x_d)$, so that $x' \in \mathbb{R}^{d-1}$ denotes the $d-1$ first components of the vector x. For $\varepsilon > 0$, $r > 0$, let

$$\begin{aligned}
D^\varepsilon_\psi(0,r) &:= \left\{(x', x_d),\ |x'| < r,\ \varepsilon\psi(x'/\varepsilon) < x_d < \varepsilon\psi(x'/\varepsilon) + r\right\},\\
\Delta^\varepsilon_\psi(0,r) &:= \left\{(x', x_d),\ |x'| < r,\ x_d = \varepsilon\psi(x'/\varepsilon)\right\},\\
D_0(0,r) &:= \left\{(x', x_d),\ |x'| < r,\ 0 < x_d < r\right\}, \qquad \Delta_0(0,r) := \left\{(x', 0),\ |x'| < r\right\},\\
\mathbb{R}^d_+ &:= \mathbb{R}^{d-1} \times (0,\infty), \qquad \Omega_+ := \{\psi(y') < y_d\},\\
\Omega_\flat &:= \{\psi(y') < y_d < 0\}, \qquad \Sigma_k := (-k,k)^{d-1},
\end{aligned}$$

where $|x'| = \max_{i=1,\dots\, d-1} |x_i|$. In the whole paper, we always use the max norm of $\mathbb{R}^{d-1}$ or $\mathbb{R}^d$, so that all the balls are square, not round. We sometimes write $D_\psi(0,r)$ and $\Delta_\psi(0,r)$ in short for $D^1_\psi(0,r)$ and $\Delta^1_\psi(0,r)$; in that situation the boundary is not highly oscillating because $\varepsilon = 1$. Let also

$$(\overline{u})_{D^\varepsilon_\psi(0,r)} := \fint_{D^\varepsilon_\psi(0,r)} u = \frac{1}{|D^\varepsilon_\psi(0,r)|}\int_{D^\varepsilon_\psi(0,r)} u.$$

The Lebesgue measure of a set is denoted by $|\cdot|$. For a positive integer m, let also I_m denote the identity matrix $M_m(\mathbb{R})$. The function $\mathbf{1}_E$ denotes the characteristic function of a set E. The notation η usually stands for a cut-off function. Ad hoc definitions are given when needed. Unless stated otherwise, the duality product $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathcal{D}',\mathcal{D}}$ always denotes the duality between $\mathcal{D}(\mathbb{R}^{d-1}) = C^\infty_0(\mathbb{R}^{d-1})$ and $\mathcal{D}'$. The space of measurable functions ψ such that

$$\|\psi\|_{L^\infty(\mathbb{R}^{d-1})} + \|\nabla\psi\|_{L^\infty(\mathbb{R}^{d-1})} < \infty$$

is denoted by $W^{1,\infty}(\mathbb{R}^{d-1})$. In the sequel, $C > 0$ is always a constant uniform in ε which may change from line to line.

2. Preliminaries

2.1. On Sobolev-Kato spaces. For $s \geq 0$, we define the Sobolev-Kato space $H^s_{uloc}(\mathbb{R}^{d-1})$ of functions of non localized H^s energy by

$$H^s_{uloc}(\mathbb{R}^{d-1}) := \left\{ u \in H^s_{loc}(\mathbb{R}^d),\ \sup_{\xi\in\mathbb{Z}^{d-1}} \|u\|_{H^s(\xi+(0,1)^{d-1})} < \infty \right\}.$$

We will mainly work with $H^{1/2}_{uloc}$. The following lemma is a useful tool to compare the $H^{1/2}_{uloc}$ norm to the $H^{1/2}$ norm of a $H^{1/2}(\mathbb{R}^{d-1})$ function.

Lemma 3. *Let $\eta \in C^\infty_c(\mathbb{R}^{d-1})$ and $v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1})$. Assume that* $\operatorname{Supp}\eta \subset B(0,R)$, *for $R > 0$. Then,*

$$\|\eta v_0\|_{H^{1/2}} \leq CR^{\frac{d-1}{2}} \|v_0\|_{H^{1/2}_{uloc}}, \tag{2.1}$$

with $C = C(d, \|\eta\|_{W^{1,\infty}})$.

For a proof, we refer to the proof of Lemma 2.26 in [DP14].

2.2. Homogenization and weak convergence. We recall the standard weak convergence result in periodic homogenization for a fixed domain Ω. As usual, the constant homogenized matrix $\overline{A} = \overline{A}^{\alpha\beta} \in M_N(\mathbb{R})$ is given by

$$\overline{A}^{\alpha\beta} := \int_{\mathbb{T}^d} A^{\alpha\beta}(y)dy + \int_{\mathbb{T}^d} A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y)dy, \tag{2.2}$$

where the family $\chi = \chi^\gamma(y) \in M_N(\mathbb{R})$, $y \in \mathbb{T}^d$, solves the cell problems

$$-\nabla_y \cdot A(y)\nabla_y \chi^\gamma = \partial_{y_\alpha} A^{\alpha\gamma},\ y \in \mathbb{T}^d \qquad \text{and} \qquad \int_{\mathbb{T}^d} \chi^\gamma(y)dy = 0. \tag{2.3}$$

Theorem 4 (weak convergence). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $u_k \in H^1(\Omega)$ be a sequence of weak solutions to*

$$-\nabla \cdot A_k(x/\varepsilon_k)\nabla u_k = f_k \in (H^1(\Omega))',$$

where $\varepsilon_k \to 0$ and the matrices $A_k = A_k(y) \in L^\infty$ satisfy (1.6) and (1.7). Assume that there exist $f \in (H^1(\Omega))'$ and $u_k \in H^1(\Omega)$, such that $f_k \longrightarrow f$ strongly in $(H^1(\Omega))'$, $u_k \to u_0$ strongly in $L^2(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u^0$ weakly in $L^2(\Omega)$. Also assume that the constant matrix $\overline{A_k}$ defined by (2.2) with A replaced by A_k converges to a constant matrix A^0. Then

$$A_k(x/\varepsilon_k)\nabla u_k \rightharpoonup A^0\nabla u^0 \quad \textit{weakly in } L^2(\Omega)$$

and

$$\nabla \cdot A^0 \nabla u^0 = f \in (H^1(\Omega))'.$$

For a proof, which relies on the classical oscillating test function argument, we refer for instance to [KLS13, Lemma 2.1]. This is an interior convergence result, since no boundary condition is prescribed on u_k.

2.3. **Uniform estimates in homogenization and applications.** We recall here the boundary Lipschitz estimate proved by Avellaneda and Lin in [AL87a].

Theorem 5 (Lipschitz estimate, [AL87a, Lemma 20]). *For all $\kappa > 0$, $0 < \mu < 1$, there exists $C > 0$ such that for all $\psi \in C^{1,\nu}(\mathbb{R}^{d-1}) \cap W^{1,\infty}(\mathbb{R}^{d-1})$, for all $A \in \mathcal{A}^\nu$, for all $r > 0$, for all $\varepsilon > 0$, for all $f \in L^{d+\kappa}(D_\psi(0,r))$, for all $F \in C^{0,\mu}(D_\psi(0,r))$, for all $u^\varepsilon \in L^\infty(D_\psi(0,r))$ weak solutions to*

$$\begin{cases} -\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon &= f + \nabla \cdot F \quad x \in D_\psi(0,r), \\ \qquad\qquad u^\varepsilon &= 0, \qquad\qquad x \in \Delta_\psi(0,r), \end{cases}$$

the following estimate holds

$$\|\nabla u^\varepsilon\|_{L^\infty(D_\psi(0,r/2))} \leq C\left\{ r^{-1}\|u^\varepsilon\|_{L^\infty(D_\psi(0,r))} + r^{1-d/(d+\kappa)}\|f\|_{L^{d+\kappa}(D_\psi(0,r))} + r^\mu\|F\|_{C^{0,\mu}(D_\psi(0,r))} \right\}. \tag{2.4}$$

Notice that $C = C(d, N, \lambda, \kappa, \mu, \|\psi\|_{W^{1,\infty}}, [\nabla\psi]_{C^{0,\nu}}, [A]_{C^{0,\nu}})$.

As stated in our earlier work [KP15], this estimate does not cover the case of highly oscillating boundaries, since the constant in (2.4) involves the $C^{0,\nu}$ semi-norm of $\nabla\psi$.

In this work, we rely on Theorem 5 to get large-scale pointwise estimates on the Poisson kernel $P = P(y, \tilde{y})$ associated to the domain $\mathbb{R}^d_+$ and to the operator $-\nabla \cdot A(y)\nabla$.

Proposition 6. *For all $d \geq 2$, there exists $C > 0$, such that for all $A \in \mathcal{A}^\nu$, we have:*

(1) for all $y \in \mathbb{R}^d_+$, for all $\tilde{y} \in \mathbb{R}^{d-1} \times \{0\}$, we have

$$|P(y,\tilde{y})| \leq \frac{Cy_d}{|y-\tilde{y}|^d}, \tag{2.5}$$

$$|\nabla_y P(y,\tilde{y})| \leq \frac{C}{|y-\tilde{y}|^d}, \tag{2.6}$$

(2) for all $y,\ \tilde{y} \in \mathbb{R}^{d-1} \times \{0\}$, $y \neq \tilde{y}$,

$$|\nabla_y P(y,\tilde{y})| \leq \frac{C}{|y-\tilde{y}|^d}. \tag{2.7}$$

Notice that $C = C(d, N, \lambda, [A]_{C^{0,\nu}})$.

The proof of those estimates starting from the uniform Lipschitz estimate of Theorem 5 is standard (see for instance [AL87a]).

3. Boundary layer corrector in a flat half-space

This section is devoted to the well-posedness of the boundary layer problem

$$\left\{ \begin{array}{rll} -\nabla\cdot A(y)\nabla v &= 0, & y_d>0,\\ v &= v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1}), & y_d=0, \end{array}\right. \tag{3.1}$$

in the flat half-space $\mathbb{R}^d_+$.

Proposition 7. *Assume $v_0\in H^{1/2}_{uloc}(\mathbb{R}^{d-1})$. Then, there exists a unique weak solution v of (3.1) such that*

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \int_{\xi+(0,1)^{d-1}}\int_0^\infty |\nabla v|^2 dy_d dy' \leq C\|v_0\|^2_{H^{1/2}_{uloc}} < \infty, \tag{3.2}$$

with $C=C(d,N,\lambda,[A]_{C^{0,\nu}})$.

The proof is in three steps: (i) we define a function v and prove it is a weak solution to (3.1), (ii) we prove that the solution we have defined satisfies the estimate (3.2), (iii) we prove uniqueness of solutions verifying (3.2).

3.1. Existence of a weak solution.

Let $\eta\in C^\infty_c(\mathbb{R})$ a cut-off function such that

$$\eta\equiv 1 \text{ on } (-1,1), \qquad 0\leq\eta\leq 1, \qquad \|\eta'\|_{L^\infty}\leq 2. \tag{3.3}$$

Let $y_*\in\mathbb{R}^d_+$ be fixed. Notice that

$$\eta(|\cdot - y'_*|)\in C^\infty_c(\mathbb{R}^{d-1}), \quad \eta(|\cdot - y'_*|)\equiv 1 \text{ on } B(y'_*,1), \quad 0\leq \eta(|\cdot - y'_*|)\leq 1 \quad \text{and} \quad \|\nabla(\eta(|\cdot - y'_*|))\|_{L^\infty}\leq 2.$$

We define

$$v(y_*) := v^\sharp(y_*) + v^\flat(y_*), \tag{3.4}$$

where for $y\in\mathbb{R}^d_+$,

$$v^\sharp(y) := \int_{\mathbb{R}^{d-1}\times\{0\}} P(y,\tilde y)(1-\eta(|\tilde y' - y'_*|))v_0(\tilde y')d\tilde y,$$

and $v^\flat = v^\flat(y)\in H^1(\mathbb{R}^d_+)$ is the unique weak solution to

$$\left\{ \begin{array}{rll} -\nabla\cdot A(y)\nabla v^\flat &= 0, & y_d>0,\\ v^\flat &= \eta(|y'-y'_*|)v_0(y') \in H^{1/2}(\mathbb{R}^{d-1}), & y_d=0, \end{array}\right.$$

satisfying

$$\int_{\mathbb{R}^d_+} |\nabla v^\flat|^2 dy' dy_d \leq C\|\eta v_0\|^2_{H^{1/2}}, \tag{3.5}$$

with $C=C(d,N,\lambda)$. First of all, one has to prove that the definition of v does not depend on the choice of the cut-off η. Let $\eta_1,\ \eta_2\in C^\infty_c(\mathbb{R})$ be two cut-off functions satisfying (3.3). We denote by $v_1(y_*)$ and $v_2(y_*)$ the associated vectors defined by

$$v_1(y_*) := \int_{\mathbb{R}^{d-1}\times\{0\}} P(y_*,\tilde y)(1-\eta_1(|\tilde y' - y'_*|)v_0(\tilde y')d\tilde y + v_1^\flat(y_*),$$

$$v_2(y_*) := \int_{\mathbb{R}^{d-1}\times\{0\}} P(y_*,\tilde y)(1-\eta_2(|\tilde y' - y'_*|))v_0(\tilde y')d\tilde y + v_2^\flat(y_*).$$

Substracting, we get

$$v_1(y_*) - v_2(y_*) = \int_{\mathbb{R}^{d-1}\times\{0\}} P(y_*,\tilde y)(\eta_2(|\tilde y' - y'_*|) - \eta_1(|\tilde y' - y'_*|))v_0(\tilde y')d\tilde y + v_1^\flat(y_*) - v_2^\flat(y_*). \tag{3.6}$$

Now since

$$y \longmapsto \int_{\mathbb{R}^{d-1}\times\{0\}} P(y,\tilde{y})(\eta_2(|\tilde{y}'-y'_*|)-\eta_1(|\tilde{y}'-y'_*|))v_0(\tilde{y}')d\tilde{y}$$

is the unique solution to

$$\begin{cases} -\nabla\cdot A(y)\nabla v^\flat &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad y_d>0,\\ \qquad\quad v^\flat &= (\eta_2(|y'-y'_*|)-\eta_1(|y'-y'_*|))v_0(y')\in H^{1/2}(\mathbb{R}^{d-1}),\quad y_d=0,\end{cases}$$

the difference in (3.6) has to be zero, which proves that our definition of v is independent of the choice of η.

It remains to prove that $v=v(y)$ defined by (3.4) is actually a weak solution to (3.1). Let $\varphi_\diamond=\varphi_\diamond(y')\in C^\infty_c(\mathbb{R}^{d-1})$ and $\varphi_d=\varphi_d(y_d)\in C^\infty_c((0,\infty))$. We choose $\eta\in C^\infty_c(\mathbb{R})$ satisfying (3.3) and such that $\eta(|\cdot|)\equiv 1$ on $\operatorname{Supp}\varphi_\diamond+B(0,1)$. We aim at proving

$$\int_{\mathbb{R}^d_+} v(y)\left(-\nabla\cdot A^*(y)\nabla(\varphi_\diamond\varphi_d)\right)dy=0.$$

This relation is clear for $v^\flat$. For $v^\sharp$, by Fubini and then integration by parts

$$\begin{aligned}&\int_{\mathbb{R}^d_+} v^\sharp(y)\left(-\nabla\cdot A^*(y)\nabla(\varphi_\diamond\varphi_d)\right)dy\\ &=\int_{\operatorname{Supp}\varphi_\diamond\times\operatorname{Supp}\varphi_d}\int_{\mathbb{R}^{d-1}\times\{0\}} P(y,\tilde{y})(1-\eta(\tilde{y}))v_0(\tilde{y}')\left(-\nabla\cdot A^*(y)\nabla\varphi_\diamond\varphi_d\right)d\tilde{y}dy\\ &=\int_{\mathbb{R}^{d-1}\times\{0\}}\int_{\operatorname{Supp}\varphi_\diamond\times\operatorname{Supp}\varphi_d} P(y,\tilde{y})\left(-\nabla\cdot A^*(y)\nabla(\varphi_\diamond\varphi_d)\right)dy(1-\eta(\tilde{y}))v_0(\tilde{y}')d\tilde{y}\\ &=\int_{\mathbb{R}^{d-1}\times\{0\}}\left\langle -\nabla\cdot A(y)\nabla P(y,\tilde{y}),\varphi_\diamond\varphi_d\right\rangle(1-\eta(\tilde{y}))v_0(\tilde{y}')d\tilde{y}=0.\end{aligned}$$

3.2. **Gradient estimate.** Let $\varphi_\diamond=\varphi_\diamond(y')\in C^\infty_c(\mathbb{R}^{d-1})$ and $\varphi_d=\varphi_d(y_d)\in C^\infty_c((0,\infty))$. We choose $R>1$ such that $\operatorname{Supp}\varphi_\diamond+B(0,1)\subset B(0,R)$. Our goal is to prove

$$\left|\int_{\mathbb{R}^d_+}\nabla v(y)\varphi_\diamond\varphi_d(y)dy\right|\leq CR^{\frac{d-1}{2}}\|v_0\|_{H^{1/2}_{uloc}}\|\varphi_\diamond\|_{L^2}\|\varphi_d\|_{L^2},$$

with $C=C(d,N,\lambda,[A]_{C^{0,\nu}})$. This estimate clearly implies the bound (3.2). Let $\eta\in C^\infty_c(\mathbb{R})$ such that (3.3)

$$\eta(|\cdot|)\equiv 1 \text{ on } B(0,R)\quad\text{and}\quad \operatorname{Supp}\eta(|\cdot|)\subset B(0,2R).$$

Combining (3.5) and the result of Lemma 3, we get

$$\int_{\mathbb{R}^d_+}|\nabla v^\flat|^2dy'dy_d\leq CR^{d-1}\|v_0\|^2_{H^{1/2}_{uloc}},$$

with $C=C(d,N,\lambda)$.

It remains to estimate

$$\int_{\mathbb{R}^d_+}\nabla v^\sharp(y)\varphi_\diamond\varphi_d(y)dy=\int_0^1\int_{\mathbb{R}^{d-1}}\nabla v^\sharp(y)\varphi_\diamond\varphi_d(y)dy'dy_d+\int_1^\infty\int_{\mathbb{R}^{d-1}}\nabla v^\sharp(y)\varphi_\diamond\varphi_d(y)dy'dy_d.$$

To estimate these terms we rely on the the bound (2.6): for all $y\in\mathbb{R}^d_+$, $\tilde{y}\in\mathbb{R}^{d-1}\times\{0\}$,

$$|\nabla_y P(y,\tilde{y})|\leq\frac{C}{|y-\tilde{y}|^d}=\frac{C}{(y_d^2+|y'-\tilde{y}'|^2)^{d/2}},$$

with $C=C(d,N,\lambda,[A]_{C^{0,\nu}})$.

We begin with two useful estimates. We have on the one hand for all $y' \in \mathbb{R}^{d-1}$ such that $y' + B(0,1) \subset B(0,R)$,

(3.7)

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \frac{1}{|y'-\tilde{y}'|^d}(1-\eta(|\tilde{y}'|))|v_0(\tilde{y}')|^2 d\tilde{y}' &= \int_{\mathbb{R}^{d-1}} \frac{1}{|\tilde{y}'|^d}(1-\eta(|y'-\tilde{y}'|))|v_0(y'-\tilde{y}')|^2 d\tilde{y}' \\
&\leq \int_{\mathbb{R}^{d-1}\setminus B(0,1)} \frac{1}{|\tilde{y}'|^d}|v_0(y'-\tilde{y}')|^2 d\tilde{y}' \\
&\leq \sum_{\xi\in\mathbb{Z}^{d-1}\setminus\{0\}} \frac{1}{|\xi|^d}\|v_0\|^2_{L^2_{uloc}} \leq C\|v_0\|^2_{L^2_{uloc}}
\end{aligned}$$

and on the other hand for all $(y', y_d) \in \mathbb{R}^d_+$,

(3.8)

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \frac{1}{(y_d^2+|y'-\tilde{y}'|^2)^{d/2}}(1-\eta(|\tilde{y}'|))|v_0(\tilde{y}')|^2 d\tilde{y}' &\leq \int_{\mathbb{R}^{d-1}} \frac{1}{(y_d^2+|y'-\tilde{y}'|^2)^{d/2}}|v_0(\tilde{y}')|^2 d\tilde{y}' \\
&\leq \int_{\mathbb{R}^{d-1}} \frac{1}{(y_d^2+|y'-\tilde{y}'|^2)^{d/2}} d\tilde{y}'\|v_0\|^2_{L^2_{uloc}} \\
&\leq \frac{C}{y_d}\|v_0\|^2_{L^2_{uloc}}.
\end{aligned}$$

Using (3.7), we get

$$\begin{aligned}
&\left|\int_0^1 \int_{\mathbb{R}^{d-1}} \nabla v^\sharp(y)\varphi_\diamond\varphi_d(y)dy'dy_d\right| = \left|\int_0^1 \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}\times\{0\}} \nabla_y P(y,\tilde{y})(1-\eta(|\tilde{y}'|))v_0(\tilde{y}')d\tilde{y}\varphi_\diamond\varphi_d(y)dy'dy_d\right| \\
&\leq \int_0^1 \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1-\eta(|\tilde{y}'|)}{|y'-\tilde{y}'|^d}d\tilde{y}'\right)^{1/2}\left(\int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1-\eta(|\tilde{y}'|)}{|y'-\tilde{y}'|^d}|v_0(\tilde{y}')|^2 d\tilde{y}\right)^{1/2} |\varphi_\diamond\varphi_d(y)|dy'dy_d \\
&\leq C\|v_0\|_{L^2_{uloc}} \int_0^1 \int_{\mathbb{R}^{d-1}} \left(\int_1^\infty \frac{1}{r^2}\right)^{1/2} |\varphi_\diamond\varphi_d(y)|dy'dy_d \\
&\leq C\|v_0\|_{L^2_{uloc}} \int_0^1 \int_{\mathbb{R}^{d-1}} |\varphi_\diamond\varphi_d(y)|dy'dy_d \leq CR^{\frac{d-1}{2}}\|v_0\|_{L^2_{uloc}}\|\varphi_\diamond\|_{L^2}\|\varphi_d\|_{L^2}.
\end{aligned}$$

Using (3.8), we infer

$$\begin{aligned}
&\left|\int_1^\infty \int_{\mathbb{R}^{d-1}} \nabla v^\sharp(y)\varphi_\diamond\varphi_d(y)dy'dy_d\right| \\
&\leq C\int_1^\infty \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1}{(y_d^2+|y'-\tilde{y}'|^2)^{d/2}}d\tilde{y}'\right)^{1/2} \\
&\qquad \left(\int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1}{(y_d^2+|y'-\tilde{y}'|^2)^{d/2}}|v_0(\tilde{y}')|^2 d\tilde{y}\right)^{1/2} |\varphi_\diamond\varphi_d(y)|dy'dy_d \\
&\leq C\|v_0\|_{L^2_{uloc}} \int_1^\infty \frac{1}{y_d}|\varphi_d(y_d)|dy_d \int_{\mathbb{R}^{d-1}} |\varphi_\diamond(y')|dy' \leq CR^{\frac{d-1}{2}}\|v_0\|_{L^2_{uloc}}\|\varphi_\diamond\|_{L^2}\|\varphi_d\|_{L^2}.
\end{aligned}$$

3.3. **Uniqueness.** By linearity, it is enough to prove uniqueness for $v = v(y)$ weak solution to

$$\begin{cases} -\nabla\cdot A(y)\nabla v &= 0, \quad y_d > 0, \\ v &= 0, \quad y_d = 0, \end{cases}$$

such that

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \int_{\xi+(0,1)^{d-1}} \int_0^\infty |\nabla v|^2 dy_d dy' \leq C < \infty. \tag{3.9}$$

Let k be a fixed integer. A rescaled version of the Lipschitz estimate of [AL87a] for the flat half-space reads for all $n > k$

$$\fint_{D(0,k)} |\nabla v|^2 \leq \fint_{D(0,n)} |\nabla v|^2 dy_d dy'.$$

Since v satisfies the bound (3.9), we get

$$\fint_{D(0,n)} |\nabla v|^2 dy \leq Cn^{d-1}/n^d = 1/n \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\fint_{D(0,k)} |\nabla v|^2 dy = 0.$$

We conclude that $v = 0$ on $D(0, k)$ by using Poincaré's inequality.

4. Estimates for a Dirichlet to Neumann operator

The Dirichlet to Neumann operator DN is crucial in the proof of the well-posedness of the elliptic system in the bumpy half-space (see section 5). The key idea there is to carry out a domain decomposition. The Dirichlet to Neumann map is the tool enabling this domain decomposition. Since we are working in spaces of infinite energy to be useful DN has to be defined on $H^{1/2}_{uloc}$. Similar studies have been carried out in [ABZ13] (context of water-waves), [GVM10] (2d Stokes system), [DP14] (3d Stokes-Coriolis system).

We first define the Dirichlet to Neumann operator on $H^{1/2}(\mathbb{R}^{d-1})$:

$$\mathrm{DN}: \ H^{1/2}(\mathbb{R}^{d-1}) \longrightarrow \mathcal{D}'(\mathbb{R}^{d-1}),$$

such that for any $v_0 \in H^{1/2}(\mathbb{R}^{d-1})$, for all $\varphi \in C^\infty_c(\mathbb{R}^{d-1})$,

$$\langle \mathrm{DN}(v_0), \varphi\rangle_{\mathcal{D}',\mathcal{D}} := \langle A(y)\nabla v \cdot e_d, \varphi\rangle_{\mathcal{D}',\mathcal{D}},$$

where v is the unique weak solution to

$$\left\{ \begin{array}{rll} -\nabla\cdot A(y)\nabla v &= 0, & y_d > 0, \\ v &= v_0 \in H^{1/2}(\mathbb{R}^{d-1}), & y_d = 0. \end{array}\right. \tag{4.1}$$

Proposition 8.

(1) For all $\varphi \in C^\infty_c(\overline{\mathbb{R}^d_+})$,

$$\langle \mathrm{DN}(v_0), \varphi|_{y_d=0}\rangle_{\mathcal{D}',\mathcal{D}} = \langle A(y)\nabla v\cdot e_d, \varphi|_{y_d=0}\rangle_{\mathcal{D}',\mathcal{D}} = -\int_{\mathbb{R}^d_+} A(y)\nabla v\cdot\nabla\varphi dy. \tag{4.2}$$

(2) For all $\varphi \in C^\infty_c(\mathbb{R}^{d-1})$,

$$\langle \mathrm{DN}(v_0), \varphi|_{y_d=0}\rangle_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}^{d-1}\times\{0\}}\int_{\mathbb{R}^{d-1}\times\{0\}} A(y)\nabla_y P(y,\tilde{y})\cdot e_d v_0(\tilde{y}) d\tilde{y}\varphi(y)dy. \tag{4.3}$$

For $y,\ \tilde{y} \in \mathbb{R}^{d-1}\times\{0\}$, let

$$K(y,\tilde{y}) := A(y)\nabla_y P(y,\tilde{y})\cdot e_d$$

be the kernel appearing in (4.3). Estimate (2.7) of Proposition 6 implies that

$$|K(y,\tilde{y})| \leq \frac{C}{|y-\tilde{y}|^d},$$

for any $y,\ \tilde{y} \in \mathbb{R}^{d-1}\times\{0\}$, $y \neq \tilde{y}$ with $C = C(d, N, \lambda, [A]_{C^{0,\nu}})$.

Both formulas in Proposition 8 follow from integration by parts. Because of (4.2), it is clear that for all $v_0 \in H^{1/2}(\mathbb{R}^{d-1})$, for all $\varphi \in C^\infty_c(\mathbb{R}^{d-1})$,

$$|\langle \mathrm{DN}(v_0), \varphi\rangle| \leq C\|v_0\|_{H^{1/2}}\|\varphi\|_{H^{1/2}}, \tag{4.4}$$

with $C = C(d, N, \lambda)$, so that $\mathrm{DN}(v_0)$ extends as a continuous operator on $H^{1/2}(\mathbb{R}^{d-1})$ into $H^{-1/2}(\mathbb{R}^{d-1})$. Another consequence of (4.2) is the following corollary.

Corollary 9. *For all* $v_0 \in H^{1/2}(\mathbb{R}^{d-1})$,

$$\langle \mathrm{DN}(v_0), v_0 \rangle = -\int_{\mathbb{R}^d_+} A(y)\nabla v \cdot \nabla v dy \leq 0,$$

where v *is the unique solution to* (4.1).

Our next goal is to extend the definition of DN to $v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1})$. We have to make sense of the duality product $\langle \mathrm{DN}(v_0), \varphi\rangle$. As for the definition of the solution to the flat half-space problem (see section 3), the basic idea is to use a cut-off function η to split the definition between one part $\langle \mathrm{DN}(\eta v_0), \varphi\rangle$ where $\eta v_0 \in H^{1/2}(\mathbb{R}^{d-1})$, and another part $\langle \mathrm{DN}((1-\eta)v_0), \varphi\rangle$ which does not see the singularity of the kernel $K(y, \tilde{y})$.

For $R > 1$, there exists $\eta \in C^\infty_c(\mathbb{R})$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } (-R, R), \quad \operatorname{Supp} \eta \subset (-R-1, R+1), \quad \|\eta'\|_{L^\infty} \leq 2.$$

Let $v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1})$. Let $R > 1$ and $\varphi \in C^\infty_c(\mathbb{R}^{d-1})$ such that $\operatorname{Supp}\varphi + B(0,1) \subset B(0,R)$. We define the action of $\mathrm{DN}(v_0)$ on φ by

$$\begin{aligned}(4.5)\quad \langle \mathrm{DN}(v_0), \varphi\rangle_{\mathcal{D}',\mathcal{D}} := {} & \langle \mathrm{DN}(\eta(|\cdot|)v_0), \varphi\rangle_{H^{-1/2},H^{1/2}} \\ & + \int_{\mathbb{R}^{d-1}\times\{0\}} \int_{\mathbb{R}^{d-1}\times\{0\}} K(y,\tilde{y})(1-\eta(|\tilde{y}'|))v_0(\tilde{y}')\varphi(y')d\tilde{y}dy.\end{aligned}$$

The fact that this definition does not depend on the cut-off $\eta \in C^\infty_c(\mathbb{R})$ follows from Proposition 8. The argument is similar to the one used in section 3.1.

The first term in the right-hand side of (4.5) is estimated using (4.4) and the bound of Lemma 3 between the $H^{1/2}$ norm of $\eta(|\cdot|)v_0$ and the $H^{1/2}_{uloc}$ norm of v_0. That yields

$$|\langle \mathrm{DN}(\eta(|\cdot|)v_0), \varphi\rangle| \leq C\|\eta(|\cdot|)v_0\|_{H^{1/2}}\|\varphi\|_{H^{1/2}} \leq CR^{\frac{d-1}{2}}\|v_0\|_{H^{1/2}_{uloc}}\|\varphi\|_{H^{1/2}},$$

with $C = C(d, N, \lambda)$.

We deal with the integral part in the right hand side of (4.5) in a way similar to the proof of the estimate (3.7). Using the fact that the supports of $(1-\eta(|y'|))v_0(y')$ on the one hand and φ on the other hand are disjoint, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^{d-1}\times\{0\}} \int_{\mathbb{R}^{d-1}\times\{0\}} K(y,\tilde{y})(1-\eta(|\tilde{y}'|))v_0(\tilde{y}')\varphi(y')d\tilde{y}dy\right| \\ &\leq C\int_{\mathbb{R}^{d-1}\times\{0\}} \int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1}{|y-\tilde{y}|^d}(1-\eta(|\tilde{y}'|))|v_0(\tilde{y}')||\varphi(y')|d\tilde{y}dy \\ &\leq C\int_{\mathbb{R}^{d-1}\times\{0\}} \left(\int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1}{|y-\tilde{y}|^d}(1-\eta(|\tilde{y}'|))d\tilde{y}\right)^{1/2} \\ &\qquad\qquad \left(\int_{\mathbb{R}^{d-1}\times\{0\}} \frac{1}{|y-\tilde{y}|^d}(1-\eta(|\tilde{y}'|))|v_0(\tilde{y}')|^2d\tilde{y}\right)^{1/2}|\varphi(y')|dy \\ &\leq C\int_{\mathbb{R}^{d-1}\times\{0\}} \left(\int_1^\infty \frac{1}{r^2}dr\right)^{1/2}|\varphi(y')|dy\|v_0\|_{L^2_{uloc}} \\ &\leq CR^{\frac{d-1}{2}}\|v_0\|_{L^2_{uloc}}\|\varphi\|_{L^2},\end{aligned}$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}})$.

These results are put together in the following proposition.

Proposition 10.

(1) For $v_0 \in H^{1/2}(\mathbb{R}^{d-1})$, for any $\varphi \in C_c^\infty(\mathbb{R}^{d-1})$, we have

$$|\langle \mathrm{DN}(v_0), \varphi\rangle| \leq C\|v_0\|_{H^{1/2}}\|\varphi\|_{H^{1/2}},$$

with $C = C(d, N, \lambda)$.

(2) For $v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1})$, for $R > 1$ and any $\varphi \in C_c^\infty(\mathbb{R}^{d-1})$ such that

$$\mathrm{Supp}\,\varphi + B(0,1) \subset B(0,R),$$

we have

$$|\langle \mathrm{DN}(v_0), \varphi\rangle| \leq CR^{\frac{d-1}{2}}\|v_0\|_{H^{1/2}_{uloc}}\|\varphi\|_{H^{1/2}}, \tag{4.6}$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}})$.

5. Boundary layer corrector in a bumpy half-space

This section is devoted to the well-posedness of the boundary layer problem

$$\begin{cases} -\nabla \cdot A(y)\nabla v &= 0, \qquad\qquad y_d > \psi(y'), \\ v &= v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1}), \quad y_d = \psi(y'), \end{cases} \tag{5.1}$$

in the bumpy half-space $\Omega_+ := \{y_d > \psi(y')\}$. For technical reasons, the boundary $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$ is assumed to be negative, i.e. $\psi(y') < 0$ for all $y' \in \mathbb{R}^{d-1}$. We prove Theorem 2 of the introduction which asserts the existence of a unique solution v in the class

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \int_{\xi+(0,1)^{d-1}} \int_{\psi(y')}^{\infty} |\nabla v|^2 dy_d dy' < \infty.$$

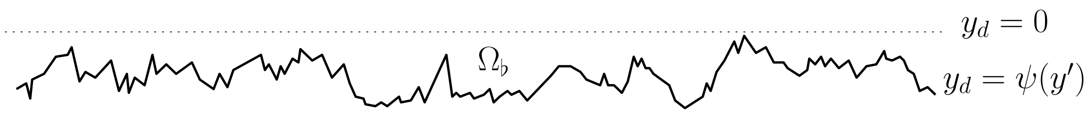

Figure 2. Splitting of the half-space Ω_+

The idea is to split the bumpy half-space into two subdomains (see Figure 2): a flat half-space $\mathbb{R}^d_+$ on the one hand and a bumpy channel $\Omega_\flat := \{\psi(y') < y_d < 0\}$ on the other hand. Both domains are connected by a transparent boundary condition involving the Dirichlet to Neumann operator DN defined in section 4. Therefore, solving (3.1) is equivalent to solving

$$\begin{cases} -\nabla \cdot A(y)\nabla v &= 0, \qquad\qquad 0 > y_d > \psi(y'), \\ v &= v_0 \in H^{1/2}_{uloc}(\mathbb{R}^{d-1}), \quad y_d = \psi(y'), \\ A(y)\nabla v \cdot e_d &= \mathrm{DN}(v|_{y_d=0}), \qquad y_d = 0. \end{cases} \tag{5.2}$$

This fact is stated in the following technical lemma.

Lemma 11. *If v is a weak solution of (5.2) in $\Omega_\flat$ such that*

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{0}|\nabla v|^2dy_ddy'<\infty,\qquad v|_{y_d=0^-}\in H^{1/2}_{uloc}(\mathbb{R}^{d-1}),\tag{5.3}$$

then $\tilde v$, defined by $\tilde v(y):=v(y)$ for $\psi(y')<y_d<0$ and $\tilde v|_{\mathbb{R}^d_+}$ is the unique solution to (3.1) with boundary condition $\tilde v|_{y_d=0^+}=v|_{y_d=0^-}$ given by Proposition 7, is a weak solution to (5.1). Moreover, the converse is also true. Namely, if v is a weak solution to (5.1) in Ω_+ such that

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{\infty}|\nabla v|^2dy_ddy'<\infty,$$

then $v|_{\{\psi(y')<y_d<0\}}$ is a weak solution to (5.2).

The main advantage of the domain decomposition is to make it possible to work in a channel, bounded in the vertical direction, in which one can rely on Poincaré type inequalities. Therefore our method is energy based, which makes it possible to deal with rough boundaries.

We now turn to the existence of a solution of (5.2) satisfying (5.3). We lift the boundary condition v_0. There exists V_0 such that

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{\infty}|V_0|^2+|\nabla V_0|^2dy_ddy'\leq C\|v_0\|^2_{H^{1/2}_{uloc}},$$

with $C=C(d,N,\|\psi\|_{W^{1,\infty}})$ and such that the trace of V_0 is v_0. Thus, $w:=v-V_0$ solves the system

$$\begin{cases}-\nabla\cdot A(y)\nabla w&=\nabla\cdot F,\qquad\qquad 0>y_d>\psi(y'),\\ \qquad\qquad w&=0,\qquad\qquad\quad y_d=\psi(y'),\\ A(y)\nabla w\cdot e_d&=\mathrm{DN}(w|_{y_d=0})+f,\quad y_d=0,\end{cases}\tag{5.4}$$

where

$$F:=A(y)\nabla V_0,$$
$$f:=\mathrm{DN}(V_0|_{y_d=0})-A(y)\nabla V_0\cdot e_d.$$

Notice that the source terms satisfy the following estimates:

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{0}|F|^2dy_ddy'\leq C\|v_0\|^2_{H^{1/2}_{uloc}},\tag{5.5}$$

with $C=C(d,N,\lambda,\|\psi\|_{W^{1,\infty}})$ and for all $\varphi\in C^\infty_c(\mathbb{R}^{d-1})$ such that $B(0,R)\subset\operatorname{Supp}\varphi\subset B(0,2R)$ for some $R>0$,

$$|\langle f,\varphi\rangle|\leq CR^{\frac{d-1}{2}}\|v_0\|_{H^{1/2}_{uloc}}\|\varphi\|_{H^{1/2}},\tag{5.6}$$

with $C=C(d,N,\lambda,[A]_{C^{0,\nu}},\|\psi\|_{W^{1,\infty}})$.

There are three steps in the proof of the well-posedness of (5.1).

Firstly, for $n\in\mathbb{N}$ we build approximate solutions $w_n=w_n(y)$ solving

$$\begin{cases}-\nabla\cdot A(y)\nabla w_n&=\nabla\cdot F,\qquad\qquad 0>y_d>\psi(y'),\\ \qquad\qquad w_n&=0,\qquad\qquad\quad \{y_d=\psi(y')\}\cup\{|y'|=n\},\\ A(y)\nabla w_n\cdot e_d&=\mathrm{DN}(w_n|_{y_d=0})+f,\quad y_d=0,\end{cases}\tag{5.7}$$

on $\Omega_{\flat,n}:=\{y'\in(-n,n)^{d-1},\ 0>y_d>\psi(y')\}$ and extend w_n by 0 on $\Omega_\flat\setminus\Omega_{\flat,n}$. We have that $w_n\in H^1(\Omega_\flat)$. For n fixed, this construction is classical. Indeed, using the positivity of the Dirichlet to Neumann operator (see (4.2)) an easy energy estimate yields the bound

$$\int_{\Omega_{\flat,n}}|\nabla w_n|^2dx\lesssim n^{d-1},$$

so that a Galerkin scheme makes it possible to conclude that w_n exists.

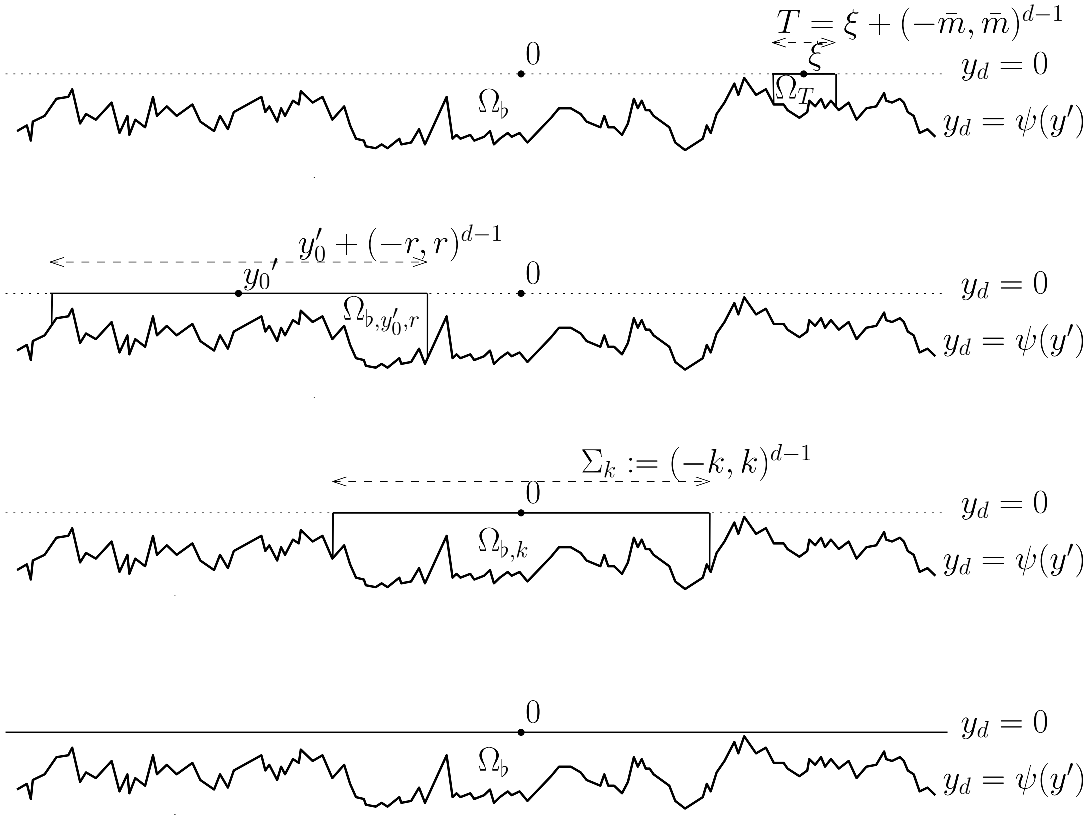

FIGURE 3. The channel $\Omega_\flat$, the subdomains $\Omega_{\flat,k}$ and $\Omega_{\flat,y_0',r}$, and the mid-size box Ω_T

Secondly, we aim at getting estimates uniform in n on w_n in the norm

$$\sup_{\xi \in \mathbb{Z}^{d-1}} \int_{\xi + (0,1)^{d-1}} \int_{\psi(y')}^{0} |\nabla w_n|^2 dy_d dy'. \tag{5.8}$$

This is done by carrying out so-called Saint-Venant estimates in the bounded channel. We close this step by using a hole-filling argument. The method has been pioneered by Ladyženskaja and Solonnikov [LS80] for the Navier-Stokes system in a bounded channel. Here the situation is more involved because of the non local operator DN on the upper boundary. The situation here is closer to [GVM10, DGV11] (2d Stokes system) and [DP14] (3d Stokes-Coriolis system).

Finally, one has to check that weak limits of w_n are indeed solutions of (5.4). This step is straightforward because of the linearity of the equations. Uniqueness follows from the Saint-Venant estimate of the second step, with zero source terms.

We focus on the second step, which is by far the most intricate one. We first introduce some notations for subdomains of $\Omega_\flat$, which are displayed on Figure 3. Let $r > 0$, $y_0' \in \mathbb{R}^{d-1}$ and

$$\Omega_{\flat,y_0',r} := \{|y' - y_0'| < r,\ 0 > y_d > \psi(y')\}.$$

Let $w_r \in H^1(\Omega_\flat)$ be a weak solution to

$$\left\{ \begin{aligned} -\nabla \cdot A(y) \nabla w_r &= \nabla \cdot F_r, && 0 > y_d > \psi(y'), \\ w_r &= 0, && y_d = \psi(y'), \\ A(y)\nabla w_r \cdot e_d &= \mathrm{DN}(w_r|_{y_d=0}) + f_r, && y_d = 0, \end{aligned} \right. \tag{5.9}$$

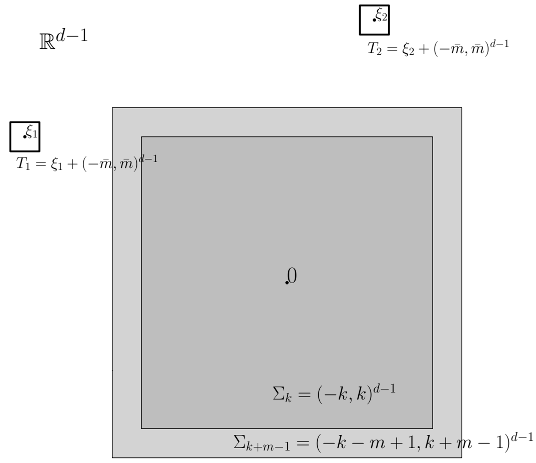

FIGURE 4. Two midsize cubes T_1 and T_2 of volume m^{d-1} belonging to $\mathcal{C}_{k,m}$

such that $w_r = 0$ on $\Omega_\flat \setminus \Omega_{\flat,y_0',r}$, and where

$$F_r := F\mathbf{1}_{\Omega_{\flat,y_0',r}}, \qquad f_r := f\mathbf{1}_{B(y_0',r)}.$$

Notice that w_n defined above (see (5.7)) is equal to w_r solution of (5.9) for $r := n$ and $y_0' = 0$. The reason w_r is introduced is that in section 5.1 we will need to translate the origin: w_r will be a translate of w_n. All estimates are carried out on w_r solving the system (5.9) so that we have the uniformity of constants both in r and y_0'.

For $k \in \mathbb{N}$, let

$$\Omega_{\flat,k} := \{y' \in (-k,k)^{d-1},\ 0 > y_d > \psi(y')\}.$$

Notice that $\Omega_{\flat,k} = \Omega_{\flat,0,k}$ with the notation above for $\Omega_{\flat,y_0',r}$. Our goal is to estimate,

$$E_k := \int_{\Omega_{\flat,k}} |\nabla w_r|^2 dy.$$

In order to deal with the non local character of the Dirichlet to Neumann operator, we rely on a careful splitting of $\mathbb{R}^{d-1}$ into midsize boxes of volume m^{d-1}.

In the following, for $k,\ m \in \mathbb{N}$, $k,\ m \geq 1$, $m = 2\bar m$ even,

$$\Sigma_k := (-k,k)^{d-1},$$

and the set $\mathcal{C}_{k,m}$ (see Figure 4) denotes the family of cubes T of volume m^{d-1} contained in $\mathbb{R}^{d-1} \setminus \Sigma_{k+m-1}$ with vertices in $\mathbb{Z}^{d-1}$, i.e.

$$\mathcal{C}_{k,m} := \left\{T = \xi + (-\bar m, \bar m)^{d-1},\ \xi \in \mathbb{Z}^{d-1} \text{ and } T \subset \mathbb{R}^{d-1} \setminus \Sigma_{k+m-1}\right\}.$$

Let also $\mathcal{C}_m$ be the family of all the cubes of volume m^{d-1} with vertices in $\mathbb{Z}^{d-1}$

$$\mathcal{C}_m := \left\{T = \xi + (-\bar m, \bar m)^{d-1},\ \xi \in \mathbb{Z}^{d-1}\right\}.$$

Notice that for $k \geq \hat k \geq \bar m$,

$$\mathcal{C}_{k,m} \subset \mathcal{C}_{\hat k,m} \subset \mathcal{C}_{\bar m,m} \subset \mathcal{C}_m.$$

For $T \in \mathcal{C}_{k,m}$,

$$E_T := \int_{\Omega_T} |\nabla w_r|^2 dy, \qquad \Omega_T := \{y' \in T,\ 0 > y_d > \psi(y')\}. \tag{5.10}$$

Proposition 12. *There exists a constant* $C^* = C^*(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}}, \|v_0\|_{H^{1/2}_{uloc}})$ *such that for all* $r > 0$*,* $y_0' \in \mathbb{R}^{d-1}$*, for all* $k,\ m \in \mathbb{N}$*,* $m \geq 3$ *and* $k \geq m/2 = \bar{m}$*, for any weak solutions* $w_r \in H^1(\Omega_\flat)$ *of* (5.9)*, the following bound holds*

$$E_k \leq C^* \left(k^{d-1} + E_{k+m} - E_k + \frac{k^{3d-5}}{m^{3d-3}} \sup_{T \in \mathcal{C}_{k,m}} E_T \right). \tag{5.11}$$

Notice that C^* *is independent of* r *and* y_0'*.*

The crucial point for the control of the large-scale energies in (5.11) is the fact that the power $3d-5$ of k is strictly smaller that the power $3d-3$ of m. Before tackling the proof of Proposition 12, let us explain how to infer from (5.11) an a priori bound uniform in n on w_n solution of (5.7).

5.1. **Proof of the a priori bound in the Sobolev-Kato space: downward induction.** Assume that the Saint-Venant estimate of Proposition 12 has been established. We come back to its proof in section 5.2 below. Our goal is to infer from (5.11) an a priori bound

$$E_k := \int_{\Omega_{\flat,k}} |\nabla w_n|^2 dy \lesssim k^{d-1} \tag{5.12}$$

uniform in n for w_n solution of (5.7). Let us stress that w_n is also a solution to (5.9) with $r = n$ and $y_0' = 0$. As explained above, this bound is all we need to get the existence of w solving (5.4). To show estimate (5.12), we prove (see Lemma 13 below) that the energy E_m contained in one elementary midsize box of volume m^{d-1} is bounded uniformly in m. The number m is an auxiliary parameter chosen thanks to the latitude allowed by the Saint-Venant estimate (5.11).

Proving an a priori bound would be much easier if the Saint-Venant estimate (5.11) did not involve the term

$$\frac{k^{3d-5}}{m^{3d-3}} \sup_{T \in \mathcal{C}_{k,m}} E_T,$$

whose reason for being there is the nonlocality of the Dirichlet to Neumann operator. For the sake of the explanation, assume temporarily that the Saint-Venant relation reads

$$E_k \leq C^* \left(k^{d-1} + E_{k+1} - E_k \right), \tag{5.13}$$

instead of (5.11). The auxiliary number which is in (5.11) to control the non local terms has no reason to appear in (5.13). For $k \geq n$, $E_{k+1} = E_k$, therefore

$$E_n \leq C^* n^{d-1}.$$

Now, using the hole filling trick, we get

$$\begin{aligned} E_{n-1} &\leq \frac{C^*}{C^*+1} \left\{ (n-1)^{d-1} + C^* n^{d-1} \right\}, \\ E_{n-k} &\leq \frac{C^*}{C^*+1} (n-k)^{d-1} + \ldots \left(\frac{C^*}{C^*+1} \right)^k (n-1)^{d-1} + \left(\frac{C^*}{C^*+1} \right)^k C^* n^{d-1}. \end{aligned}$$

Eventually,

$$E_1 \leq \left\{ \left(\frac{C^*}{C^*+1} \right)^{n-1} C^* n^{d-1} + \sum_{k=1}^{n} \left(\frac{C^*}{C^*+1} \right)^k k^{d-1} \right\} \tag{5.14}$$

which is bounded uniformly in n.

Let us go back to our actual estimate (5.11). The general idea of the downward induction is the same as in the simple case above. As expected yet, the non local terms make the

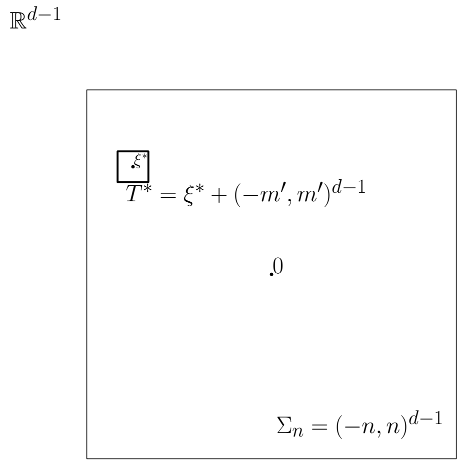

FIGURE 5. The channel $\Omega_\flat$ seen from *above*: midsize cube T^* of volume m^{d-1} such that Ω_{T^*} concentrates most energy among the midsize cubes $T \in \mathcal{C}_m$

induction trickier. Before going into the details, we fix some notation, and define the auxiliary integer m once for all.

Let C^* be given by Proposition 12, and let

(5.15)
$$A := (C^*+1)\sum_{k=1}^{\infty}\left(\frac{C^*}{C^*+1}\right)^k (2k-1)^{d-1} < \infty, \qquad B := \sum_{k=1}^{\infty}\left(\frac{C^*}{C^*+1}\right)^k (2k-1)^{3d-5} < \infty.$$

These two numbers are the analogues of the terms appearing in the right hand side of (5.14).

We now choose an integer m so that

$$m \geq 3, \quad m \text{ is even} \quad \text{and} \quad 1 - 2^{5-3d}\frac{B}{m^2} > \frac{1}{2}. \tag{5.16}$$

Notice that $m = m(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}}, \|v_0\|_{H^{1/2}_{uloc}})$, but is independent of r and y_0'. The reason for taking m even is technical; it is only used in the translation argument below. The reason for imposing the condition

$$1 - 2^{5-3d}\frac{B}{m^2} > \frac{1}{2}$$

is to be able to swallow the right hand side into the left hand side at the end of the iteration. We also take $n = lm = 2l\bar{m}$, with $l \in \mathbb{N}$, $l \geq 1$ and let w_n be the solution of (5.7).

How do the non local terms make the downward induction more complicated? The trouble comes from the fact that when iterating downward the boxes $\Omega_{\flat,k}$, on which the energy E_k is computed, are always centered at 0. Nevertheless, at the end of the iteration (down to $k = \bar{m}$) the energy $E_{\bar{m}}$ may not be comparable in any way to

$$\sup_{T \in \mathcal{C}_{\bar{m},m}} E_T,$$

which appears on the right hand side. The way out of this deadlock is to iterate downward taking into account another center ξ^* defined as follows. There exists $T^* \in \mathcal{C}_m$ such that $T^* \subset \Sigma_n$ and $E_{T^*} = \sup_{T\in\mathcal{C}_m} E_T$ (see Figure 5). Of course for all $k \geq \bar{m}$

$$\sup_{T\in\mathcal{C}_{k,m}} E_T \leq \sup_{T\in\mathcal{C}_{\bar{m},m}} E_T \leq \sup_{T\in\mathcal{C}_m} E_T = E_{T^*}.$$

By definition, there is $\xi^* \in \mathbb{Z}^{d-1}$ for which $T^* = \xi^* + (-\bar{m}, \bar{m})^{d-1}$.

In order to write the iteration, it is more convenient to first center T^* at zero. So we now simply translate the origin. Doing so, $w_n^*(y) := w_n(y' + \xi^*, y_d)$ is a solution of (5.9) with $y_0' := -\xi^*$, $r = n$ and

$$A^*(y) := A(y' + \xi^*, y_d), \quad \psi^*(y') := \psi(y' + \xi^*), \quad v_0^*(y') := v_0(y' + \xi^*),$$
$$F^*(y) := F(y' + y^*, y_d) \quad \text{and} \quad f^*(y') := f(y' + \xi^*).$$

Notice that

$$[A^*]_{C^{0,\mu}} = [A]_{C^{0,\nu}}, \quad \|\psi^*\|_{W^{1,\infty}} = \|\psi\|_{W^{1,\infty}} \quad \text{and} \quad \|v_0^*\|_{H^{1/2}_{uloc}} = \|v_0\|_{H^{1/2}_{uloc}},$$

so that w_n^* satisfies the Saint-Venant estimate (5.11) with the same constant C^*. Here we use the key fact that the Saint-Venant estimate (5.11) is uniform in y_0'. Furthermore, $E_{\bar{m}} = E_{T^*}$.

We are now ready to state the a priori bound and to prove it by downward induction.

Lemma 13. *We have the following a priori bound*

$$E_{\bar{m}} \le 2^{2-d} A m^{d-1},$$

where A is defined by (5.15).

The Lemma is obtained by downward induction, using a hole-filling type argument. Since w_n^* is supported in $\Omega_{\flat,2n}$, we start from k sufficiently large in (5.11). For $k = 2n + \bar{m} = (4l+1)\bar{m}$, estimate (5.11) implies

$$E_{(4l+1)\bar{m}} \le C^*((4l+1)\bar{m})^{d-1},$$

because $E_T = 0$ for any $T \in \mathcal{C}_{(4l+1)\bar{m},m}$. Then,

$$E_{(2(2l-1)+1)\bar{m}} = E_{(4l+1)\bar{m}-m} \le \frac{C^*}{C^*+1}(2(2l-1)+1)^{d-1}\bar{m}^{d-1} + \left(\frac{C^*}{C^*+1}\right) C^*(4l+1)^{d-1}\bar{m}^{d-1}.$$

Let $p \in \{0, \dots\ 2l-1\}$. We then have

$$\begin{aligned} E_{(2p+1)\bar{m}} \le{}& \frac{C^*}{C^*+1}(2p+1)^{d-1}\bar{m}^{d-1} \\ &+ \dots \left(\frac{C^*}{C^*+1}\right)^{2l-p-1}(4l-1)^{d-1}\bar{m}^{d-1} + \left(\frac{C^*}{C^*+1}\right)^{2l-p-1} C^*(4l+1)^{d-1}\bar{m}^{d-1} \\ &\qquad + \frac{2^{5-3d}}{m^2}\left[\frac{C^*}{C^*+1}(2p+1)^{3d-5} + \dots \left(\frac{C^*}{C^*+1}\right)^{2l-p}(4l+1)^{3d-5}\right] E_{\bar{m}}. \end{aligned}$$

Eventually, for $p = 0$

$$\begin{aligned} E_{\bar{m}} \le{}& \frac{C^*}{C^*+1}\bar{m}^{d-1} + \left(\frac{C^*}{C^*+1}\right)^2 (3\bar{m})^{d-1} \\ &+ \dots \left(\frac{C^*}{C^*+1}\right)^{2l-p-1}(4l-1)^{d-1}\bar{m}^{d-1} + \left(\frac{C^*}{C^*+1}\right)^{2l-p-1} C^*(4l+1)^{d-1}\bar{m}^{d-1} \\ &+ \frac{2^{5-3d}}{m^2}\left[\frac{C^*}{C^*+1} + \dots \left(\frac{C^*}{C^*+1}\right)^{2l-p}(4l+1)^{3d-5}\right] E_{\bar{m}} \le 2^{1-d} A m^{d-1} + \frac{2^{5-3d}}{m^2} B E_{\bar{m}}. \end{aligned}$$

Therefore,

$$\frac{E_{\bar{m}}}{2} < \left(1 - 2^{5-3d}\frac{B}{m^2}\right) E_{\bar{m}} \le 2^{1-d} A m^{d-1},$$

which proves Lemma 13.

Finally,

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{0}|\nabla w_n|^2 dy_d dy' \leq E_{\bar m} \leq 2^{2-d}Am^{d-1},$$

which proves the a priori bound in the norm (5.8) uniformly in n.

5.2. **Proof of Proposition 12: the Saint-Venant estimate.** We proceed in four steps:

(1) we construct a cut-off η_k with bounds uniform in k to carry out the local estimates,
(2) we carry out energy estimates on system (5.9), and seperate the large scales (non local effects) from the small scales,
(3) we show a control of the non local terms,
(4) we gather the estimates to get (5.11).

Construction of a cut-off. Let $\eta \in C^\infty(B(0,1/2))$ such that $\eta \geq 0$ and $\int_{\mathbb{R}^d}\eta = 1$. For all $k \in \mathbb{N}$, let $\eta_k = \eta_k(y')$ be defined by

$$\eta_k(y') = \int_{\mathbb{R}^{d-1}} \mathbf{1}_{[-k-1/2,k+1/2]^{d-1}}(y'-\tilde y')\eta(\tilde y')d\tilde y' = \int_{[-k-1/2,k+1/2]^{d-1}} \eta(y'-\tilde y')d\tilde y'.$$

For all $k\in\mathbb{N}$, we have the following properties:

$$\eta_k \equiv 1 \text{ on } [-k,k]^{d-1}, \quad \operatorname{Supp}\eta_k \subset [-k-1,k+1]^{d-1}, \quad \eta_k \in C_c^\infty(\mathbb{R}^{d-1})$$

and most importantly, we have the control

$$\|\nabla\eta_k\|_{L^\infty} \leq \|\nabla\eta\|_{L^1}$$

uniformy in k.

Energy estimate. Testing the system (5.9) against $\eta_k^2 w_r$ we get

$$\int_{\Omega_\flat}\eta_k^2 A(y)\nabla w_r\cdot\nabla w_r dy = -\int_{\Omega_\flat} 2\eta_k A(y)\nabla w_r \cdot \nabla\eta_k w_r dy + \langle\nabla\cdot F_r, \eta_k^2 w_r\rangle + \langle f_r,\eta_k^2 w_r\rangle + \langle \mathrm{DN}(w_r|_{y_d=0}),\eta_k^2 w_r\rangle. \tag{5.17}$$

By ellipticity, we have

$$\lambda\int_{\Omega_\flat}\eta_k^2|\nabla w_r|^2 dy \leq \int_{\Omega_\flat}\eta_k^2 A(y)\nabla w_r\cdot\nabla w_r dy.$$

Using that $\eta_k^2 w_r$ vanishes on the lower oscillating boundary, the fundamental theorem of calculus yields a Poincaré type inequality

$$\|\eta_k^2 w_r\|_{L^2(\Omega_\flat)} \leq C\|\partial_{y_d}(\eta_k^2 w_r)\|_{L^2(\Omega_\flat)}, \tag{5.18}$$

where the constant C only depends on the height of the channel $\Omega_\flat$, and therefore not on k. The following estimate (or variations of it) is of constant use: by the trace theorem and the Poincaré inequality (5.18)

(5.19)

$$\begin{aligned}\left(\int_{\Sigma_{k+1}}\eta_k^4|w_r(y',0)|^2dy'\right)^{1/2} &\leq \|\eta_k^2 w_r\|_{H^{1/2}} \leq C\left\{\|\eta_k^2 w_r\|_{L^2(\Omega_\flat)} + \|\nabla(\eta_k^2 w_r)\|_{L^2(\Omega_\flat)}\right\}\\ &\leq C\left\{\|\partial_{y_d}(\eta_k^2 w_r)\|_{L^2(\Omega_\flat)} + \|\nabla(\eta_k^2 w_r)\|_{L^2(\Omega_\flat)}\right\} \leq C\|\nabla(\eta_k^2 w_r)\|_{L^2(\Omega_\flat)}\\ &\leq C(E_{k+1}-E_k)^{1/2} + C'\left(\int_{\Omega_\flat}\eta_k^4|\nabla w_r|^2 dy\right)^{1/2},\end{aligned}$$

with $C = C(d, \|\psi\|_{W^{1,\infty}}, \|\eta\|_{L^1})$ and $C' = C'(d)$. We now estimate every term on the right hand side of (5.17). We have,

$$\left|\int_{\Omega_\flat} 2\eta_k A(y)\nabla w_r \cdot \nabla\eta_k w_r dy\right| \leq \frac{2}{\lambda}\left(\int_{\Omega_\flat} \eta_k^2|\nabla w_r|^2 dy\right)^{1/2}\left(\int_{\Omega_\flat} |\nabla\eta_k|^2|w_r|^2 dy\right)^{1/2}$$
$$\leq C\left(\int_{\Omega_\flat} \eta_k^2|\nabla w_r|^2 dy\right)^{1/2}(E_{k+1} - E_k)^{1/2},$$

with $C = C(\lambda, \|\eta\|_{L^1})$. We also have,

$$|\langle\nabla\cdot F_r, \eta_k^2 w_r\rangle| = |\langle\nabla\cdot F, \eta_k^2 w_r\rangle| = |\langle F, \nabla(\eta_k^2 w_r)\rangle|$$
$$\leq Ck^{\frac{d-1}{2}}(E_{k+1} - E_k)^{1/2} + C'k^{\frac{d-1}{2}}\left(\int_{\Omega_\flat} \eta_k^4|\nabla w_r|^2 dy\right)^{1/2},$$

where $C = C(\|v_0\|_{H^{1/2}_{uloc}}, \|\nabla\eta\|_{L^1})$ and $C' = C'(\|v_0\|_{H^{1/2}_{uloc}})$, and by the trace theorem and Poincaré inequality

$$|\langle f_r, \eta_k^2 w_r\rangle| = |\langle f, \eta_k^2 w_r\rangle| \leq Ck^{\frac{d-1}{2}}\|\eta_k^2 w_r\|_{H^{1/2}} \leq Ck^{\frac{d-1}{2}}\|\nabla(\eta_k^2 w_r)\|_{L^2}$$
$$\leq Ck^{\frac{d-1}{2}}(E_{k+1} - E_k)^{1/2} + C'k^{\frac{d-1}{2}}\left(\int_{\Omega_\flat} \eta_k^4|\nabla w_r|^2 dy\right)^{1/2},$$

with $C = C(d, \|\psi\|_{W^{1,\infty}}, \|v_0\|_{H^{1/2}_{uloc}}, \|\nabla\eta\|_{L^1})$ and $C' = C'(\|v_0\|_{H^{1/2}_{uloc}})$. We have now to tackle the non local term involving the Dirichlet to Neumann operator. We split this term into

$$\langle \mathrm{DN}(w_r|_{y_d=0}), \eta_k^2 w_r\rangle = \langle \mathrm{DN}((1 - \eta_{k+m-1}^2)w_r|_{y_d=0}), \eta_k^2 w_r\rangle$$
$$+ \langle \mathrm{DN}((\eta_{k+m-1}^2 - \eta_k^2)w_r|_{y_d=0}), \eta_k^2 w_r\rangle + \langle \mathrm{DN}(\eta_k^2 w_r), \eta_k^2 w_r\rangle.$$

By Corollary 9,

$$\langle \mathrm{DN}(\eta_k^2 w_r), \eta_k^2 w_r\rangle \leq 0.$$

Relying on Proposition 10 and on estimate (4.6), we get

$$|\langle \mathrm{DN}((\eta_{k+m-1}^2 - \eta_k^2)w_r|_{y_d=0}), \eta_k^2 w_r\rangle| \leq Ck^{\frac{d-1}{2}}\|(\eta_{k+m-1}^2 - \eta_k^2)w_r|_{y_d=0}\|_{H^{1/2}_{uloc}}\|\eta_k^2 w_r\|_{H^{1/2}}$$
$$\leq C(E_{k+m} - E_k)^{1/2}\left(\int_{\Omega_\flat} \eta_k^4|\nabla w_r|^2 dy\right)^{1/2} + C(E_{k+m} - E_k)^{1/2}(E_{k+1} - E_k)^{1/2},$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}}, \|\eta\|_{L^1})$. Notice that the bound (4.4) for the Dirichlet to Neumann operator in $H^{1/2}$ here is actually enough, since w_r is compactly supported. However, when dealing with solutions not compactly supported, as for the uniqueness proof in section 5.3, we have to use the result of Proposition 10.

Control of the non local term.

Lemma 14. *For all $m \geq 3$, all $k \geq \bar{m} = m/2$, we have*

$$\int_{\Sigma_{k+1}}\left(\int_{\mathbb{R}^{d-1}} \frac{1}{|y' - \tilde{y}'|^d}(1 - \eta_{k+m-1}^2)|w_r(\tilde{y}', 0)|d\tilde{y}'\right)^2 dy' \leq C\frac{k^{3d-5}}{m^{3d-3}}\sup_{T\in\mathcal{C}_{k,m}} E_T, \tag{5.20}$$

where $C = C(d)$.

Let $y' \in \Sigma_{k+1}$ be fixed. We have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \frac{1}{|y'-\tilde{y}'|^d}(1-\eta_{k+m-1}^2)|w_r(\tilde{y}',0)|d\tilde{y}' \\
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^{d-1}} \frac{1}{|y'-\tilde{y}'|^d}(\eta_{k+(j+1)(m-1)}^2 - \eta_{k+j(m-1)}^2)|w_r(\tilde{y}',0)|d\tilde{y}' \\
&\leq \sum_{j=1}^{\infty} \int_{\Sigma_{k+(j+1)(m-1)+1}\setminus\Sigma_{k+j(m-1)}} \frac{1}{|y'-\tilde{y}'|^d}|w_r(\tilde{y}',0)|d\tilde{y}' \\
&= \sum_{j=1}^{\infty} \sum_{T\in\mathcal{C}_{k,j,m}} \int_T \frac{1}{|y'-\tilde{y}'|^d}|w_r(\tilde{y}',0)|d\tilde{y}',
\end{aligned}$$

where $\mathcal{C}_{k,j,m}$ is a family of disjoint cubes $T = \xi + (-\bar{m},\bar{m})^{d-1}$ such that $T \subset \Sigma_{k+(j+1)(m-1)+1} \setminus \Sigma_{k+j(m-1)}$ and

$$\bigsqcup_{T\in\mathcal{C}_{k,j,m}} T = \Sigma_{k+(j+1)(m-1)+1} \setminus \Sigma_{k+j(m-1)}.$$

For all $T \in \mathcal{C}_{k,j,m}$, by Cauchy-Schwarz, trace theorem and Poincaré inequality

$$\begin{aligned}
\int_T \frac{1}{|y'-\tilde{y}'|^d}|w_r(\tilde{y}',0)|d\tilde{y}' &\leq \left(\int_T \frac{1}{|y'-\tilde{y}'|^{2d}}d\tilde{y}'\right)^{1/2}\left(\int_T |w_r(\tilde{y}',0)|^2 d\tilde{y}'\right)^{1/2} \\
&\leq C\left(\int_T \frac{1}{|y'-\tilde{y}'|^{2d}}d\tilde{y}'\right)^{1/2}\left(\int_{\Omega_T} |\nabla w_r|^2 d\tilde{y}'\right)^{1/2} \\
&\leq C\left(\int_T \frac{1}{|y'-\tilde{y}'|^{2d}}d\tilde{y}'\right)^{1/2}\left(\sup_{T\in\mathcal{C}_{k,j,m}} E_T\right)^{1/2},
\end{aligned}$$

where Ω_T and E_T are defined in (5.10). Notice that the constant C in the last inequality only depends on d and on $\|\psi\|_{W^{1,\infty}}$. Moreover, for any $T \in \mathcal{C}_{k,j,m}$,

$$\left(\int_T \frac{1}{|y'-\tilde{y}'|^{2d}}d\tilde{y}'\right)^{1/2} \leq \frac{m^{\frac{d-1}{2}}}{(k+j(m-1)-|y'|)^d},$$

and the number of elements of $\mathcal{C}_{k,j,m}$ is bounded by

$$\#\mathcal{C}_{k,j,m} = \frac{\left|\Sigma_{k+(j+1)(m-1)+1} \setminus \Sigma_{k+j(m-1)}\right|}{m^{d-1}} \lesssim \frac{(k+j(m-1))^{d-2}}{m^{d-2}}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \frac{1}{|y'-\tilde{y}'|^d}(1-\eta_{k+m-1}^2)|w_r(\tilde{y}',0)|d\tilde{y}' \\
&\leq C\left(\sup_{T\in\mathcal{C}_{k,j,m}} E_T\right)^{1/2} \sum_{j=1}^{\infty}\sum_{T\in\mathcal{C}_{k,j,m}} \frac{m^{\frac{d-1}{2}}}{(k+j(m-1)-|y'|)^d} \\
&\leq C\left(\sup_{T\in\mathcal{C}_{k,j,m}} E_T\right)^{1/2} \sum_{j=1}^{\infty} \frac{1}{m^{\frac{d-3}{2}}} \frac{(k+j(m-1))^{d-2}}{(k+j(m-1)-|y'|)^d} \\
&\leq C\left(\sup_{T\in\mathcal{C}_{k,j,m}} E_T\right)^{1/2} \frac{(k+m-1)^{d-2}}{m^{\frac{d-1}{2}}(k+m-1-|y'|)^{d-1}},
\end{aligned}$$

with $C = C(d)$. Eventually, we get for $m \geq 3$

$$\int_{\Sigma_{k+1}} \left(\int_{\mathbb{R}^{d-1}} \frac{1}{|y' - \tilde{y}'|^d} (1 - \eta_{k+m-1}^2) |w_r(\tilde{y}', 0)| d\tilde{y}' \right)^2 dy'$$
$$\leq C \left(\sup_{T \in \mathcal{C}_{k,j,m}} E_T \right) \frac{(k+m-1)^{2d-4}}{m^{d-1}} \int_{\Sigma_{k+1}} \frac{1}{(k+m-1-|y'|)^{2d-2}} dy'$$
$$\leq C \left(\sup_{T \in \mathcal{C}_{k,j,m}} E_T \right) \frac{(k+m-1)^{2d-4}}{m^{d-1}} \frac{(k+1)^{d-1}}{(m-2)^{2d-2}} \leq C \frac{k^{3d-5}}{m^{3d-3}} \sup_{T \in \mathcal{C}_{k,j,m}} E_T,$$

with $C = C(d)$, the last inequality being only true on condition that $k \geq m/2 = \bar{m}$. This proves Lemma 14.

In particular, by the definition of DN in (4.5), by the fact that $(1 - \eta_{k+m-1}^2) w_r(\tilde{y}', 0)$ and $\eta_k^2 w_r(y', 0)$ have disjoint support, by estimate (5.20) and by the bound (5.19) we get

$$|\langle \mathrm{DN}((1 - \eta_{k+m-1}^2) w_r|_{y_d=0}), \eta_k^2 w_r \rangle|$$
$$\leq C \left| \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{1}{|y' - \tilde{y}'|^d} (1 - \eta_{k+m-1}^2(\tilde{y}')) |w_r(\tilde{y}', 0)| \eta_k^2(y') |w_r(y', 0)| d\tilde{y}' dy' \right|$$
$$\leq C \left(\int_{\Sigma_{k+1}} \eta_k^4 |w_r(y', 0)|^2 dy' \right)^{1/2} \left(\int_{\Sigma_{k+1}} \left(\int_{\mathbb{R}^{d-1}} \frac{1 - \eta_{k+m-1}^2(\tilde{y}')}{|y' - \tilde{y}'|^d} |w_r(y', 0)| d\tilde{y}' \right)^2 dy' \right)^{1/2},$$
$$\leq C \frac{k^{\frac{3d-5}{2}}}{m^{\frac{3d-3}{2}}} \left(\int_{\Sigma_{k+1}} \eta_k^4 |w_r(y', 0)|^2 dy' \right)^{1/2} \left(\sup_{T \in \mathcal{C}_{k,j,m}} E_T \right)^{\frac{1}{2}}$$
$$\leq C \frac{k^{\frac{3d-5}{2}}}{m^{\frac{3d-3}{2}}} \left[(E_{k+1} - E_k)^{1/2} + \left(\int_{\Omega_\flat} \eta_k^4 |\nabla w|^2 dy \right)^{1/2} \right] \left(\sup_{T \in \mathcal{C}_{k,j,m}} E_T \right)^{\frac{1}{2}},$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}})$.

End of the proof of the Saint-Venant estimate. Combining all our bounds and using

$$E_{k+1} - E_k \leq E_{k+m} - E_k, \qquad \eta_k^4 \leq C(\|\eta\|_{L^\infty}) \eta_k^2$$

whenever possible, we get from (5.17) the following estimate

$$\lambda \int_{\Omega_\flat} \eta_k^2 |\nabla w_r|^2 dy \leq C \left(\int_{\Omega_\flat} \eta_k^2 |\nabla w_r|^2 dy \right)^{1/2} (E_{k+m} - E_k)^{1/2} + C k^{\frac{d-1}{2}} (E_{k+m} - E_k)^{1/2} + C(E_{k+m} - E_k)$$
$$+ C k^{\frac{d-1}{2}} \left(\int_{\Omega_\flat} \eta_k^2 |\nabla w_r|^2 dy \right)^{1/2} + C(E_{k+m} - E_k)^{1/2} \left(\int_{\Omega_\flat} \eta_k^2 |\nabla w_r|^2 dy \right)^{1/2}$$
$$+ C \frac{k^{\frac{3d-5}{2}}}{m^{\frac{3d-3}{2}}} \left[(E_{k+1} - E_k)^{1/2} + \left(\int_{\Omega_\flat} \eta_k^2 |\nabla w|^2 dy \right)^{1/2} \right] \left(\sup_{T \in \mathcal{C}_{k,j,m}} E_T \right)^{1/2},$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|v_0\|_{H^{1/2}_{uloc}}, \|\psi\|_{W^{1,\infty}})$. Swallowing every term of the type

$$\int_{\Omega_\flat} \eta_k^2 |\nabla w|^2 dy$$

in the left hand side, we end up with the Saint-Venant estimate (5.11). This concludes the proof of Proposition 12.

5.3. **End of the proof of Theorem 2: uniqueness.** Extracting subsequences using a classical diagonal argument and passing to the limit in the weak formulation of (5.7) relying on the continuity of the Dirichlet to Neumann map asserted in estimate (4.4) yields the existence of a weak solution w to the system (5.4). In addition, the weak solution satisfies the bound

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{\infty}|\nabla w|^2 dy_d dy' \leq 2^{2-d}Am^{d-1}<\infty. \tag{5.21}$$

Let us turn to the uniqueness of the solution to (5.4) satisfying the bound (5.21). By linearity of the problem, it is enough to prove the uniqueness for zero source terms. Assume $w\in H^1_{loc}(\Omega_\flat)$ is a weak solution to (5.4) with $f=F=0$ satisfying

$$\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi(y')}^{0}|\nabla w|^2 dy_d dy' \leq C_0<\infty. \tag{5.22}$$

Repeating the estimates leading to Proposition 12 (see section 5.2), we infer that for the same constant C^* appearing in the Saint-Venant estimate (5.11) and for m defined by (5.16), for $k\in\mathbb{N}$, $k\geq m/2=\bar{m}$,

$$E_k\leq C^*\left(E_{k+m}-E_k+\frac{k^{3d-5}}{m^{3d-3}}\sup_{T\in\mathcal{C}_{k,m}}E_T\right). \tag{5.23}$$

The fact that w, unlike w_n, does not vanish outside $\Omega_{\flat,n}$ does not lead to any difference in the proof of this estimate.

Since

$$\sup_{T\in\mathcal{C}_m}E_T<\infty,$$

for any ε, there exists $T^*_\varepsilon\in\mathcal{C}_m$ such that

$$\sup_{T\in\mathcal{C}_m}E_T-\varepsilon\leq E_{T^*_\varepsilon}\leq\sup_{T\in\mathcal{C}_m}E_T. \tag{5.24}$$

Again, $T^*_\varepsilon:=\xi^*_\varepsilon+(-\bar{m},\bar{m})^{d-1}$ for $\xi^*_\varepsilon\in\mathbb{Z}^{d-1}$, and we can translate T^*_ε so that it is centered at the origin as has been done in section 5.1. Estimate (5.23) still holds. For any $n\in\mathbb{N}$, $E_n\leq C_0n^{d-1}$ where C_0 is defined by (5.22). The idea is now to carry out a downward iteration. For any $n=(2l+1)\bar{m}$ with $l\in\mathbb{N}$, $l\geq1$ fixed, for $p\in\{1,\dots\ l-1\}$ one can show that

$$\begin{aligned}
E_{(2p+1)\bar{m}}\leq{}&\left[\frac{C^*}{C^*+1}+\left(\frac{C^*}{C^*+1}\right)^2+\dots\ \left(\frac{C^*}{C^*+1}\right)^{l-p}\right]E_n\\
&+\frac{2^{5-3d}}{m^2}\left[\frac{C^*}{C^*+1}(2p+1)^{3d-5}+\dots\ \left(\frac{C^*}{C^*+1}\right)^{l-p}(2l+1)^{3d-5}\right]\sup_{T\in\mathcal{C}_m}E_T\\
\leq{}&C_0\frac{C^*+1}{2C^*+1}\left(\frac{C^*}{C^*+1}\right)^{l-p+1}n^{d-1}\\
&+\frac{2^{5-3d}}{m^2}\left[\frac{C^*}{C^*+1}(2p+1)^{3d-5}+\dots\ \left(\frac{C^*}{C^*+1}\right)^{l-p}(2l+1)^{3d-5}\right]\sup_{T\in\mathcal{C}_m}E_T.
\end{aligned}$$

Thus,

$$\begin{aligned}
E_{\bar{m}}&\leq C_0\frac{C^*+1}{2C^*+1}\left(\frac{C^*}{C^*+1}\right)^{l}(2l+1)^{d-1}\bar{m}^{d-1}+\frac{2^{5-3d}}{m^2}B\sup_{T\in\mathcal{C}_m}E_T\\
&\leq C_0\frac{C^*+1}{2C^*+1}\left(\frac{C^*}{C^*+1}\right)^{2l+1}(2l+1)^{d-1}\bar{m}^{d-1}+\frac{2^{5-3d}}{m^2}B(E_{\bar{m}}+\varepsilon).
\end{aligned}$$

From this we infer using (5.16) that

$$E_{\bar m}\leq 2C_0\frac{C^*+1}{2C^*+1}\left(\frac{C^*}{C^*+1}\right)^{2l+1}(2l+1)^{d-1}\bar m^{d-1}+\frac{2^{6-3d}}{m^2}B\varepsilon\overset{l\to\infty}{\longrightarrow}\frac{2^{6-3d}}{m^2}B\varepsilon.$$

Therefore, from equation (5.24)

$$\sup_{T\in\mathcal{C}_m}E_T\leq\left(1+\frac{2^{6-3d}}{m^2}B\right)\varepsilon,$$

which eventually leads to $\sup_{T\in\mathcal{C}_m}E_T=0$, or in other words $w=0$.

Combining this existence and uniqueness result for the system (5.4) in the bumpy channel $\Omega_\flat$ with Lemma 11 and Proposition 7 about the well-posedness in the flat half-space finishes the proof of Theorem 2.

6. Improved regularity over Lipschitz boundaries

The goal in this section is to prove Theorem 1 of the introduction. Let us recall the result we prove in the following proposition.

Proposition 15. *For all $\nu>0$, $\gamma>0$, there exist $C>0$ and $\varepsilon_0>0$ such that for all $\psi\in W^{1,\infty}(\mathbb{R}^{d-1})$, $-1<\psi<0$ and $\|\nabla\psi\|_{L^\infty}\leq\gamma$, for all $A\in\mathcal{A}^\nu$, for all $0<\varepsilon<(1/2)\varepsilon_0$, for all weak solutions u^ε to* (1.1)*, for all $r\in[\varepsilon/\varepsilon_0,1/2]$*

$$\fint_{D^\varepsilon_\psi(0,r)}|\nabla u^\varepsilon|^2\leq C\fint_{D^\varepsilon_\psi(0,1)}|\nabla u^\varepsilon|^2,\tag{6.1}$$

or equivalently,

$$\fint_{D^\varepsilon_\psi(0,r)}|u^\varepsilon|^2\leq Cr^2\fint_{D^\varepsilon_\psi(0,1)}|u^\varepsilon|^2,$$

with $C=C(d,N,\lambda,\nu,\gamma,[A]_{C^{0,\nu}})$.

We rely on a compactness argument inspired by the pioneering work of Avellaneda and Lin [AL87a, AL89b], and our recent work [KP15]. The proof is in two steps. Firstly, we carry out the compactness argument. Secondly, we iterate the estimate obtained in the first step, to get an estimate down to the microscopic scale $O(\varepsilon)$.

A key step in the proof of boundary Lipschitz estimates is to estimate boundary layer correctors, which is done by combining the classical Lipschitz estimate with a uniform Hölder estimate, as in [AL87a, Lemma 17] or [KP15, Lemma 10]. Here, we are able to relax the regularity assumption on ψ. This progress is enabled by our new estimate (1.4) for the boundary layer corrector, which holds for Lipschitz boundaries ψ.

We begin with an estimate which is of constant use in this part of our work. Take $\psi\in W^{1,\infty}(\mathbb{R}^{d-1})$ and $A\in\mathcal{A}^\nu$. By Caccioppoli's inequality, there exists $C>0$ such that for all $\varepsilon>0$, for all weak solutions u^ε to

$$\begin{cases}-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon&=0,\quad x\in D^\varepsilon_\psi(0,1),\\ u^\varepsilon&=0,\quad x\in\Delta^\varepsilon_\psi(0,1),\end{cases}\tag{6.2}$$

for all $0<\theta<1$,

$$\begin{aligned}\left|(\overline{\partial_{x_d}u^\varepsilon})_{D_\psi(0,\theta)}\right|&=\left|\fint_{D_\psi(0,\theta)}\partial_{x_d}u^\varepsilon\right|\leq\left(\fint_{D_\psi(0,\theta)}|\partial_{x_d}u^\varepsilon|^2\right)^{1/2}\\&\leq\frac{C_0}{\theta^{d/2}(1-\theta)}\left(\fint_{D_\psi(0,1)}|u^\varepsilon|^2\right)^{1/2}.\end{aligned}\tag{6.3}$$

Notice that C_0 in (6.3) only depends on λ.

Proposition 15 is a consequence of the two following lemmas. The first one contains the compactness argument. The second one is the iteration lemma. In order to alleviate the statement of the following lemma, the definition of the boundary layer v is given straight after the lemma.

Lemma 16. *For all $\nu > 0$, $\gamma > 0$, there exists $\theta > 0$, $0 < \mu < 1$, $\varepsilon_0 > 0$, such that for all $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$, $-1 < \psi < 0$ and $\|\nabla\psi\|_{L^\infty} \leq \gamma$, for all $A \in \mathcal{A}^\nu$, for all $0 < \varepsilon < \varepsilon_0$, for all weak solutions u^ε to (6.2) we have*

$$\fint_{D^\varepsilon_\psi(0,1)} |u^\varepsilon|^2 \leq 1$$

implies

$$\fint_{D^\varepsilon_\psi(0,\theta)} \left| u^\varepsilon(x) - (\overline{\partial_{x_d} u^\varepsilon})_{D^\varepsilon_\psi(0,\theta)} \left[x_d + \varepsilon\chi^d(x/\varepsilon) + \varepsilon v(x/\varepsilon) \right] \right|^2 \leq \theta^{2+2\mu}.$$

The boundary layer $v = v(y)$ is the unique solution given by Theorem 2 to the system

$$\left\{ \begin{array}{rll} -\nabla\cdot A(y)\nabla v & = 0, & y_d > \psi(y'), \\ v & = -y_d - \chi^d(y), & y_d = \psi(y'). \end{array} \right. \tag{6.4}$$

The estimate of Theorem 2 implies

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \int_{\xi+(0,1)^{d-1}} \int_{\psi(y')}^\infty |\nabla v|^2 dy_d dy' \leq C \left\{ \|\psi\|_{H^{1/2}_{uloc}(\mathbb{R}^{d-1})} + \|\chi(\cdot,\psi(\cdot))\|_{H^{1/2}_{uloc}(\mathbb{R}^{d-1})} \right\},$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}})$. Now, by Sobolev injection $W^{1,\infty}(\mathbb{R}^{d-1}) \hookrightarrow H^{1/2}(\mathbb{R}^{d-1})$

$$\|\psi\|_{H^{1/2}_{uloc}(\mathbb{R}^{d-1})} \leq C\|\psi\|_{W^{1,\infty}(\mathbb{R}^{d-1})},$$

with $C = C(d)$ and by classical interior Lipschitz regularity

$$\|\chi(\cdot,\psi(\cdot))\|_{H^{1/2}_{uloc}} \leq C\|\chi(\cdot,\psi(\cdot))\|_{W^{1,\infty}(\mathbb{R}^{d-1})} \leq C\|\chi\|_{W^{1,\infty}(\mathbb{R}^d)} \leq C,$$

with in the last inequality $C = C(d, N, \lambda, [A]_{C^{0,\nu}})$. Eventually,

$$\sup_{\xi\in\mathbb{Z}^{d-1}} \int_{\xi+(0,1)^{d-1}} \int_{\psi(y')}^\infty |\nabla v|^2 dy_d dy' \leq C, \tag{6.5}$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}})$ uniform in ε.

The interior corrector χ^d as well as the boundary layer corrector v are crucial in the iteration procedure, which is the second step of the method.

Lemma 17. *Let θ, ε_0 and γ be given as in Lemma 16. For all $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$, $-1 < \psi < 0$ and $\|\nabla\psi\|_{L^\infty} \leq \gamma$, for all $A \in \mathcal{A}^\nu$, for all $k \in \mathbb{N}$, $k > 0$, for all $0 < \varepsilon < \theta^{k-1}\varepsilon_0$, for all weak solutions u^ε to (6.2) there exists $a^\varepsilon_k \in \mathbb{R}^N$ satifying*

$$|a^\varepsilon_k| \leq C_0 \frac{1 + \theta^\mu + \ldots\ \theta^{\mu(k-1)}}{\theta^{d/2}(1-\theta)},$$

such that

$$\fint_{D^\varepsilon_\psi(0,1)} |u^\varepsilon|^2 \leq 1$$

implies

$$\fint_{D^\varepsilon_\psi(0,\theta^k)} \left| u^\varepsilon(x) - a^\varepsilon_k \left[x_d + \varepsilon\chi^d(x/\varepsilon) + \varepsilon v(x/\varepsilon) \right] \right|^2 \leq \theta^{(2+2\mu)k}, \tag{6.6}$$

where $v = v(y)$ is the solution, given by Theorem 2, to the boundary layer system (6.4).

The condition $\varepsilon < \theta^{k-1}\varepsilon_0$ can be seen as giving a lower bound on the scales θ^k for which one can prove the regularity estimate: $\theta^{k-1} > \varepsilon/\varepsilon_0$. In that perspective, estimate (6.6) is an improved $C^{1,\mu}$ estimate down to the microscale $\varepsilon/\varepsilon_0$.

For fixed $0 < \varepsilon/\varepsilon_0 < 1/2$ and $r \in [\varepsilon/\varepsilon_0, 1/2]$, there exists $k \in \mathbb{N}$ such that $\theta^{k+1} < r \leq \theta^k$. We aim at estimating

$$\fint_{D^\varepsilon_\psi(0,r)} |u^\varepsilon(x)|^2$$

using the bound (6.6). We have

(6.7)

$$\begin{aligned}
&\left(\fint_{D^\varepsilon_\psi(0,r)} |u^\varepsilon(x)|^2\right)^{1/2} \leq \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |u^\varepsilon(x)|^2\right)^{1/2} \\
&\leq \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} \left|u^\varepsilon(x) - a^\varepsilon_k\left[x_d + \varepsilon\chi^d(x/\varepsilon) + \varepsilon v(x/\varepsilon)\right]\right|^2\right)^{1/2} \\
&\qquad + |a^\varepsilon_k| \left\{ \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |x_d|^2\right)^{1/2} + \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\varepsilon\chi^d(x/\varepsilon)|^2\right)^{1/2} + \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\varepsilon v(x/\varepsilon)|^2\right)^{1/2} \right\}.
\end{aligned}$$

Let us focus on the term involving the boundary layer. Let $\eta = \eta(y_d) \in C^\infty_c(\mathbb{R})$ be a cut-off such that $\eta \equiv 1$ on $(-1,1)$ and $\operatorname{Supp}\eta \subset (-2,2)$. The triangle inequality yields

$$\begin{aligned}
\left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\varepsilon v(x/\varepsilon)|^2\right)^{1/2} &\leq \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\varepsilon v(x/\varepsilon) - (x_d + \varepsilon\chi^d(x/\varepsilon))\eta(x_d/\varepsilon)|^2\right)^{1/2} \\
&\qquad + \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |(x_d + \varepsilon\chi^d(x/\varepsilon))\eta(x_d/\varepsilon)|^2\right)^{1/2}.
\end{aligned}$$

Poincaré's inequality implies

$$\begin{aligned}
&\left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\varepsilon v(x/\varepsilon) - (x_d + \varepsilon\chi^d(x/\varepsilon))\eta(x_d/\varepsilon)|^2\right)^{1/2} \\
&\leq \theta^k \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} \left|\nabla\left(\varepsilon v(x/\varepsilon) - (x_d + \varepsilon\chi^d(x/\varepsilon))\eta(x_d/\varepsilon)\right)\right|^2\right)^{1/2} \\
&\leq \theta^k \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\nabla v(x/\varepsilon)|^2\right)^{1/2} + (1 + \|\nabla\chi\|^2_{L^\infty})\theta^k \left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\eta(x_d/\varepsilon)|^2\right)^{1/2} \\
&\qquad + \frac{\theta^k}{\varepsilon}\left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |(x_d + \varepsilon\chi^d(x/\varepsilon))\eta'(x_d/\varepsilon)|^2\right)^{1/2}.
\end{aligned}$$

Estimate (6.5) now yields

$$\fint_{D^\varepsilon_\psi(0,\theta^k)} |\nabla v(x/\varepsilon)|^2 \leq C\varepsilon\theta^{-k},$$

so that eventually using $\varepsilon/\varepsilon_0 \leq r \leq \theta^k$,

$$\left(\fint_{D^\varepsilon_\psi(0,\theta^k)} |\varepsilon v(x/\varepsilon) - (x_d + \varepsilon\chi^d(x/\varepsilon))\eta(x_d/\varepsilon)|^2\right)^{1/2} \leq C\left(\varepsilon^{1/2}\theta^{k/2} + \theta^k + \frac{\theta^k}{\varepsilon}(\varepsilon + \varepsilon)\right) \leq C\theta^k$$

with $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}})$. It follows from (6.7) and (6.6) that

$$\left(\fint_{D^\varepsilon_\psi(0,r)} |u^\varepsilon(x)|^2\right)^{1/2} \leq \theta^{(1+\mu)k} + C\theta^k \leq C\theta^k \leq Cr,$$

which is the estimate of Proposition 15.

6.1. **Proof of Lemma 16.** Let $0 < \theta < 1/8$ and $u^0 \in H^1(D_0(0, 1/4))$ be a weak solution of

$$\left\{ \begin{array}{rll} -\nabla \cdot A^0 \nabla u^0 & = 0, & x \in D_0(0, 1/4), \\ u^0 & = 0, & x \in \Delta_0(0, 1/4), \end{array} \right. \tag{6.8}$$

such that

$$\fint_{D_0(0,1/4)} |u^0|^2 \leq 4^d.$$

The classical regularity theory yields $u^0 \in C^2(\overline{D_0(0, 1/8)})$. Using that for all $x \in D_0(0, \theta)$

$$\begin{aligned} u^0(x) - \left(\overline{\partial_{x_d} u^0}\right)_{0,\theta} x_d &= u^0(x) - u^0(x', 0) - \left(\overline{\partial_{x_d} u^0}\right)_{0,\theta} x_d \\ &= \frac{1}{|D^0(0,\theta)|} \int_0^1 \int_{D_0(0,\theta)} \left(\partial_{x_d} u^0(x', tx_d) - \partial_{x_d} u^0(y)\right) x_d dx dt. \end{aligned}$$

we get

$$\fint_{D_0(0,\theta)} \left| u^0(x) - (\overline{\partial_{x_d} u^0})_{D_0(0,\theta)} x_d \right|^2 \leq \widehat{C}\theta^4, \tag{6.9}$$

where $\widehat{C} = \widehat{C}(d, N, \lambda)$. Fix $0 < \mu < 1$. Choose $0 < \theta < 1/8$ sufficiently small such that

$$\theta^{2+2\mu} > \widehat{C}\theta^4. \tag{6.10}$$

The rest of the proof is by contradiction. Fix $\gamma > 0$. Assume that for all $k \in \mathbb{N}$, there exists $\psi_k \in W^{1,\infty}(\mathbb{R}^{d-1})$,

$$-1 < \psi < 0 \quad \text{and} \quad \|\psi_k\|_{L^\infty} \leq \gamma, \tag{6.11}$$

there exists $A_k \in \mathcal{A}^\nu$, there exists $0 < \varepsilon_k < 1/k$, there exists $u_k^{\varepsilon_k}$ solving

$$\left\{ \begin{array}{rll} -\nabla \cdot A_k(x/\varepsilon_k) \nabla u_k^{\varepsilon_k} & = 0, & x \in D^{\varepsilon_k}_{\psi_k}(0, 1), \\ u_k^{\varepsilon_k} & = 0, & x \in \Delta^{\varepsilon_k}_{\psi_k}(0, 1), \end{array} \right.$$

such that

$$\fint_{D^{\varepsilon_k}_{\psi_k}(0,1)} |u_k^{\varepsilon_k}|^2 \leq 1 \tag{6.12}$$

and

$$\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)} \left| u_k^{\varepsilon_k}(x) - (\overline{\partial_{x_d} u_k^{\varepsilon_k}})_{D^{\varepsilon_k}_{\psi_k}(0,\theta)} \left[x_d + \varepsilon_k \chi_k^d(x/\varepsilon_k) + \varepsilon_k v_k(x/\varepsilon_k) \right] \right|^2 > \theta^{2+2\mu}. \tag{6.13}$$

Notice that χ_k^d is the cell corrector associated to the operator $-\nabla \cdot A_k(y)\nabla$ and v_k is the boundary layer corrector associated to $-\nabla \cdot A_k(y)\nabla$ and to the domain $y_d > \psi_k(y')$.

First of all, for technical reasons, let us extend $u_k^{\varepsilon_k}$ by zero below the boundary, on $\{x' \in (-1, 1)^{d-1},\ x_d \leq \varepsilon_k \psi_k(x'/\varepsilon_k)\}$. The extended functions are still denoted the same, and $u_k^{\varepsilon_k}$ is a weak solution of

$$-\nabla \cdot A_k(x/\varepsilon_k)\nabla u_k^{\varepsilon_k} = 0$$

on $\{x' \in (-1, 1)^{d-1},\ x_d \leq \varepsilon_k \psi_k(x'/\varepsilon_k) + 1\}$.

For k sufficiently large, by Caccioppoli's inequality,

$$\int_{(-1/4,1/4)^d} |\nabla u_k^{\varepsilon_k}|^2 dx \leq C \int_{(-1/2,1/2)^d} |u_k^{\varepsilon_k}|^2 dx \leq C,$$

where $C = C(d, N, \lambda)$. Therefore, up to a subsequence, which we denote again by $u_k^{\varepsilon_k}$, we have

$$\begin{aligned} u_k^{\varepsilon_k} &\overset{k\to\infty}{\longrightarrow} u^0, \quad \text{strongly in} \quad L^2((-1/4,1/4)^{d-1} \times (-1,1/4)), \\ \nabla u_k^{\varepsilon_k} &\overset{k\to\infty}{\rightharpoonup} \nabla u^0, \quad \text{weakly in} \quad L^2((-1/4,1/4)^{d-1} \times (-1,1/4)). \end{aligned} \tag{6.14}$$

Moreover, $\varepsilon_k \psi_k(\cdot/\varepsilon_k)$ converges to 0 because ψ_k is bounded uniformly in k (see (6.11)). Let $\varphi \in C_c^\infty(D_0(0,1/4))$. Theorem 4 implies that

$$\int_{D_{\psi_k}^{\varepsilon_k}(0,1/4)} A_k(x/\varepsilon_k)\nabla u_k^{\varepsilon_k} \cdot \nabla\varphi dx \overset{k\to\infty}{\longrightarrow} \int_{D_0(0,1/4)} A^0 \nabla u^0 \nabla \varphi dx,$$

so that u^0 is a weak solution to

$$-\nabla \cdot A^0 \nabla u^0 = 0 \quad \text{in} \quad D_0(0,1/4).$$

Furthermore, for all $\varphi \in C_c^\infty((-1/4,1/4)^{d-1} \times (-1,0))$,

$$0 = \int_{\{x' \in (-1/4,1/4)^{d-1},\ -1 \leq x_d \leq \varepsilon_k \psi_k(x'/\varepsilon_k)\}} u_k^{\varepsilon_k} \varphi dx \overset{k\to\infty}{\longrightarrow} \int_{(-1/4,1/4)^{d-1}\times(-1,0)} u^0 \varphi dx,$$

so that $u^0(x) = 0$ for all $x \in (-1/4,1/4)^{d-1} \times (-1,0)$. In particular, $u^0 = 0$ in $H^{1/2}(\Delta_0(0,1/4))$. Thus, u^0 is a solution to (6.8) and satisfies the estimate (6.9).

It remains to pass to the limit in (6.13) to reach a contradiction. Since $|D_{\psi_k}^{\varepsilon_k}(0,\theta)| = |D_0(0,\theta)|$, we have

$$\begin{aligned} \left| (\overline{\partial_{x_d} u_k^{\varepsilon_k}})_{D_{\psi_k}^{\varepsilon_k}(0,\theta)} - (\overline{\partial_{x_d} u^0})_{D_0(0,\theta)} \right| \leq \frac{1}{|D_0(0,\theta)|} &\left[\left| \int_{D_{\psi_k}^{\varepsilon_k}(0,\theta) \cap D_0(0,\theta)} \left(\partial_{x_d} u_k^{\varepsilon_k} - \partial_{x_d} u^0 \right) dx \right| \right. \\ &\left. + \int_{\left(D_{\psi_k}^{\varepsilon_k}(0,\theta) \setminus D_0(0,\theta)\right) \cup \left(D_0(0,\theta) \setminus D_{\psi_k}^{\varepsilon_k}(0,\theta)\right)} \left| \partial_{x_d} u_k^{\varepsilon_k} - \partial_{x_d} u^0 \right| dx \right]. \end{aligned} \tag{6.15}$$

The first term in the right hand side of (6.15) tends to 0 thanks to the weak convergence of $\nabla u_k^{\varepsilon_k}$ in (6.14). The second term in the right hand side of (6.15) goes to 0 when $k \to \infty$ because of the L^2 bound on the gradient, and the fact that

$$\left| \left(D_{\psi_k}^{\varepsilon_k}(0,\theta) \setminus D_0(0,\theta) \right) \cup \left(D_0(0,\theta) \setminus D_{\psi_k}^{\varepsilon_k}(0,\theta) \right) \right| \overset{k\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\fint_{D_{\psi_k}^{\varepsilon_k}(0,\theta) \cap D_0(0,\theta)} \left| (\overline{\partial_{x_d} u_k^{\varepsilon_k}})_{D_{\psi_k}^{\varepsilon_k}(0,\theta)} \left[x_d + \varepsilon_k \chi_k^d(x/\varepsilon_k) \right] - (\overline{\partial_{x_d} u^0})_{D_0(0,\theta)} x_d \right|^2 \overset{k\to\infty}{\longrightarrow} 0.$$

Moreover, the strong L^2 convergence in (6.14) implies

$$\fint_{D_{\psi_k}^{\varepsilon_k}(0,\theta) \cap D_0(0,\theta)} |u_k^{\varepsilon_k} - u^0|^2 \overset{k\to\infty}{\longrightarrow} 0.$$

The last thing we have to check is the convergence

$$\fint_{D_{\psi_k}^{\varepsilon_k}(0,\theta)} |\varepsilon_k v_k(x/\varepsilon_k)|^2 \overset{k\to\infty}{\longrightarrow} 0.$$

Let $\eta=\eta(y_d)\in C_c^\infty(\mathbb{R})$ such that $\eta\equiv 1$ on $(-1,1)$ and $\mathrm{Supp}\,\eta\subset(-2,2)$. We have
(6.16)
$$\begin{aligned}&\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|\varepsilon_k v_k(x/\varepsilon_k)|^2\\ &\leq \fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|\varepsilon_k v_k(x/\varepsilon_k)-(x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k))\eta(x_d/\varepsilon_k)|^2+\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|(x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k))\eta(x_d/\varepsilon_k)|^2.\end{aligned}$$
The last term in the right hand side of (6.16) goes to 0 when $k\to\infty$. Now by Poincaré's inequality,
$$\begin{aligned}&\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|\varepsilon_k v_k(x/\varepsilon_k)-(x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k))\eta(x_d/\varepsilon_k)|^2\\ &\leq C\theta^2\left[\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|\nabla v_k(x/\varepsilon_k)|^2+\fint_{D_{\psi_k}(0,\theta)}\left|\nabla\left((x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k))\eta(x_d/\varepsilon_k)\right)\right|^2\right].\end{aligned}$$
On the one hand by estimate (6.5)
$$\begin{aligned}\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|\nabla v_k(x/\varepsilon_k)|^2&\leq\frac{C\varepsilon_k^d}{\theta^d}\int_{D^1_{\psi_k}(0,\theta/\varepsilon_k)}|\nabla v_k(y)|^2dx\\ &\leq C\varepsilon_k\sup_{\xi\in\mathbb{Z}^{d-1}}\int_{\xi+(0,1)^{d-1}}\int_{\psi_k(y')}^\infty|\nabla v_k|^2dx\leq C\varepsilon_k\overset{k\to\infty}{\longrightarrow}0\end{aligned}$$
with in the last inequality $C=C(d,N,\lambda,[A]_{C^{0,\nu}})$ uniform in ε, and on the other hand
$$\begin{aligned}\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}\left|\nabla\left((x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k))\eta(x_d/\varepsilon_k)\right)\right|^2&\leq(1+\|\nabla\chi_k\|^2_{L^\infty})\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|\eta(x_d/\varepsilon_k)|^2\\ &+\frac{1}{\varepsilon_k^2}\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}|(x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k))\eta'(x_d/\varepsilon_k)|^2\leq C\varepsilon_k\overset{k\to\infty}{\longrightarrow}0,\end{aligned}$$
with in the last inequality $C=C(d,N,\lambda,[A]_{C^{0,\nu}})$. These convergence results imply that passing to the limit in (6.13) we get
$$\begin{aligned}\theta^{2+2\mu}&\leq\fint_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}\left|u_k^{\varepsilon_k}(x)-(\overline{\partial_{x_d}u_k^{\varepsilon_k}})_{D^{\varepsilon_k}_{\psi_k}(0,\theta)}\left[x_d+\varepsilon_k\chi_k^d(x/\varepsilon_k)+\varepsilon_k v_k(x/\varepsilon_k)\right]\right|^2dy\\ &\overset{k\to\infty}{\longrightarrow}\fint_{D_0(0,\theta)}\left|u^0(x)-(\overline{\partial_{x_d}u^0})_{D_0(0,\theta)}x_d\right|^2\leq\widehat{C}\theta^4,\end{aligned}$$
which contradicts (6.10).

6.2. **Proof of Lemma 17.** The proof is by induction on k. The result for $k=1$ is true because of Lemma 16. Let $k\in\mathbb{N}$, $k\geq 1$. Assume that for all $\psi\in W^{1,\infty}(\mathbb{R}^{d-1})$ such that $-1<\psi<0$ and $\|\nabla\psi\|_{L^\infty}\leq\gamma$, for all $A\in\mathcal{A}^\nu$, for all $k\in\mathbb{N}$, $k>0$, for all $0<\varepsilon<\theta^{k-1}\varepsilon_0$, for all weak solutions u^ε to (6.2) there exists $a_k^\varepsilon\in\mathbb{R}^N$ satifying
$$|a_k^\varepsilon|\leq C_0\frac{1+\theta^\mu+\ldots\ \theta^{\mu(k-1)}}{\theta^{d/2}(1-\theta)},$$
such that
$$\fint_{D^\varepsilon_\psi(0,1)}|u^\varepsilon|^2\leq 1$$
implies
$$\fint_{D^\varepsilon_\psi(0,\theta^k)}\left|u^\varepsilon(x)-a_k^\varepsilon\left[x_d+\varepsilon\chi^d(x/\varepsilon)+\varepsilon v(x/\varepsilon)\right]\right|^2dy\leq\theta^{(2+2\mu)k}. \tag{6.17}$$

This is our induction hypothesis.

Given $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$, $-1 < \psi < 0$ and $\|\nabla\psi\|_{L^\infty} \leq \gamma$ and $A \in \mathcal{A}^\nu$, $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ and a solution u^ε to (6.2) such that

$$\fint_{D_\psi^\varepsilon(0,1)} |u^\varepsilon|^2 \leq 1$$

we define

$$U^\varepsilon(x) := \frac{1}{\theta^{(1+\mu)k}} \left\{ u^\varepsilon(\theta^k x) - a_k^\varepsilon \left[\theta^k x_d + \varepsilon\chi^d(\theta^k x/\varepsilon) + \varepsilon v(\theta^k x/\varepsilon) \right] \right\}$$

for all $x \in D_\psi^{\varepsilon/\theta^k}(0,1)$. The goal is to apply the estimate of Lemma 17 to U^ε. By the induction estimate (6.17), we have

$$\fint_{D_\psi^{\varepsilon/\theta^k}(0,1)} |U^\varepsilon|^2 \leq 1.$$

Moreover, U^ε solves the system

$$\left\{ \begin{aligned} -\nabla \cdot A(\theta^k x/\varepsilon)\nabla U^\varepsilon &= 0, \quad x \in D_\psi^{\varepsilon/\theta^k}(0,1), \\ U^\varepsilon &= 0, \quad x \in \Delta_\psi^{\varepsilon/\theta^k}(0,1). \end{aligned} \right. \tag{6.18}$$

The boundary layer v solving (6.4) has been designed for U^ε to solve (6.18). It follows that U^ε satisfies the assumptions of Lemma 16. Therefore, for all $\varepsilon/\theta^k < \varepsilon_0$, we have

$$\fint_{D_\psi^{\varepsilon/\theta^k}(0,\theta)} \left| U^\varepsilon(x) - (\overline{\partial_{x_d} U^\varepsilon})_{D_\psi^{\varepsilon/\theta^k}(0,\theta)} \left[x_d + \frac{\varepsilon}{\theta^k}\chi^d(\theta^k x/\varepsilon) + \frac{\varepsilon}{\theta^k} v(\theta^k x/\varepsilon) \right] \right|^2 \leq \theta^{2+2\mu}.$$

Eventually,

$$\fint_{D_\psi^\varepsilon(0,\theta^{k+1})} \left| u^\varepsilon(x) - a_{k+1}^\varepsilon \left[x_d + \varepsilon\chi^d(x/\varepsilon) + \varepsilon v(x/\varepsilon) \right] \right|^2 \leq \theta^{(2+2\mu)(k+1)},$$

with

$$a_{k+1}^\varepsilon := a_k^\varepsilon + \theta^{\mu k} (\overline{\partial_{x_d} U^\varepsilon})_{D_\psi^{\varepsilon/\theta^k}(0,\theta)}$$

satisfying the estimate

$$|a_{k+1}^\varepsilon| \leq C_0 \frac{1 + \theta^\mu + \ldots\ \theta^{\mu(k-1)}}{\theta^{d/2}(1-\theta)} + C_0 \frac{\theta^{\mu k}}{\theta^{d/2}(1-\theta)} \leq C_0 \frac{1 + \theta^\mu + \ldots\ \theta^{\mu k}}{\theta^{d/2}(1-\theta)}.$$

This concludes the iteration step and proves Lemma 17.

As a concluding remark, let us notice that Theorem the improved Lipschitz estimate of Theorem 1 can be extended to the system

$$\left\{ \begin{aligned} -\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon &= f + \nabla \cdot F, \quad x \in D_\psi^\varepsilon(0,1), \\ u^\varepsilon &= 0, \qquad\qquad\quad x \in \Delta_\psi^\varepsilon(0,1). \end{aligned} \right. \tag{6.19}$$

The proof goes through exactly as in the paper [KP15].

Theorem 18. *Let $0 < \mu < 1$ and $\kappa > 0$. There exists $C > 0$, such that for all $\psi \in W^{1,\infty}(\mathbb{R}^{d-1})$, for all matrix $A = A(y) = (A_{ij}^{\alpha\beta}(y)) \in \mathbb{R}^{d^2 \times N^2}$, elliptic with constant λ, 1-periodic and Hölder continuous with exponent $\nu > 0$, for all $0 < \varepsilon < 1/2$, for all $f \in L^{d+\kappa}(D_\psi^\varepsilon(0,1))$, for all $F \in C^{0,\mu}(D_\psi^\varepsilon(0,1))$, for all u^ε weak solution to (6.19), for all*

$r \in [\varepsilon, 1/2]$,

$$r^{-d} \int_{(-r,r)^{d-1}} \int_{\varepsilon\psi(x'/\varepsilon)}^{\varepsilon\psi(x'/\varepsilon)+r} |\nabla u^\varepsilon|^2 dx_d dx'$$
$$\leq C \left\{ \int_{(-1,1)^{d-1}} \int_{\varepsilon\psi(x'/\varepsilon)}^{\varepsilon\psi(x'/\varepsilon)+1} |\nabla u^\varepsilon|^2 dx_d dx' + \|f\|_{L^{d+\kappa}(D^\varepsilon(0,1))} + \|F\|_{C^{0,\mu}(D^\varepsilon(0,1))} \right\}.$$

Notice that $C = C(d, N, \lambda, [A]_{C^{0,\nu}}, \|\psi\|_{W^{1,\infty}}, \kappa, \mu)$.

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