Lecture 6: Regularity Theory by Compactness Methods

Christophe Prange*

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This lecture is concerned with the regularity theory for (quasilinear) elliptic equations (or systems). We handle two different situations:

(1) the case of quasilinear elliptic equations

$$-\nabla\cdot A(x,u)\nabla u=0,\quad x\in\Omega,\tag{0.1}$$

where the nonlinearity is gentle,

(2) the case of linear elliptic equations with highly oscillating coefficients

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon=0,\quad x\in\Omega.\tag{0.2}$$

We focus on interior regularity, away from boundaries. The issue of regularity for elliptic equations is local. It is based on estimates which prove that the solution gets flatter and flatter, when zooming in to the smaller scales. We address the issue of regularity via compactness methods. There are two key points. The starting point is always an *improvement of flatness* estimate. The second point consists in *iterating* this estimate down to smaller scales.

As for the improvement of flatness, the two situations highlight different phenomena. In the first case of the nonlinear equation (0.1), the idea is roughly speaking that the more you zoom in the closest you get to an equation with constant coefficients (thus linear). In the second case (0.2), when ε goes to zero, i.e. when you zoom out, homogenization takes place, and one gets closer to the situation with constant homogenized coefficients. In the end, regularity for the elliptic equations with non constant coefficients follows from the regularity theory for equations with constant coefficients.

The main assumptions on $A=(A^{\alpha\beta}(x,u))$ are:

(A1) A is elliptic, i.e. there exists $\lambda>0$, such that for all $\xi\in\mathbb{R}^d$, for all $x\in\mathbb{R}^d$, $u\in\mathbb{R}$,

$$\lambda|\xi|^2\leq A(x,u)\xi\cdot\xi,\tag{0.3}$$

*Université de Bordeaux, 351 cours de la Libération, 33405 Talence. *E-mail address:* christophe.prange@math.cnrs.fr

(A2) A is bounded measurable, i.e. there exists $L > 0$ such that

$$\|A\|_{L^\infty(\mathbb{R}^d\times\mathbb{R})} \le L. \tag{0.4}$$

Denote by $\mathcal{A}(\lambda, L)$ the set of A satisfying (0.3) and (0.4). Since we only consider interior regularity, no (smoothness) assumption is made on the domain $\Omega \subset \mathbb{R}^d$.

1 Caccioppoli's inequality

The setup of the regularity results is like this: consider a family of $H^1(\Omega)$ weak solutions u to (0.1) or u^ε to (0.2) bounded in $L^2(\Omega)$. This $L^2(\Omega)$ bound yields compactness of the family of solutions, in particular via the Caccioppoli inequality, which is the starting point of the regularity theory for elliptic equations.

Proposition 1 (Caccioppoli's inequality)**.** *Assume that $A \in \mathcal{A}(\lambda, L)$. Let $u \in H^1_{loc}(\Omega)$ be a weak solution to* (0.1)*, i.e. for all $\varphi \in \mathcal{C}^\infty_c(\Omega)$,*

$$\int_\Omega A(x,u)\nabla u \cdot \nabla\varphi = 0. \tag{1.1}$$

Then there exists $C = C(d,\lambda,L)$ such that for all $x_0 \in \Omega$, for all $0 < \rho < R <$ $\operatorname{dist}(x_0,\partial\Omega)$, for all $m \in \mathbb{R}$, the following estimate holds

$$\int_{B(0,\rho)} |\nabla u|^2 \le \frac{C}{(R-\rho)^2}\int_{B(0,R)\setminus B(0,\rho)} |u-m|^2.$$

Proof. Let $x_0 \in \Omega$ and $R < \operatorname{dist}(x_0,\partial\Omega)$.

(1) We first choose a cut-off function $\eta \in C^\infty_c(B(x_0,R))$ such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } B(x_0,R), \quad |\nabla\eta| \lesssim \frac{1}{R-\rho}.$$

Existence of such an η is standard.

(2) Let $m \in \mathbb{R}$. Without loss of generality, we may assume $m = 0$. Indeed if u is a weak solution to (0.1), then $u - m$ is a weak solution to

$$-\nabla\cdot A_m(x,u)\nabla(u-m) = 0, \quad x \in \Omega,$$

where $A_m(x,u) := A(x,u+m)$ is elliptic with constant λ and bounded by L. We now test the equation against $\eta^2 u \in H^1_0(\Omega)$. It is clearly an admissible test function, because the class of test functions in (1.1) can be extended by density to $H^1_0(\Omega)$. We get

$$0 = \int_\Omega A(x,u)\nabla u\cdot\nabla(\eta^2 u) = \int_\Omega A(x,u)\nabla u\cdot\left(2\eta\nabla\eta u + \eta^2\nabla u\right),$$

hence with Cauchy-Schwarz's and Young's inequalities

$$\begin{aligned}
\lambda\int_\Omega \eta^2|\nabla u|^2 &\le \int_\Omega \eta^2 A(x,u)\nabla u\cdot\nabla u = -\int_\Omega 2\eta A(x,u)\nabla u\cdot(\nabla\eta)u \\
&\le 2\|A\|_{L^\infty}\left(\int_\Omega \eta^2|\nabla u|^2\right)^{1/2}\left(\int_\Omega |\nabla\eta|^2|u|^2\right)^{1/2} \\
&\le \frac{\lambda}{2}\int_\Omega \eta^2|\nabla u|^2 + \frac{2}{\lambda}\left(2\|A\|_{L^\infty}\right)^2\int_\Omega |\nabla\eta|^2|u|^2.
\end{aligned}$$

Eventually,

$$\int_{B(0,\rho)} |\nabla u|^2 \le \int_\Omega \eta^2 |\nabla u|^2 \le \frac{C}{(R-\rho)^2} \int_{B(0,R)\setminus B(0,\rho)} |u|^2,$$

with $C = C(d, \lambda, L)$, which ends the proof. □

Caccioppoli's inequality also clearly holds for the $H^1(\Omega)$ solutions u^ε to (0.2), with a constant C uniform in ε, since the only properties of A which are involved are the ellipticity and the boundedness. If A is constant, one can of course reiterate Caccioppoli's inequality and get estimates on higher-order derivatives. Though simple, this inequality is the essence of the regularity theory for elliptic equations of the type considered here. In the end, the regularity results considered in this lecture boil down to Caccioppoli's inequality.

Corollary 2. *Let $u \in H^1_{loc}(\Omega)$ be a weak solution to*

$$-\nabla \cdot B \nabla u = 0, \quad x \in \Omega.$$

Assume that B is constant (independent of x and u) and belongs to $\mathcal{A}(\lambda, L)$. Then, for all $x_0 \in \Omega$, for all $0 < 2\rho \le R < \mathrm{dist}(x_0, \partial\Omega)$,

$$\fint_{B(x_0,\rho)} |u|^2 \le C \fint_{B(x_0,R)} |u|^2 \tag{1.2}$$

$$\fint_{B(x_0,\rho)} |u - (\bar u)_{x_0,\rho}|^2 \le C \left(\frac{\rho}{R}\right)^2 \fint_{B(x_0,R)} |u - (\bar u)_{x_0,R}|^2, \tag{1.3}$$

where

$$(\bar u)_{x_0,\rho} := \fint_{B(0,\rho)} u$$

and $C = C(d, \lambda, L)$.

Proof. For all $x_0 \in \Omega$, for all $0 < 4\rho/3 \le R < \mathrm{dist}(x_0, \partial\Omega)$, for $m > d/2$,

$$\fint_{B(x_0,\rho)} |u|^2 \le C\|u\|_{L^\infty(B(x_0,\rho))} \le C\|u\|_{L^\infty(B(x_0,3R/4))} \le C\|u\|_{H^m(B(x_0,3R/4))} \le C \fint_{B(x_0,R)} |u|^2$$

by (in this order) Sobolev's imbedding $H^m(B(x_0,\rho))$ into $L^\infty(B(x_0,\rho))$ and reiterated Caccioppoli's inequality. Hence (1.2). For (1.3), it is enough to notice that ∇u is a weak solution to the equation. Therefore, for all $x_0 \in \Omega$, for all $0 < 2\rho \le R < \mathrm{dist}(x_0, \partial\Omega)$,

$$\fint_{B(x_0,\rho)} |\nabla u|^2 \le C \fint_{B(x_0,5R/6)} |\nabla u|^2,$$

because $\rho \le \frac{R}{2} \le \frac{3}{4}\frac{5}{6}R$. Applying Poincaré's inequality on the left and Caccioppoli's inequality on the right

$$\fint_{B(x_0,\rho)} |u - (\bar u)_{x_0,\rho}|^2 \le C\rho^2 \fint_{B(x_0,\rho)} |\nabla u|^2$$
$$\le C\rho^2 \fint_{B(x_0,5R/6)} |\nabla u|^2 \le C \left(\frac{\rho}{R}\right)^2 \fint_{B(x_0,R)} |u - (\bar u)_{x_0,R}|^2$$

yields the result. All constants above only depend on d, λ and L. □

Remark 1 (For what purpose do we use Caccioppoli's inequality?). Besides regularity for equations with constant coefficients as in Corollary 2, we use Caccioppoli's inequality to upgrade weak convergence in $L^2(B(0,1))$ into strong convergence in $L^2(B(0,1/2))$. Indeed, if a sequence of solutions is bounded in $L^2(B(0,1))$, by Caccioppoli's inequality, it is bounded in $H^1(B(0,1/2))$. Rellich's compact injection theorem yields the conclusion.

2 Measures of regularity

With state here a characterization of Hölder continuity in terms of oscillation measured in L^2 norm. For details and proofs, we refer to [Gia83, Chapter III.1].

Let Ω be a open domain in $\mathbb{R}^d$. We define the Campanato spaces for $p = 2$. By $\mathcal{L}^{2,\alpha}(\Omega)$ we denote the space of functions $u \in L^2(\Omega)$ such that

$$[u]_{2,\alpha}^2 := \sup_{x_0 \in \Omega,\ \rho > 0} \rho^{-\alpha} \fint_{B(x_0,\rho)\cap\Omega} |u(x) - (\bar{u})_{x_0,\rho}|^2 < \infty.$$

Theorem 3 (Theorem 1.2, Chapter III.1 in [Gia83]). *Let Ω be a Lipschitz domain and $0 < \alpha < 2$. Then,*

$$\mathcal{L}^{2,\alpha}(\Omega) \simeq \mathcal{C}^{0,\frac{\alpha}{2}}(\Omega).$$

In particular the semi-norm $[\cdot]_{2,\alpha}$ of $\mathcal{L}^{2,\alpha}(\Omega)$ is equivalent to $[\cdot]_{\frac{\alpha}{2}}$ of $\mathcal{C}^{0,\frac{\alpha}{2}}(\Omega)$.

3 Hölder regularity for quasilinear elliptic equations

This part of the lecture is inspired by [Gia83, Chapter IV.1] and [Eva90, Chapter 3.D]

Lemma 4. *Let $A_n = (A_n^{\alpha\beta}(x))$ be a family (indexed by n) of $\mathcal{A}(\lambda, L)$. Let $v_n = v_n(x)$ be a family of weak solutions in $H^1_{loc}(B(0,1)) \cap L^2(B(0,1))$, i.e. for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\int_{B(0,1)} A_n(x)\nabla v_n \cdot \nabla\varphi = 0.$$

Assume that

(1) the matrix

$$A_n^{\alpha\beta}(x) \quad \textit{converges almost everywhere to} \quad A^{\alpha\beta}(x) \in \mathcal{A}(\lambda, L),$$

(2) the sequence

$$v_n \rightharpoonup v \quad \textit{weakly in} \quad L^2(B(0,1)).$$

Then, $v \in H^1_{loc}(B(0,1))$ and for all $0 < \rho < 1$,

$$v_n \to v \quad \textit{strongly in} \quad L^2(B(0,\rho)),$$
$$\nabla v_n \rightharpoonup \nabla v \quad \textit{weakly in} \quad L^2(B(0,\rho)),$$

and for all $\varphi \in \mathcal{C}_c^\infty(B(0,1))$,

$$\int_{B(0,1)} A(x)\nabla v \cdot \nabla\varphi = 0.$$

Proof. Since the sequence v_n is bounded uniformly in $L^2(B(0,1))$, it follows from Caccioppoli's inequality that for all $0<\rho<1$, v_n is bounded uniformly in $H^1(B(0,\rho))$, so that

$$\begin{aligned} v_n \to v \quad &\text{strongly in} \quad L^2(B(0,\rho)),\\ \nabla v_n \rightharpoonup \nabla v \quad &\text{weakly in} \quad L^2(B(0,\rho)). \end{aligned}$$

Now, let us prove that v is a weak solution. Let $\varphi \in \mathcal{C}_c^\infty(B(0,1))$. We have,

$$\begin{aligned} \int_{B(0,1)} A_n(x)\nabla v_n \cdot \nabla\varphi - \int_{B(0,1)} A(x)\nabla v \cdot \nabla\varphi = &\int_{B(0,1)} \left(A_n(x) - A(x)\right) \nabla v_n \cdot \nabla\varphi\\ &+ \int_{B(0,1)} A(x)\nabla(v_n - v)\cdot\nabla\varphi. \end{aligned}$$

The convergence to zero of the second term in the right hand side is easy to handle thanks to the weak convergence of ∇v_n. For the first term, we have

$$\begin{aligned} &\int_{B(0,1)} \left(A_n(x) - A(x)\right)\nabla v_n \cdot \nabla\varphi\\ &\qquad \leq \|\nabla\varphi\|_{L^\infty(B(0,1))} \left(\int_{B(0,1)} |A_n(x) - A(x)|^2\right)^{1/2} \left(\int_{\mathrm{Supp}(\varphi)} |\nabla v_n|^2\right)^{1/2}. \end{aligned}$$

Now, ∇v_n is bounded uniformly in n in $L^2(\mathrm{Supp}(\varphi))$ and the $L^2(B(0,1))$ norm of $A_n - A$ goes to zero by dominated convergence. □

Denote by $\mathcal{A}(\lambda, L, \nu)$ the class of matrices $A \in \mathcal{A}(\lambda, L)$ such that in addition A is Hölder continuous with exponent ν and $[A]_\nu \leq L$.

Theorem 5. *Let $A \in \mathcal{A}(\lambda, L, \nu)$. Let $u \in H^1_{loc}(\Omega)_{loc}$ be a weak solution to the quasilinear elliptic equation 0.1, i.e. for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$,*

$$\int_\Omega A(x,u)\nabla u \cdot \nabla\varphi = 0.$$

Then, there exists an open set $\Omega_0 \subset \Omega$ such that u is Hölder continuous in Ω_0, $u \in \mathcal{C}^{0,\mu}(\Omega_0)$ for all $0 < \mu < 1$, and

$$|\Omega \setminus \Omega_0| = 0.$$

Remark 2. Notice that the set Ω_0 is independent of the exponent μ of the class of Hölder functions. The set Ω_0 only depends on d, λ and L.

In the following two lemmas, we state that if the excess function

$$\fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx$$

is sufficiently small for a given ball $B(x_0,R)$, then

$$\fint_{B(x_0,\rho)} |u - (\bar{u})_{x_0,\rho}|^2 dx$$

decays as a power of ρ. The proof is in two steps:

(1) a first step based on the idea that if the excess function is small enough the blow-up profiles are close to solving a linear equation with constant coefficients, for which improved regularity is known,

(2) a second step which consists in iteration the estimate obtained in the first step down to smaller scales.

Lemma 6 (Improvement). *Let $L > 0$ and $\lambda > 0$ be fixed. There exists a constant $C > 0$, for all $0 < \theta < 1/2$, there exists $\eta_0 > 0$ and $R_0 > 0$ such that for all $A \in \mathcal{A}(\lambda, L, \nu)$, A is continuous, for all weak solution $u \in H^1_{loc}(\Omega)$ to the quasilinear elliptic equation* (0.1)*, for all $x_0 \in \Omega$, for all $0 < R < \inf(R_0, \operatorname{dist}(x_0, \partial\Omega))$, the bound*

$$\fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx < \eta_0^2$$

implies

$$\fint_{B(x_0,\theta R)} |u - (\bar{u})_{x_0,\theta R}|^2 dx \leq C\theta^2 \fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx.$$

The following lemma is an iteration of the estimate in the first lemma.

Lemma 7 (Iteration). *Let $L > 0$ and $\lambda > 0$ be fixed. Then for all $0 < \mu < 1$, there exists $0 < \theta < 1/2$ such that for all $A \in \mathcal{A}(\lambda, L, \nu)$, A is continuous, for all weak solution $u \in H^1_{loc}(\Omega)$ to the quasilinear elliptic equation* (0.1)*, for all $x_0 \in \Omega$, for all $0 < R < \inf(R_0, \operatorname{dist}(x_0, \partial\Omega))$, for all integer $k \geq 1$, the bound*

$$\fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx < \eta_0^2$$

implies

$$\fint_{B(x_0,\theta^k R)} |u - (\bar{u})_{x_0,\theta^k R}|^2 dx \leq \theta^{2k\mu} \fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx.$$

Proof of Lemma 6. The proof is by contradiction. Let $C_0 = C_0(d, \lambda, L)$ be the constant in Corollary 2. Assume now, by contradiction, that there exists $0 < \theta < 1/2$, there exist sequences

$$\begin{aligned}
&\eta_n \to 0, \quad x_n \in \Omega, \quad 0 < R_n < \operatorname{dist}(x_n, \partial\Omega), \quad R_n \to 0, \\
&A_n = A_n(x,u) \quad \text{a family of} \quad \mathcal{A}(\lambda, L, \nu), \\
&u_n \in H^1(\Omega) \quad \text{a family of weak solutions to} \quad -\nabla \cdot A_n(x,u)\nabla u_n = 0, \quad x \in \Omega,
\end{aligned}$$

so that

$$\fint_{B(x_n,R_n)} |u_n - (\overline{u_n})_{x_n,R_n}|^2 dx = \eta_n^2 \tag{3.1}$$

and

$$\fint_{B(x_n,\theta R_n)} |u_n - (\overline{u_n})_{x_n,\theta R_n}|^2 dx > 2C_0\theta^2 \fint_{B(x_n,R_n)} |u_n - (\overline{u_n})_{x_n,R_n}|^2 dx. \tag{3.2}$$

Let us rescale u_n: for $z \in B(0,1)$,

$$v_n(z) := \frac{u_n(x_n + R_n z) - (\overline{u_n})_{x_n,R_n}}{\eta_n}.$$

Then, $v_n \in H^1(B(0,1))$,

$$\fint_{B(0,1)} v_n = 0,$$

and by assumption (3.1),

$$\fint_{B(0,1)} |v_n(z)|^2 dz = 1.$$

Moreover by (3.2),

$$\fint_{B(0,\theta)} |v_n(z) - (\overline{v_n})_{0,\theta}|^2 dz > 2C_0\theta^2, \tag{3.3}$$

and and for all $\varphi \in \mathcal{C}_c^\infty(B(0,1))$,

$$\int_{B(0,1)} A_n\left(x_n + R_n z, \eta_n v_n(z) + (\overline{u_n})_{x_n,R_n}\right) \nabla v_n \nabla \varphi = 0.$$

Now, the uniform $L^2(B(0,1))$ bound on v_n implies that

$$\eta_n v_n \to 0 \quad \text{strongly in} \quad L^2(B(0,1)),$$

thus, up to a subsequence (still denoted the same),

$$\eta_n v_n \to 0 \quad \text{almost everywhere in} \quad B(0,1).$$

Moreover, up to a subsequence (still denoted the same),

$$v_n \rightharpoonup v \quad \text{weakly in} \quad L^2(B(0,1)),$$

and

$$\fint_{B(0,1)} |v|^2 \leq \fint_{B(0,1)} |v_n|^2 = 1.$$

The sequence of matrices $A_n\left(x_n, (\overline{u_n})_{x_n,R_n}\right)$ is bounded by L, so that a subsequence (still denoted the same) converges to a constant matrix $B \in \mathcal{A}(\lambda, L)$:

$$A_n\left(x_n, (\overline{u_n})_{x_n,R_n}\right) \stackrel{n\to\infty}{\longrightarrow} B.$$

Now, the uniform equi-continuity of A_n implies

$$A_n\left(x_n + R_n z, \eta_n v_n(z) + (\overline{u_n})_{x_n,R_n}\right) \stackrel{n\to\infty}{\longrightarrow} B \quad \text{almost everywhere in} \quad B(0,1).$$

One can therefore rely on Lemma 4 and conclude that the limit v is a weak solution to a constant coefficient elliptic equation, i.e. for all $\varphi \in \mathcal{C}_c^\infty(B(0,1))$,

$$\int_\Omega B\nabla u \cdot \nabla \varphi = 0.$$

We have on the one hand by (1.3),

$$\fint_{B(0,\theta)} |v - (\bar{v})_{0,\theta}|^2 \leq C_0\theta^2 \fint_{B(0,1)} |v|^2 \leq C_0\theta^2,$$

and on the other hand passing to the limit in (3.3)

$$\fint_{B(0,\theta)} |v - (\bar{v})_{0,\theta}|^2 \geq 2C_0\theta^2 \fint_{B(0,1)} |v|^2 = 2C_0\theta^2,$$

which is a contradiction. Hence the lemma with $C = 2C_0$. $\square$

Proof of Lemma 7. Let the $C_0 = C_0(d, \lambda, L)$ be the constant in Corollary 2. Let $0 < \mu < 1$. We choose $0 < \theta < 1/2$ so that

$$2C_0\theta^2 = \theta^{2\mu}.$$

This θ being fixed, we have $\eta_0 > 0$ and $R_0 > 0$ given by Lemma 6. The proof is by iteration on the integer k. For $k = 1$, the estimate is exactly the one of Lemma 6. Let $k \geq 1$ be an integer and assume the lemma is true at rank k. Let $A \in \mathcal{A}(\lambda, L, \nu)$, let $u \in H^1_{loc}(\Omega)$ be a weak solution to the quasilinear elliptic equation (0.1), let $x_0 \in \Omega$, $0 < R < \inf(R_0, \mathrm{dist}(x_0, \partial\Omega))$. Assume that

$$\fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx < \eta_0^2.$$

Then, by iteration hypothesis

$$\fint_{B(x_0,\theta^k R)} |u - (\bar{u})_{x_0,\theta^k R}|^2 dx \leq \theta^{2k\mu} \fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx. \tag{3.4}$$

Consider the properly rescaled function

$$U(z) := \frac{u(x_0 + \theta^k z) - (\bar{u})_{x_0,\theta^k R}}{\theta^{k\mu}}.$$

Then, estimate (3.4) implies

$$\fint_{B(0,R)} |U - (\bar{U})_{0,R}|^2 dx \leq \fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx < \eta_0^2.$$

Therefore, we can apply Lemma 6 to U and get

$$\begin{aligned} \frac{1}{\theta^{2k\mu}} \fint_{B(x_0,\theta^{k+1}R)} |u - (\bar{u})_{x_0,\theta^{k+1}R}|^2 dx &= \fint_{B(0,\theta R)} |U - (\bar{U})_{0,\theta R}|^2 dx \\ &\leq \theta^{2\mu} \fint_{B(0,R)} |U - (\bar{U})_{0,R}|^2 dx = \theta^{2\mu} \fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx, \end{aligned}$$

hence the lemma. □

End of the Proof of Theorem 5. (1) First of all, for any $0 < 2\rho \leq \theta R$, there exists a positive integer k such that

$$\theta^{k+1} R < 2\rho \leq \theta^k R,$$

therefore it is easy to infer from Lemma 7 that

$$\fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx \leq \eta_0^2$$

implies via estimate (1.3) of Corollary (2)

$$\fint_{B(x_0,\rho)} |u - (\bar{u})_{x_0,\rho}|^2 dx \leq C \left(\frac{\rho}{R}\right)^{2k\mu} \fint_{B(x_0,R)} |u - (\bar{u})_{x_0,R}|^2 dx,$$

with $C = C(d, \lambda, L, \theta)$.

(2) By dominated convergence, one can see that for any $u \in L^2_{loc}(\mathbb{R}^d)$, for $R > 0$, fixed,

$$\fint_{B(x,R)} \left| u - \fint_{B(x,R)} u \right|^2 = \fint_{B(x,R)} |u|^2 - \left(\fint_{B(x,R)} u \right)^{1/2},$$

is a continuous function of $x \in \mathbb{R}^d$. Therefore, the set $\Omega_0 \subset \Omega$ such that

$$\fint_{B(x_0,R)} \left| u - \fint_{B(x_0,R)} u \right|^2 < \eta_0^2$$

for $x \in \Omega_0$, is open.

(3) By Lebesgue's differentiation theorem, for almost every $x_0 \in \Omega$,

$$\fint_{B(x_0,R)} \left| u - \fint_{B(x_0,R)} u \right|^2 \xrightarrow{R \to 0} 0.$$

Thus $\Omega \setminus \Omega_0$ is of measure 0. $\square$

4 Improved Lipschitz regularity in homogenization

In this part of the lecture, we investigate the possibility of getting Hölder and Lipschitz estimates for

$$-\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = 0, \quad x \in B(0,1),$$

uniform in ε. This is an other instance where the compactness method is effective, though here the improvement of flatness does not come from zooming in to the small scale, but rather from zooming out to the large scales, i.e. letting $\varepsilon \to 0$. The results are based on the pioneering work by Avellaneda and Lin [AL87].

In addition to the ellipticity (0.3) and to the boundedness (0.4), we assume that

$$A = A(y) \quad \text{is} \quad \mathbb{Z}^d - \text{periodic}.$$

The class of such matrices is denoted by $\mathcal{A}_{per}(\lambda, L)$. We have seen (cf. Lecture 1 and Lecture 4 in particular) that under this assumption homogenization holds. Homogenization is enough to get the following Hölder estimate down to the scale ε.

Proposition 8 (Hölder regularity). *Let $\lambda > 0$ and $L > 0$ be fixed. For all $0 < \mu < 1$, there exists a constant $C > 0$, such that for all $A \in \mathcal{A}_{per}(\lambda, L)$, for all families of weak solutions $u^\varepsilon \in H^1(B(0,1))$ to*

$$-\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = 0, \quad x \in B(0,1),$$

the bound

$$\fint_{B(0,1)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,1}|^2 \leq 1,$$

implies for all $0 < \varepsilon < 1/2$, $\varepsilon < r < 1/2$,

$$\fint_{B(0,r)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,r}|^2 \leq C r^{2\mu}.$$

Remark 3. This estimate can be read as a Hölder estimate for the large scales $\varepsilon < r < 1/2$. On these scales, u^ε inherits the regularity of the limit equation with constant (homogenized) coefficients. On the smaller scales, $0 < r \leq \varepsilon$, the regularity is the classical regularity of the equation with non highly oscillating coefficients. For scalar equations, $A \in L^\infty$ is enough for the classical Hölder regularity, with some exponent $0 < \mu < 1$ to hold (De Giorgi, Nash, Moser estimate). For systems, classical Hölder regularity holds for all $0 < \mu < 1$ as long as $A \in \mathcal{C}^{0,\mu'}$, with some $\mu' > 0$.

Homogenization is enough for Proposition 8 to hold. In order to establish Lipschitz estimates, we will need the existence of bounded cell correctors χ. The proof will be written for periodic structures, but everything carries over mutatis mutandis to quasiperiodic structures with the diophantine condition.

Proposition 9 (Lipschitz regularity). *Let $\lambda > 0$ and $L > 0$ be fixed. There exists a constant $C > 0$, such that for all $A \in \mathcal{A}_{per}(\lambda, L)$, for all families of weak solutions $u^\varepsilon \in H^1(B(0,1))$ to*

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = 0, \quad x \in B(0,1),$$

the bound

$$\fint_{B(0,1)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,1}|^2 \leq 1,$$

implies for all $0 < \varepsilon < 1/2$, $\varepsilon < r < 1/2$,

$$\fint_{B(0,r)} |\nabla u^\varepsilon - (\overline{\nabla u^\varepsilon})_{0,r}|^2 \leq C.$$

Lemma 10 (Improvement). *Let $L > 0$ and $\lambda > 0$ be fixed. For all $0 < \mu < 1$, there exists $0 < \theta < 1/4$, there exists $\varepsilon_0 > 0$ such that for all $A \in \mathcal{A}_{per}(\lambda, L)$, for all family of weak solutions $u^\varepsilon \in H^1(\Omega)$ to*

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = 0, \quad x \in B(0,1),$$

the bound

$$\fint_{B(0,1)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,1}|^2 \leq 1,$$

implies for all $0 < \varepsilon < \varepsilon_0$,

$$\fint_{B(0,\theta)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,\theta} - (\overline{\nabla u^\varepsilon})_{0,\theta} \cdot (x + \varepsilon\chi(x/\varepsilon))|^2 \leq \theta^{2+2\mu}.$$

Lemma 11 (Iteration). *Let $L > 0$ and $\lambda > 0$ be fixed. Let also $\mu > 0$, $\theta > 0$ and $\varepsilon_0 > 0$ be given by Lemma 10. For all $A \in \mathcal{A}_{per}(\lambda, L)$, for all family of weak solutions $u^\varepsilon \in H^1(\Omega)$ to*

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = 0, \quad x \in B(0,1),$$

the bound

$$\fint_{B(0,1)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,1}|^2 \leq 1,$$

implies for all positive integer k, for all $0 < \varepsilon < \theta^{k-1}\varepsilon_0$, there exists

$$a_k^\varepsilon \in \mathbb{R}^d, \quad |a_k^\varepsilon| \leq C(1 + \theta + \dots \theta^{k-1}),$$

with $C = C(d, \lambda, L, \mu)$ such that

$$\fint_{B(0,\theta^k)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,\theta^k} - a_k^\varepsilon \cdot (x + \varepsilon\chi(x/\varepsilon))|^2 \leq \theta^{2k(1+\mu)}.$$

Remark 4. The condition $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ gives a lower bound

$$\frac{\varepsilon}{\varepsilon_0} < \theta^{k-1}$$

on the scales one can reach uniformly in ε.

Proof of Lemma 10. Let $\lambda > 0$, $L > 0$ and $0 < \mu < 1$ be fixed. Let $B = (B^{\alpha\beta})$ a constant matrix belonging to $\mathcal{A}(\lambda, L)$. Let $u \in H^1(B(0, 1/2))$ be a weak solution to

$$-\nabla \cdot B\nabla u = 0, \quad x \in B(0, 1/2).$$

such that

$$\fint_{B(0,1/2)} |u - (\bar{u})_{0,1/2}|^2 \leq C\fint_{B(0,1/2)} |\nabla u|^2 \leq C_0\fint_{B(0,1)} |u - (\bar{u})_{0,1}|^2 \leq C_0. \qquad (4.1)$$

Then, ∇u is also a weak solution. Therefore, estimate (1.3) of Corollary 2 implies that

$$\fint_{B(0,\theta)} |\nabla u - (\overline{\nabla u})_{0,\theta}|^2 \leq C\theta^2\fint_{B(0,1/4)} |\nabla u - (\overline{\nabla u})_{0,1/4}|^2,$$

so that by Poincaré's and Caccioppoli's inequalities,

$$\begin{aligned}\fint_{B(0,\theta)} |u - (\overline{u})_{0,\theta} - (\overline{\nabla u})_{0,\theta} \cdot x|^2 &\leq C\theta^2\fint_{B(0,\theta)} |\nabla u - (\overline{\nabla u})_{0,\theta}|^2 \\ &\leq C\theta^4\fint_{B(0,1/4)} |\nabla u - (\overline{\nabla u})_{0,1/4}|^2 = C\theta^4\fint_{B(0,1/4)} |\nabla(u - (\overline{\nabla u})_{0,1/4} \cdot x)|^2 \\ &\leq C_1\theta^4\fint_{B(0,1/2)} |u - (\bar{u})_{0,1/2}|^2 \leq C_0C_1\theta^4, \qquad (4.2)\end{aligned}$$

where $C_0 = C_0(d, \lambda, L)$ is the constant in (4.1) and $C_1 = C_1(d, \lambda, L)$. We choose for the rest of the proof $0 < \theta < 1/4$ such that

$$C_0C_1\theta^4 < \theta^{2+2\mu}.$$

The proof is by contradiction. Assume that there exists sequences

$$\varepsilon_n \to 0, \quad A_n \in \mathcal{A}_{per}(\lambda, L) \quad \text{and} \quad u_n^{\varepsilon_n} \in H^1(B(0, 1)),$$

weak solution to

$$-\nabla \cdot A_n(x/\varepsilon_n)\nabla u_n^{\varepsilon_n} = 0, \quad x \in B(0, 1),$$

such that

$$\fint_{B(0,1)} |u_n^{\varepsilon_n} - (\overline{u_n^{\varepsilon_n}})_{0,1}|^2 = 1,$$

and

$$\fint_{B(0,\theta)} |u_n^{\varepsilon_n} - (\overline{u_n^{\varepsilon_n}})_{0,\theta} - (\overline{\nabla u_n^{\varepsilon_n}})_{0,\theta} \cdot (x + \varepsilon_n \chi_n(x/\varepsilon_n))|^2 > \theta^{2+2\mu}. \tag{4.3}$$

Without loss of generality, we can assume

$$(\overline{u_n^{\varepsilon_n}})_{0,1} = 0.$$

Then, up to a subsequence,

$$\begin{aligned} u_n^{\varepsilon_n} &\to u \quad \text{strongly in} \quad L^2(B(0,1/2)), \\ \nabla u_n^{\varepsilon_n} &\rightharpoonup \nabla u \quad \text{weakly in} \quad L^2(B(0,1/2)), \end{aligned}$$

and $(\bar{u})_{0,1}$. By the homogenization theorem, there exists a constant matrix $B \in \mathcal{A}(\lambda, L)$ such that

$$-\nabla \cdot B \nabla u = 0, \quad x \in B(0,1/2).$$

Moreover,

$$\begin{aligned} \fint_{B(0,1/2)} |u - (\bar{u})_{0,1/2}|^2 &\le \liminf_n \fint_{B(0,1/2)} |u_n^{\varepsilon_n} - (\overline{u_n^{\varepsilon_n}})_{0,1/2}|^2 \\ &\le C_0 \liminf_n \fint_{B(0,1)} |u_n^{\varepsilon_n} - (\overline{u_n^{\varepsilon_n}})_{0,1}|^2 \le C_0, \end{aligned}$$

where $C_0 = C_0(d, \lambda, L)$ is the constant in (4.1). Now, we can pass to the limit in (4.3) and get

$$\fint_{B(0,\theta)} |u - (\bar{u})_{0,\theta} - (\overline{\nabla u})_{0,\theta} \cdot x|^2 > \theta^{2+2\mu},$$

which contradicts (4.2). $\square$

Proof of Lemma 11. The proof is by iteration on the integer k. The estimate for $k = 1$ is exactly the one of Lemma 10. Let $k \ge 1$. Let $A \in \mathcal{A}_{per}(\lambda, L)$, and $u^\varepsilon \in H^1(\Omega)$ such that

$$-\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = 0, \quad x \in B(0,1),$$

and

$$\fint_{B(0,1)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,1}|^2 \le 1.$$

Assume that for $0 < \varepsilon < \theta^{k-1}\varepsilon_0$, there exists

$$a_k^\varepsilon \in \mathbb{R}^d, \quad |a_k^\varepsilon| \le C(1 + \theta + \ldots \theta^{k-1}),$$

with $C = C(d, \lambda, L, \mu)$ such that

$$\fint_{B(0,\theta^k)} |u^\varepsilon - (\overline{u^\varepsilon})_{0,\theta^k} - a_k^\varepsilon \cdot (x + \varepsilon \chi(x/\varepsilon))|^2 \le \theta^{2k(1+\mu)}. \tag{4.4}$$

Then, consider the rescaled function

$$U^\varepsilon(z) := \frac{1}{\theta^{2k(1+\mu)}} \left\{ u^\varepsilon(\theta^k z) - (\overline{u^\varepsilon})_{0,\theta^k} - a_k^\varepsilon \cdot (\theta^k z + \varepsilon\chi(\theta^k z/\varepsilon)) \right\}.$$

We have

$$(\overline{U^\varepsilon})_{0,1} = \fint_{B(0,\theta^k)} \left\{ u^\varepsilon(x) - (\overline{u^\varepsilon})_{0,\theta^k} - a_k^\varepsilon \cdot (x + \varepsilon\chi(x/\varepsilon) \right\} dx = 0,$$

and $U^\varepsilon \in H^1(B(0,1))$ is a weak solution to

$$-\nabla \cdot A(\theta^k x/\varepsilon)\nabla U^\varepsilon = 0, \quad x \in B(0,1).$$

Moreover by (4.4),

$$\fint_{B(0,1)} |U^\varepsilon(z)|^2 \leq 1.$$

Therefore, by Lemma 10, for $\varepsilon/\theta^k < \varepsilon_0$,

$$\fint_{B(0,\theta)} |U^\varepsilon - (\overline{U^\varepsilon})_{0,\theta} - (\overline{\nabla U^\varepsilon})_{0,\theta} \cdot (x + \varepsilon/\theta^k \chi(\theta^k x/\varepsilon))|^2 \leq \theta^{2+2\mu}.$$

After rescaling, this yields exactly the estimate at rank $k+1$. $\square$

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