Lecture 5: Interior and Boundary Layer Correctors

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In this lecture, we consider divergence form elliptic equations with coefficients $A = (A^{\alpha\beta}(y))$, though all the results carry over to elliptic systems. The coefficients are not assumed to be symmetric and, in addition to ellipticity, we assume $A \in L^\infty(\mathbb{T}^d)$. If smoothness on A is required, it will be stated clearly.

1 Bounded correctors in quasiperiodic homogenization

In this part of the lecture, we consider the homogenization of equations with quasiperiodic coefficients

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f, \quad x \in \Omega. \tag{1.1}$$

A quasiperiodic function is the restriction of a periodic function to a linear subspace. More precisely, we assume that there exist an integer $m \geq d$, a $\mathbb{Z}^m$-periodic matrix $B = B(\theta)$, $\theta \in \mathbb{T}^m$, and a constant matrix $N \in M_{md}(\mathbb{R})$, such that

$$A(y) = B(Ny), \quad \forall y \in \mathbb{R}^d. \tag{1.2}$$

Assume, in the whole section, that

$$\forall k \in \mathbb{Z}^m, k \neq 0, \forall 1 \leq \alpha \leq d, \quad (N^T k)_\alpha = \sum_{\beta=1}^m N_{\beta\alpha} k_\beta \neq 0. \tag{1.3}$$

We have explained in the fourth lecture that the homogenization of (1.1) can be recast into the framework of stochastic homogenization. Under assumption (1.3), the dynamical system T is ergodic. This gives the homogenization for μ-almost every realization $\omega \in \Sigma$, which is of course a weaker result than individual homogenization, i.e. for any given quasiperiodic microstructure. Moreover, we have proved the existence of a random variable $\gamma_\alpha^\varepsilon = \gamma_\alpha(\omega)$ so that

$$\gamma_\alpha \in L^2_{pot}(\Sigma), \quad A(\omega)\gamma_\alpha \in L^2_{sol}(\Sigma) \quad \text{and} \quad \mathbb{E}(\gamma_\alpha) = e_\alpha.$$

For fixed $\omega \in \Sigma$, there exists a potential function $v_\alpha = v_\alpha(\cdot, \omega)$ for the stationary field $\gamma_\alpha(T(\cdot)\omega)$: for all $y \in \mathbb{R}^d$,

$$\gamma_\alpha(T(y)\omega) = \nabla_y v_\alpha(y, \omega).$$

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Therefore,

$$-\nabla_y \cdot A(T(y)\omega)\nabla_y v_\alpha(y,\omega) = 0,$$

and $\chi^\alpha(y,\omega) := v_\alpha(y,\omega) - y_\alpha$ solves the cell problem (corrector problem)

$$-\nabla \cdot A(T(y)\omega)\nabla_y \chi^\alpha(y,\omega) = \partial_{y_\gamma} A^{\gamma\alpha}(T(y)\omega), \quad y \in \mathbb{R}^d.$$

Here are the facts:

(1) We know that $\gamma_\alpha(T(y)\omega)$ is stationary, thus for all $y \in \mathbb{R}^d$,

$$\gamma_\alpha(T(y)\omega) = \gamma_\alpha(\omega + Ny),$$

which means that $\gamma_\alpha(T(\cdot)\omega)$ is quasiperiodic.

(2) The potential $v_\alpha(y,\omega)$ on the contrary has no reason to be stationary. In particular it may not be quasiperiodic. We have in general no clue on how this potential grows at infinity.

The second point is crucial in order to get error estimates in homogenization.

We focus on the (deterministic) quasiperiodic corrector problem

$$-\nabla \cdot A(y)\nabla_y \chi^\alpha(y) = \partial_{y_\gamma} A^{\gamma\alpha} A(y), \quad y \in \mathbb{R}^d, \tag{1.4}$$

assuming that A satisfies (1.2). The first results on the existence of bounded quasiperiodic solutions to (1.4) are due to Kozlov [Koz78].

Auxiliary problem on the torus

Proving the existence of quasiperiodic correctors to (1.4) cannot be done in full generality on N even satisfying the non degeneracy condition (1.3). In the course of the proof we will add a *diophantine condition* on N. Use of this assumption will in turn cause a loss of derivatives in the estimates. We therefore work assuming in addition that

$$A \in \mathcal{C}^\infty(\mathbb{R}^d).$$

Let us look for $\chi^\alpha = \chi^\alpha(y)$ for which

there exists $\quad X^\alpha = X^\alpha(\theta),\ \theta \in \mathbb{T}^m, \quad$ so that for all $\quad y \in \mathbb{R}^d, \quad \chi^\alpha(y) = X^\alpha(Ny).$

If there exists an X^α solving

$$-N^T\nabla_\theta \cdot B(\theta)N^T\nabla_\theta X^\alpha = N^T\nabla_\theta \cdot B^{\cdot\alpha}, \quad \theta \in \mathbb{T}^m, \quad \text{and} \quad \fint_{\mathbb{T}^m} X^\alpha(\theta)d\theta = 0, \tag{1.5}$$

and if in addition X^α is $\mathcal{C}^\infty$, then χ^α defined by for all $y \in \mathbb{R}^d$, $\chi^\alpha(y) = X^\alpha(Ny)$, is a solution of (1.4).

The drawback of the problem (1.5) is that the gradient $N^T\nabla_\theta$ is degenerate, so that the equation is not elliptic any more. For any $\varphi = \varphi(\theta)$, $\theta \in \mathbb{T}^m$,

$$\lambda \int_{\mathbb{T}^m} \left|N^T\nabla_\theta\varphi\right|^2 \leq \int_{\mathbb{T}^m} B(\theta)N^T\nabla_\theta\varphi \cdot N^T\nabla_\theta\varphi,$$

so that we only control the $L^2(\mathbb{T}^m)$ norm of the degenerate gradient $N^T\nabla_\theta\varphi$, not the whole gradient $\nabla_\theta\varphi$. For example take $m=2$, $d=1$, $N^T=(n_1,n_2)$ and $\varphi(\theta)=e^{ik\cdot\theta}$ for fixed $k\in\mathbb{Z}^2$. Then

$$\int_{\mathbb{T}^m}\left|N^T\nabla_\theta\varphi\right|^2=|n_1k_1+n_2k_2|^2=|n\cdot k|^2$$

which does not bound

$$\int_{\mathbb{T}^m}|\nabla_\theta\varphi|^2=k_1^2+k_2^2\geq 1,\quad \forall k\neq 0.$$

uniformly in k_1 and k_2 in $\mathbb{Z}$. Indeed, either

$$n_1k_1+n_2k_2=0$$

for some $k\in\mathbb{Z}^2\setminus\{0\}$ (if $N\in\mathbb{R}\mathbb{Z}^2$), or for any $\varepsilon>0$, there exists $k\in\mathbb{Z}^2\setminus\{0\}$ such that

$$|n_1k_1+n_2k_2|<\varepsilon.$$

The issue of non ellipticity in (1.5) can be handled by adding a viscous regularization to the equation

$$-N^T\nabla_\theta\cdot B(\theta)N^T\nabla_\theta X_\kappa^\alpha-\kappa\Delta X_\kappa^\alpha=N^T\nabla_\theta\cdot B^{\cdot\alpha},\quad \theta\in\mathbb{T}^m.$$

The key is to get a priori estimates on X_κ^α uniform in κ.

The main advantage of (1.5) is that it is posed on the torus $\mathbb{T}^m$ and not on the whole space. We have thus gained compactness.

A priori estimates

We carry out a priori estimates on (1.5).

Energy estimate Multiplying by X^α and integrating by parts on $\mathbb{T}^m$ we get

$$\int_{\mathbb{T}^m}B(\theta)N^T\nabla_\theta X^\alpha\cdot N^T\nabla_\theta X^\alpha=-\int_{\mathbb{T}^m}B^{\cdot\alpha}\cdot N^T\nabla_\theta X^\alpha.$$

Therefore

$$\lambda\int_{\mathbb{T}^m}\left|N^T\nabla_\theta X^\alpha\right|^2\leq\|B^{\cdot\alpha}\|_{L^\infty(\mathbb{T}^m)}\left(\int_{\mathbb{T}^m}\left|N^T\nabla_\theta X^\alpha\right|^2\right)^{1/2},$$

so that

$$\lambda\left(\int_{\mathbb{T}^m}\left|N^T\nabla_\theta X^\alpha\right|^2\right)^{1/2}\leq\|B^{\cdot\alpha}\|_{L^\infty(\mathbb{T}^m)}.\tag{1.6}$$

Estimates on the derivatives Let $1 \leq \gamma \leq m$. Differentiating (1.5) with respect to ∂_{θ_γ}, we get

$$-N^T\nabla_\theta \cdot B(\theta)N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha = N^T\nabla_\theta \cdot \partial_{\theta_\gamma}B^{\cdot\alpha} + N^T\nabla_\theta \cdot \partial_{\theta_\gamma}B(\theta)N^T\nabla_\theta X^\alpha.$$

We now integrate toward $\partial_{\theta_\gamma}X^\alpha$ and get

$$\int_{\mathbb{T}^m} B(\theta)N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha \cdot N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha = -\int_{\mathbb{T}^m} B^{\cdot\alpha} \cdot N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha - \int_{\mathbb{T}^m} \partial_{\theta_\gamma}B(\theta)N^T\nabla_\theta X^\alpha \cdot N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha.$$

Consequently,

$$\lambda\int_{\mathbb{T}^m}\left|N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha\right|^2 \leq \|B^{\cdot\alpha}\|_{L^\infty(\mathbb{T}^m)}\left(\int_{\mathbb{T}^m}\left|\partial_{\theta_\gamma}N^T\nabla_\theta X^\alpha\right|^2\right)^{1/2} + \|\partial_{\theta_\gamma}B(\theta)\|_{L^\infty(\mathbb{T}^m)}\left(\int_{\mathbb{T}^m}\left|N^T\nabla_\theta X^\alpha\right|^2\right)^{1/2}\left(\int_{\mathbb{T}^m}\left|N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha\right|^2\right)^{1/2},$$

so that combining with the a priori estimate (1.6) on X^α we get

$$\lambda\left(\int_{\mathbb{T}^m}\left|N^T\nabla_\theta\partial_{\theta_\gamma}X^\alpha\right|^2\right)^{1/2} \leq C,$$

with $C = C\left(\|B^{\cdot\alpha}\|_{L^\infty(\mathbb{T}^m)}, \|\partial_{\theta_\gamma}B(\theta)\|_{L^\infty(\mathbb{T}^m)}\right)$.

By iteration, one can therefore prove the following proposition.

Proposition 1. *For all multi-index $\gamma \in \mathbb{N}^m$, the weak solution X^α of* (1.5) *satisfies*

$$\left(\int_{\mathbb{T}^m}\left|N^T\nabla_\theta\partial_\theta^\gamma X^\alpha\right|^2\right)^{1/2} \leq C,$$

with $C = C\left(\lambda, \|B\|_{\mathcal{C}^{|\gamma|}}\right)$.

Poincaré inequality

We have already seen that the L^2 norm of $N^T\nabla_\theta\varphi$ does not control $\nabla_\theta\varphi$.

Question. At what condition on the matrix N does

$$\int_{\mathbb{T}^m}\left|N^T\nabla_\theta\varphi\right|^2$$

control lower-order derivatives of φ?

We aim for some sort of Poincaré inequality for $\varphi = \varphi(\theta)$ of mean zero.

Assume that the following diophantine condition, *small divisors condition*, is satisfied:

$$\begin{aligned} &\text{there exist} \quad C>0, \quad \tau>0, \quad \text{such that for all} \quad k \in \mathbb{Z}^m\setminus\{0\}, \\ &\text{for all} \quad 1\leq\alpha\leq d, \quad \left|(N^Tk)_\alpha\right| \geq \frac{C}{|\xi|^{d+\tau}}. \end{aligned} \tag{1.7}$$

Notice that for all $1 \leq \alpha \leq d$, $(N^Tk)_\alpha = N_{\gamma\alpha}k_\gamma$. We stress that the diophantine condition holds on every component of the d-dimensional vector N^Tk.

Lemma 2. *There exists a constant $C > 0$ such that for any $\varphi = \varphi(\theta)$, $\varphi \in H^{d+\tau}(\mathbb{T}^m)$,*

$$\int_{\mathbb{T}^m} |\varphi - \overline{\varphi}|^2 \leq C\|\varphi\|_{\dot{H}^{d+\tau}(\mathbb{T}^m)} \left(\int_{\mathbb{T}^m} \left|N^T \nabla_\theta \varphi\right|^2\right)^{1/2}.$$

Here $\dot{H}^{d+\tau}(\mathbb{T}^m)$ denotes the homogeneous Sobolev norm.

Proof. Without loss of generality, we assume that $\overline{\varphi} = 0$. By expansion of φ in Fourier series and use of Parseval-Plancherel identity, we have

$$\begin{aligned}
\int_{\mathbb{T}^m} |\varphi(\theta)|^2 d\theta &= \sum_{k\neq 0} |\hat{\varphi}(k)|^2 = \sum_{\beta=1}^{d} \sum_{k\neq 0} \frac{1}{\left|(N^T k)_\beta\right|} |\hat{\varphi}(k)| \left|(N^T k)_\beta\right| |\hat{\varphi}(k)| \\
&\leq C \sum_{\beta=1}^{d} \sum_{k\neq 0} |k|^{d+\tau} |\hat{\varphi}(k)| \left|(N^T k)_\beta\right| |\hat{\varphi}(k)| \\
&\leq C \sum_{\beta=1}^{d} \left(\sum_{k\neq 0} |k|^{2(d+\tau)} |\hat{\varphi}(k)|^2\right)^{1/2} \left(\sum_{k\neq 0} \left|(N^T k)_\beta\right|^2 |\hat{\varphi}(k)|^2\right)^{1/2} \\
&\leq C \left(\sum_{k\neq 0} |k|^{2(d+\tau)} |\hat{\varphi}(k)|^2\right)^{1/2} \sum_{\beta=1}^{d} \left(\sum_{k\neq 0} \left|(N^T k)_\beta\right|^2 |\hat{\varphi}(k)|^2\right)^{1/2} \\
&\leq C\|\varphi\|_{\dot{H}^{d+\tau}(\mathbb{T}^m)} \left(\int_{\mathbb{T}^m} \left|N^T \nabla_\theta \varphi\right|^2\right)^{1/2}.
\end{aligned}$$

Hence the lemma. □

Conclusion

Let l be an integer larger than $d + \tau$. Then

$$\left(\int_{\mathbb{T}^m} \left\|N^T \nabla_\theta \nabla^l X^\alpha\right\|^2\right)^{1/2} \tag{1.8}$$

is controlled thanks to Proposition 1. Now, since for all $0 \leq s \leq \lfloor l - d - \tau \rfloor$,

$$\begin{aligned}
\int_{\mathbb{T}^m} \left|N^T \nabla_\theta \nabla^l X^\alpha\right|^2 &= \sum_{\beta=1}^{d} \sum_{k\neq 0} |k|^{2l} \left|(N^T k)_\beta\right|^2 \left|\widehat{X^\alpha}(k)\right|^2 \\
&\geq C \sum_{k\neq 0} |k|^{2(l-d-\tau)} \left|\widehat{X^\alpha}(k)\right|^2 \\
&\geq C \left\|X^\alpha - \overline{X^\alpha}\right\|_{H^{\lfloor l-d-\tau \rfloor}(\mathbb{T}^m)} = \|X^\alpha\|_{H^{\lfloor l-d-\tau \rfloor}(\mathbb{T}^m)}.
\end{aligned} \tag{1.9}$$

Here $\lfloor l - d - \tau \rfloor$ is the integer part of $l - d - \tau$ and $H^{\lfloor l-d-\tau \rfloor}(\mathbb{T}^m)$ denotes the inhomogeneous (full) Sobolev norm.

Existence under the diophantine assumption

Let us consider the following viscous regularization of (1.5):

$$-N^T\nabla_\theta\cdot B(\theta)N^T\nabla_\theta X^\alpha_\kappa-\kappa\Delta X^\alpha_\kappa=N^T\nabla_\theta\cdot B^{\cdot\alpha},\quad \theta\in\mathbb{T}^m\quad\text{and}\quad \fint_{\mathbb{T}^m}X^\alpha_\kappa=0. \tag{1.10}$$

Standard methods (Lax Milgram) enable to prove the existence of a unique solution $X^\alpha\in\mathcal{C}^\infty(\mathbb{T}^m)$ to (1.10) satisfying in addition the a priori bounds uniform in κ of Proposition 1. It follows from the diophantine condition (1.7) and from (1.9) that for all nonnegative integer s,

$$X^\alpha_\kappa\quad\text{is bounded uniformly in}\quad\kappa\quad\text{in}\quad H^s(\mathbb{T}^m).$$

Existence is thus easy. Uniqueness follows from the Poincaré inequality for the degenerate gradient of Lemma 2. Therefore, we have proved the following theorem.

Theorem 3. *Let $1\leq\alpha\leq d$. Assume that $B\in\mathcal{C}^\infty(\mathbb{T}^m)$, that B is elliptic and periodic, and that the small divisors assumption* (1.7) *holds. Then, there exist a unique solution $X^\alpha\in\mathcal{C}^\infty(\mathbb{T}^m)$ to the problem* (1.5).

Corollary 4. *Let $1\leq\alpha\leq d$. Assume that $A\in\mathcal{C}^\infty(\mathbb{R}^d)$, that A is elliptic and quasiperiodic, and that the small divisors assumption* (1.7) *holds. Then, there exists a unique quasiperiodic (so bounded) corrector $\chi^\alpha\in\mathcal{C}^\infty(\mathbb{R}^d)$ solving* (1.4).

Remark 1 (Loss of derivatives). What happens if A is not infinitely differentiable? Assume A is quasiperiodic, $A(\cdot)=B(N\cdot)$ and N satisfies the diophantine condition (1.7). Let l be a integer such that $l>d+\tau$, τ being given by (1.7). Assume that $B\in\mathcal{C}^l(\mathbb{T}^m)$. Then, X^α satisfies the bounds of Proposition 1 for any $\gamma\in\mathbb{N}^m$, $|\gamma|\leq l$. Now, by (1.9) we know that the inhomogeneous Sobolev norm $H^{\lfloor l-d-\tau\rfloor}(\mathbb{T}^m)$ of X^α is controlled. Therefore, one can prove the existence and uniqueness of X^α in $H^{\lfloor l-d-\tau\rfloor}(\mathbb{T}^m)$. If

$$l>\frac{m}{2}+d+\lceil\tau\rceil+2,\qquad H^{\lfloor l-d-\tau\rfloor}(\mathbb{T}^m)\hookrightarrow\mathcal{C}^2(\mathbb{T}^m),$$

so that we also get a quasiperiodic solution $\chi^\alpha\in\mathcal{C}^2(\mathbb{R}^d)$ to the cell problem (1.4).

2 Boundary layer correctors in homogenization

In this part of the lecture, we investigate the existence of boundary layer correctors:

$$\begin{cases}-\nabla\cdot A(y)\nabla v&=0,\quad y\cdot n>0,\\ \qquad\qquad v&=v_0,\quad y\cdot n=0,\end{cases}\tag{2.1}$$

where $n\in\mathbb{S}^{d-1}$ and $A=A(y)$ as well as $v_0=v_0(y)$ are $\mathbb{Z}^d$-periodic. These correctors come from the study of the asymptotics (blow-up near a boundary point) of systems with highly oscillating coefficients and boundary data, typically

$$\begin{cases}-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon_{bl}&=0,\qquad\qquad\qquad\quad x\in\Omega,\\ \qquad\qquad\quad u^\varepsilon_{bl}&=-\chi(x/\varepsilon)\cdot\nabla u^0(x),\quad x\in\partial\Omega.\end{cases}$$

Contrary to interior correctors which solve an equation posed in $\mathbb{R}^d$, the boundary layer correctors equations are posed in a half-space.

The analysis of (2.1) is contingent on whether the normal $n \in \mathbb{S}^{d-1}$:

(1) has rational coordinates $n \in \mathbb{R}\mathbb{Z}^d$,

(2) or is irrational $n \notin \mathbb{R}\mathbb{Z}^d$.

In the first case, one is typically led to study the system (see [AA99])

$$\begin{cases} -\nabla \cdot A(y)\nabla v &= 0, \qquad (y', y_d) \in \mathbb{T}^{d-1} \times (0,\infty), \\ v &= v_0(y', 0), \quad y' \in \mathbb{T}^{d-1}, \end{cases} \tag{2.2}$$

where $v(\cdot, y_d)$ is $\mathbb{Z}^{d-1}$-periodic. In the second case, the boundary $y \cdot n = 0$ breaks the periodic structure. Therefore, one is led to study a corrector problem with a quasiperiodic structure in the tangential variable. One has to analyze the following degenerate elliptic problem in the half-space (see [GVM11])

$$\begin{cases} -\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B(\theta, t) \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V &= 0, \qquad t > 0, \\ V &= V_0(\theta), \quad t = 0, \end{cases} \tag{2.3}$$

with $N \in M_{d,d-1}(\mathbb{R})$. Two questions are relevant regarding these systems: (i) well-posedness, (ii) asymptotic behavior away from the boundary.

Theorem 5. *There exists a unique solution v to* (2.2) *such that*

$$\int_0^\infty \int_{\mathbb{T}^{d-1}} |\nabla v(y', y_d)|^2 dy' dy_d \leq C\|v_0\|^2_{H^{1/2}(\mathbb{T}^{d-1})},$$

with $C = C(d, \lambda)$. Moreover, if A is in addition $\mathcal{C}^{0,\mu}(\mathbb{T}^d)$, with $\mu > 0$, then there exists $\kappa = \kappa(d, \lambda)$ such that

$$\int_T^\infty \int_{\mathbb{T}^{d-1}} |\nabla v(y', y_d)|^2 dy' dy_d \leq C\|v_0\|^2_{H^{1/2}(\mathbb{T}^{d-1})} \exp(-\kappa T),$$

with $C = C(d, \lambda)$.

Theorem 6. *Assume that $B \in \mathcal{C}^\infty(\mathbb{T}^{d-1} \times (0,\infty))$ and $V_0 \in \mathcal{C}^\infty(\mathbb{T}^d)$. Then there exists a unique solution V to* (2.3) *such that for all multi-index $\gamma \in \mathbb{N}^d$, for all $l \geq 0$,*

$$\int_{\mathbb{T}^d} \int_0^\infty \left|N^T\nabla_\theta \partial_\theta^\gamma V\right|^2 + \left|\partial_t^{l+1} V\right|^2 dt d\theta \leq C\|V_0\|^2_{\mathcal{C}^{|\gamma|+1}}, \tag{2.4}$$

where $C = C\left(\lambda, d, \|B\|_{\mathcal{C}^{|\gamma|+l}}\right)$. Moreover, if the diophantine condition (1.7) *holds for $N \in M_{d,d-1}(\mathbb{R})$, then for all multi-index $\gamma \in \mathbb{N}^d$, for all $l \geq 0$, for all $1 < \kappa < \infty$, there exists $C = C(\lambda, d, \tau, p, B, V_0)$ such that*

$$\int_{\mathbb{T}^d} \int_T^\infty \left|N^T\nabla_\theta \partial_\theta^\gamma V\right|^2 + \left|\partial_t^{l+1} V\right|^2 dt d\theta \leq CT^{-\kappa}.$$

Remark 2. As for the existence of bounded correctors in homogenization, the use of the small divisors assumption to get the asymptotic behavior of V is responsible for a loss of derivatives, which prompts the need for more regularity on A; to make things simple, we take $B \in \mathcal{C}^\infty(\mathbb{T}^{d-1} \times (0,\infty))$. Notice however that the loss of derivatives takes place in the tangential direction, so that it would be enough to only assume smoothness of B in θ.

Proof of Theorem 5

For the existence, we first lift the boundary data. Existence to the problem with homogeneous Dirichlet data and source term compactly supported in y_d in the Hilbert space

$$H_0^1(\mathbb{T}^{d-1} \times (0,\infty)) \quad \text{endowed with the norm} \quad \int_0^\infty \int_{\mathbb{T}^{d-1}} |\nabla v(y', y_d)|^2 dy' dy_d,$$

then follows from the Lax-Milgram lemma.

We now go into more details for the asymptotic behavior of v away from the boundary. The idea is to get a Saint-Venant estimate (also called Phragmen-Lindelöf) estimate on

$$F(T) := \int_T^\infty \int_{\mathbb{T}^{d-1}} |\nabla v(y', y_d)|^2 dy' dy_d.$$

In other words, what we aim for is a differential inequality on $F(T)$. To be able to differentiate F, we need to assume some smoothness of v away from the boundary. Assuming $A \in \mathcal{C}^{0,\mu}(\mathbb{T}^d)$ yields that v is (at least) of class $\mathcal{C}^{1,\mu'}$, $0 < \mu' < \mu$ in the interior of the domain (not necessarily up to the boundary). Let $T > 0$ be fixed. Consider

$$w(y', y_d) := v(y', y_d) - \int_{\mathbb{T}^{d-1}} v(y', T) dy',$$

which is a weak solution to

$$-\nabla \cdot A(y) \nabla w = 0 \quad \text{in} \quad \mathbb{T}^{d-1} \times (0,\infty).$$

Thus, testing against w and integrating by parts on $\mathbb{T}^{d-1} \times (T,\infty)$ one infers

$$\begin{aligned} \lambda \int_T^\infty \int_{\mathbb{T}^{d-1}} |\nabla w|^2 dy' dy_d &\leq -\int_{\mathbb{T}^{d-1}} \nabla w(y', T) \cdot e_d w(y', T) dy' \\ &\leq \left(\int_{\mathbb{T}^{d-1}} |\nabla w(y', T)|^2 dy' \right)^{1/2} \left(\int_{\mathbb{T}^{d-1}} |w(y', T)|^2 dy' \right)^{1/2}. \end{aligned}$$

By the Poincaré-Wirtinger inequality on $\mathbb{T}^{d-1}$, we get

$$\int_{\mathbb{T}^{d-1}} |w(y', T)|^2 dy' \leq C \int_{\mathbb{T}^{d-1}} |\nabla_{y'} w(y', T)|^2 dy' \leq \int_{\mathbb{T}^{d-1}} |\nabla_y w(y', T)|^2 dy'.$$

Since

$$F'(T) = -\int_{\mathbb{T}^{d-1}} |\nabla w(y', T)|^2 dy',$$

we have

$$F(T) \leq C(-F'(T))^{1/2}(-F'(T))^{1/2} = -CF'(T),$$

which implies

$$\int_T^\infty \int_{\mathbb{T}^{d-1}} |\nabla v(y', y_d)|^2 dy' dy_d \leq C \exp(-\kappa T) \int_0^\infty \int_{\mathbb{T}^{d-1}} |\nabla v(y', y_d)|^2 dy' dy_d,$$

hence the result. Notice that the constant C does not depend on the $\mathcal{C}^{0,\mu}$ semi-norm of A.

Proof of Theorem 6

Existence basically comes from the following lemma (see [GVM11, Lemma 3]). Let us notice that the small divisors condition is not needed.

Lemma 7. *Let $Y = Y(\theta, t)$ be a smooth weak solution to*

$$\begin{cases} -\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B(\theta,t) \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} Y & = H + \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot G, \quad t > 0, \\ V & = 0, \quad t = 0, \end{cases} \tag{2.5}$$

with $tH,\ G \in L^2(\mathbb{T}^d \times \mathbb{R}_+)$. Then

$$\int_{\mathbb{T}^d}\int_0^\infty \left|N^T\nabla_\theta Y\right|^2 + |\partial_t Y|^2\, dtd\theta \leq C \int_{\mathbb{T}^d}\int_0^\infty |tH|^2 + |G|^2\, dtd\theta.$$

Proof. The key ingredient is the one-dimensional Hardy inequality (see [Gal11, Inequality II.4.14]) applied in the vertical direction. It takes advantage of the fact that $Y(\cdot, 0) = 0$. Somehow, Hardy's inequality is a good alternative to Poincaré's inequality, since the constant does not depend on the size of the domain: for all $\theta \in \mathbb{T}^d$,

$$\int_0^\infty \left|\frac{Y(\theta,t)}{t}\right|^2 dt \leq C \int_0^\infty |\partial_t Y|^2\, dt.$$

Testing (2.5) against Y and integrating by parts yields

$$\begin{aligned} &\lambda \int_{\mathbb{T}^d}\int_0^\infty \left|N^T\nabla_\theta Y\right|^2 + |\partial_t Y|^2\, dtd\theta \\ &\leq \int_{\mathbb{T}^d}\int_0^\infty tH\frac{Y}{t} dtd\theta - \int_{\mathbb{T}^d}\int_0^\infty G\cdot \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} Y\, dtd\theta \\ &\leq C\left(\int_{\mathbb{T}^d}\int_0^\infty |tH|^2 dtd\theta\right)^{1/2} \left(\int_{\mathbb{T}^d}\int_0^\infty |\partial_t Y|^2 dtd\theta\right)^{1/2} \\ &\quad + \left(\int_{\mathbb{T}^d}\int_0^\infty |G| dtd\theta\right)^{1/2} \left(\int_{\mathbb{T}^d}\int_0^\infty \left|N^T\nabla_\theta Y\right|^2 + |\partial_t Y|^2\, dtd\theta\right)^{1/2}, \end{aligned}$$

hence the result. □

The study of the asymptotic behavior relies on the following Poincaré inequality, which is an improvement on Lemma 2.

Lemma 8. *For all $1 < p < \infty$, there exists a constant $C > 0$ such that for any $\varphi = \varphi(\theta)$, $\varphi \in H^{d+\tau}(\mathbb{T}^d)$,*

$$\int_{\mathbb{T}^d} |\varphi - \overline{\varphi}|^2 \leq C\|\varphi\|^{2-2/p}_{\dot{H}^{\frac{d+\tau}{p-1}}(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} \left|N^T\nabla_\theta\varphi\right|^2\right)^{1/p}.$$

Here $\dot{H}^{\frac{d+\tau}{p-1}}(\mathbb{T}^d)$ denotes the homogeneous Sobolev norm.

Proof. This generalization is just due to another choice of exponents in Hölder's inequality. Without loss of generality, we assume that $\overline{\varphi} = 0$. By expansion of φ in Fourier series and use of Parseval-Plancherel identity, we have for $\alpha := 1/p$, $1/p' := 1 - 1/p$,

$$\begin{aligned}
\int_{\mathbb{T}^d} |\varphi(\theta)|^2 d\theta &= \sum_{k\neq 0} |\hat{\varphi}(k)|^2 = \sum_{\beta=1}^{d} \sum_{k\neq 0} \frac{1}{\left|(N^T k)_\beta\right|^{2\alpha}} |\hat{\varphi}(k)|^{2(1-\alpha)} \left|(N^T k)_\beta\right|^{2\alpha} |\hat{\varphi}(k)|^{2\alpha} \\
&\leq C \sum_{\beta=1}^{d} \sum_{k\neq 0} |k|^{2\alpha(d+\tau)} |\hat{\varphi}(k)|^{2(1-\alpha)} \left|(N^T k)_\beta\right|^{2\alpha} |\hat{\varphi}(k)|^{2\alpha} \\
&\leq C \sum_{\beta=1}^{d} \left(\sum_{k\neq 0} |k|^{\frac{2(d+\tau)}{p-1}} |\hat{\varphi}(k)|^2 \right)^{1-1/p} \left(\sum_{k\neq 0} \left|(N^T k)_\beta\right|^2 |\hat{\varphi}(k)|^2 \right)^{1/p} \\
&\leq C \left(\sum_{k\neq 0} |k|^{\frac{2(d+\tau)}{p-1}} |\hat{\varphi}(k)|^2 \right)^{1-1/p} \sum_{\beta=1}^{d} \left(\sum_{k\neq 0} \left|(N^T k)_\beta\right|^2 |\hat{\varphi}(k)|^2 \right)^{1/p} \\
&\leq C \|\varphi\|^{2-2/p)}_{\dot{H}^{\frac{d+\tau}{p-1}}(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} \left|N^T \nabla_\theta \varphi\right|^2 \right)^{1/p}.
\end{aligned}$$

Hence the lemma. □

Let $T > 0$ and $1 < p < \infty$ be fixed. Consider

$$W(\theta, t) := V(\theta, t) - \int_{\mathbb{T}^d} V(\theta, T) d\theta,$$

which is a smooth weak solution to

$$-\begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B(\theta, t) \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W = 0, \quad t > 0.$$

Testing this equation against W and integrating between T and ∞ we get

$$\begin{aligned}
&\lambda \int_{\mathbb{T}^d} \int_T^\infty \left|N^T \nabla_\theta W\right|^2 + |\partial_t W|^2 \, dt d\theta \\
&\leq - \int_{\mathbb{T}^d} \begin{pmatrix} -N^T \nabla_\theta \\ \partial_t \end{pmatrix} W(\theta, T) \cdot e_{d+1} W(\theta, T) d\theta \\
&\leq \left(\int_{\mathbb{T}^d} \left|N^T \nabla_\theta W(\theta, T)\right|^2 + |\partial_t W(\theta, T)|^2 \, d\theta \right)^{1/2} \left(\int_{\mathbb{T}^d} |W(\theta, T)|^2 \right)^{1/2} \\
&\leq (-F'(T))^{1/2} \left(\int_{\mathbb{T}^d} |W(\theta, T)|^2 \right)^{1/2}.
\end{aligned}$$

Now by the Poincaré inequality of Lemma 8, we have

$$\left(\int_{\mathbb{T}^d} |W(\theta, T)|^2 \right)^{1/2} \leq C \|W(\theta, T)\|^{1-1/p}_{\dot{H}^{\frac{d+\tau}{p-1}}(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} \left|N^T \nabla_\theta W(\theta, T)\right|^2 \right)^{1/(2p)} \leq C(-F'(T))^{1/(2p)},$$

with $C = C(\lambda, d, \tau, p, B, V_0)$. We have used the bounds on the higher-order derivatives (2.4) to bound the $\dot{H}^{\frac{d+\tau}{p-1}}(\mathbb{T}^d)$ norm of $W(\cdot, T)$. Eventually,

$$F(T) \leq C(-F'(T))^{\frac{p+1}{2p}},$$

so that

$$F(T) \leq CT^{\frac{p+1}{1-p}}.$$

As p ranges between 1 and ∞, the power $\frac{p+1}{1-p}$ takes all the values between -1 and $-\infty$. Therefore,

$$\int_{\mathbb{T}^d} \int_T^\infty \left|N^T \nabla_\theta \partial_\theta^\gamma V\right|^2 + \left|\partial_t^{l+1} V\right|^2 dt d\theta$$

decays faster than any power of T when $T \to \infty$.

References

[AA99] Grégoire Allaire and Micol Amar. Boundary layer tails in periodic homogenization. *ESAIM Control Optim. Calc. Var.*, 4:209–243, 1999.

[Gal11] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: steady-states problems*. Springer Monographs in Mathematics. Springer, second edition edition, 2011.

[GVM11] David Gérard-Varet and Nader Masmoudi. Homogenization in polygonal domains. *J. Eur. Math. Soc.*, 13:1477–1503, 2011.

[Koz78] S. M. Kozlov. Averaging of differential operators with almost periodic rapidly oscillating coefficients. *Mat. Sb. (N.S.)*, 107(149)(2):199–217, 317, 1978.