Lecture 4: Stochastic Homogenization

Christophe Prange*

February 12, 2016

1 Review of the periodic case

Let us shed a new light on the homogenization of periodic microstructures thanks to the div-curl lemma. We use the notations introduced in the first lecture. For $1 \leq \alpha \leq d$,

$$\begin{aligned}
\xi^\varepsilon &:= A(x/\varepsilon)\nabla u^\varepsilon,\\
v^\varepsilon_\alpha &:= x_\alpha + \varepsilon\chi^{*,\alpha}(x/\varepsilon),\\
\gamma^{*,\varepsilon}_\alpha &:= \nabla v^\varepsilon_\alpha = e_\alpha + \nabla_y\chi^{*,\alpha}(x/\varepsilon),\\
\eta^{*,\varepsilon}_\alpha &:= A^*(x/\varepsilon)\nabla v^\varepsilon_\alpha = A^*(x/\varepsilon)\gamma^{*,\varepsilon}_\alpha = A^*(x/\varepsilon)(e_\alpha + \nabla_y\chi^{*,\alpha}(x/\varepsilon)).
\end{aligned}$$

We have for all $x \in \Omega$,

$$\xi^\varepsilon \cdot \gamma^{*,\varepsilon}_\alpha = A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla v^\varepsilon_\alpha = \nabla u^\varepsilon \cdot A^*(x/\varepsilon)\nabla v^\varepsilon_\alpha = \nabla u^\varepsilon \cdot \eta^{*,\varepsilon}_\alpha = \eta^{*,\varepsilon}_\alpha \cdot \nabla u^\varepsilon. \quad (1.1)$$

Notice that,

$$\nabla u^\varepsilon,\ \xi^\varepsilon,\ \gamma^{*,\varepsilon}_\alpha \text{ and } \eta^{*,\varepsilon}_\alpha \quad \text{are bounded uniformly in} \quad L^2(\Omega).$$

Moreover,

$$\mathrm{div}(\xi^\varepsilon) = f, \quad \mathrm{curl}(\gamma^{*,\varepsilon}_\alpha) = 0, \quad \mathrm{div}(\eta^{*,\varepsilon}_\alpha) = 0 \quad \text{and} \quad \mathrm{curl}(\nabla u^\varepsilon) = 0.$$

Therefore, by the div-curl lemma, we can pass to the limit in (2.9) and get

$$\xi^0\,(e_\alpha + \overline{\chi^{*,\alpha}}) = \overline{A^*(y)(e_\alpha + \nabla_y\chi^{*,\alpha})}\nabla u^0,$$

that is after simplification

$$\xi^{0,\alpha} = \left(\overline{A^{*\gamma\alpha}} + \overline{A^{*\gamma\beta}\partial_{y_\beta}\chi^{*,\alpha}}\right)\partial_{x_\gamma}u^0.$$

To put it in a nutshell, we recover the expression for the homogenized matrix given in the first lecture.

Remark 1. In this proof, we only use the weak convergence of ∇u^ε, ξ^ε, $\gamma^{*,\varepsilon}_\alpha$ and $\eta^{*,\varepsilon}_\alpha$. Thus resorting to the div-curl lemma enables to bypass in particular the use of the strong convergence of v^ε. As far as homogenization is concerned, we do not need bounded correctors.

*Université de Bordeaux, 351 cours de la Libération, 33405 Talence. *E-mail address:* christophe.prange@math.cnrs.fr

2 Stochastic homogenization

In this section we will free ourselves from the periodic setting. The abstract framework of stochastic homogenization will cover as a particular case homogenization of periodic and quasiperiodic structures.

As an insight into what follows, let us say that we consider the homogenization of

$$-\nabla \cdot A(x/\varepsilon, \omega)\nabla u^\varepsilon(\cdot, \omega) = f, \quad x \in \Omega,$$

with $\omega \in \Sigma$ and $(\Sigma, \mathcal{F}, \mu)$ a probability space. Picking an ω amounts at choosing some structure $A(\cdot, \omega)$. The results we aim for are of the following type: there exists a constant matrix A^0 (independent of the realization) so that for almost every $\omega \in \Sigma$, $u^\varepsilon(\cdot, \omega)$ converges to a solution u^0 of

$$-\nabla \cdot A^0 \nabla u^0 = f, \quad x \in \Omega.$$

More precisely, we will assume that the d-dimensional random process A is of the following form (with a slight abuse of notation)

$$A(y, \omega) = A(T(y)\omega)$$

where $A = A(\omega)$ is a random variable on Σ and $T = (T(y))_{y \in \mathbb{R}^d}$ is an ergodic dynamical system acting on Σ. Our homogenization problem can be rewritten as

$$-\nabla \cdot A(T(x/\varepsilon)\omega)\nabla u^\varepsilon(x, \omega) = f, \quad x \in \Omega.$$

This generalization comes at two expenses:

(1) We get a homogenization theorem for almost every realization (not for all realizations as in the periodic setting developed in the first lecture). In particular, it is not possible to know given a specific realization if the convergence holds or not.

(2) We lose the existence of bounded correctors χ. The correctors χ still exist, but only properties of their gradient will be used.

There are two main interests in introducing the framework of stochastic homogenization in this lecture:

(1) It enables to highlight the key properties needed for homogenization to hold.

(2) By introducing a dynamical system point of view, it unifies the presentation of homogenization and emphasizes the role of the underlying dynamics in the homogenization process.

The last point is particularly relevant in the perspective of constructing interior and boundary layer correctors.

The main source of inspiration for this lecture is Chapter 7 of Jikov, Kozlov and Oleĭnik's book [JKO94].

The setting

Let $(\Sigma, \mathcal{F}, \mu)$ be a probability space. Let $T = (T(y))_{y\in\mathbb{R}^d}$, be a family of random variables (a random process), acting on Σ: for all $y \in \mathbb{R}^d$, $T(y)$ is a random variable on Σ

$$T(y): \ \omega \in \Sigma \longmapsto T(y)\omega \in \Sigma.$$

Actually, $(T(y))_{y\in\mathbb{R}^d}$ is a *dynamical system* acting on the probability space Σ with, in addition, the following properties:

Group property $T(0) = \mathrm{id}: \ \Sigma \longrightarrow \Sigma$ and

$$T(y+\tilde{y}) = T(y)T(\tilde{y}) \quad \text{for all} \quad y,\ \tilde{y} \in \mathbb{R}^d. \tag{2.1}$$

Measure preserving $T(y): \ \Sigma \to \Sigma$ is measure preserving i.e.

$$\forall F \in \mathcal{F},\ \forall y \in \mathbb{R}^d, \quad \mu(T(y)F) = \mu(F). \tag{2.2}$$

In other terms, the image measure μ_T is equal to μ.

Measurability For all measurable (random variable) f on Σ (with values in $\mathbb{R}^m$),

$$(y,\omega) \in \mathbb{R}^d \times \Sigma \longmapsto f(T(y)\omega)$$

is measurable.

Let $L^p(\Sigma; \mathbb{R}^m)$, $1 \le p < \infty$ be the space of measurable μ integrable functions on Σ with exponent p, quotiented by equality almost everywhere; let $L^\infty(\Omega; \mathbb{R}^m)$ be the space of measurable essentially bounded functions. With the terminology of probabilities, the elements of $L^p(\Sigma)$ and $L^\infty(\Sigma)$ are random variables.

Let $\zeta = (\zeta(y,\cdot))_{y\in\mathbb{R}^d}$ be a family of random variables on Σ with values in $\mathbb{R}^m$ (a d-dimensional random process):

$$(y,\omega) \in \mathbb{R}^d \times \Sigma \longmapsto \zeta(y,\omega) \in \mathbb{R}^m.$$

We say that ζ is *stationary* if for any integer k, for any $y^{(1)}, \dots\ y^{(k)} \in \mathbb{R}^d$, for any $h \in \mathbb{R}^d$, the distribution (law) of the random vector

$$\omega \in \Sigma \longmapsto \left(\zeta(y^{(1)}+h,\omega), \zeta(y^{(2)}+h,\omega), \dots\ \zeta(y^{(k)}+h,\omega)\right) \in \mathbb{R}^{mk}$$

is independent of h.

Here we will work with random processes of the form:

$$\text{there exists} \quad Z: \ \Sigma \to \mathbb{R}^m \quad \text{such that for all} \quad y \in \mathbb{R}^d,\ \omega \in \Sigma, \quad \zeta(y,\omega) = Z(T(y)\omega). \tag{2.3}$$

Random processes of the form (2.3) are stationary. This follows from the group property (2.1) and the measure preservation (2.2). Let k be a fixed integer and $y^{(1)}, \dots\ y^{(k)} \in \mathbb{R}^d$. For $(F^{(1)}, \dots\ F^{(k)}) \in \otimes_{i=1}^k \mathcal{F}$, for $h \in \mathbb{R}^d$,

$$\begin{aligned}
&\mu \otimes \mu \otimes \dots\ \mu \left\{\zeta(y^{(1)}+h,\omega^{(1)}) \in F^{(1)}, \dots\ \zeta(y^{(k)}+h,\omega^{(k)}) \in F^{(k)}\right\} \\
&= \mu\left(Z(T(y^{(1)}+h)\omega^{(1)}) \in F^{(1)}\right) \dots\ \mu\left(Z(T(y^{(k)}+h)\omega^{(k)}) \in F^{(k)}\right) \\
&= \mu\left(Z(T(y^{(1)})T(h)\omega^{(1)}) \in F^{(1)}\right) \dots\ \mu\left(Z(T(y^{(k)})T(h)\omega^{(k)}) \in F^{(k)}\right) \\
&= \mu\left(Z(T(y^{(1)})\omega^{(1)}) \in F^{(1)}\right) \dots\ \mu\left(Z(T(y^{(k)})\omega^{(k)}) \in F^{(k)}\right) \\
&= \mu \otimes \mu \otimes \dots\ \mu \left\{\zeta(y^{(1)},\omega^{(1)}) \in F^{(1)}, \dots\ \zeta(y^{(k)},\omega^{(k)}) \in F^{(k)}\right\}.
\end{aligned}$$

A measurable function f on Σ is *invariant* if

$$f(T(y)\omega) = f(\omega) \quad \text{and almost surely in} \quad \omega, \quad \text{for all} \quad y \in \mathbb{R}^d.$$

The dynamical system $T = (T(y))_{y\in\mathbb{R}^d}$ is called *ergodic* if any invariant random variable f is constant almost surely.

Examples

Periodic setting Let $\Sigma = \mathbb{T}^d$, $\mathcal{F}$ be its Borel σ-algebra and μ the Lebesgue measure on $\mathbb{T}^d$. Let $T = T(y)_{y\in\mathbb{R}^d}$ be the dynamical system defined by for all $y \in \mathbb{R}^d$,

$$T(y) : \ \omega \in \mathbb{T}^d \longmapsto \omega + y \in \mathbb{T}^d.$$

The Lebesgue measure being invariant under translation, so is μ under T. And for an $f = f(\omega)$ defined on $\Sigma = \mathbb{T}^d$, f invariant under T i.e. for almost every $\omega \in \Sigma$, for all $y \in \mathbb{R}^d$,

$$f(T(y)\omega) = f(\omega + y) = f(\omega),$$

implies that f is constant. Therefore, T is ergodic.

Quasiperiodic setting Let m be an interger, $m \geq d$. Let $\Sigma = \mathbb{T}^m$, $\mathcal{F}$ be its Borel σ-algebra and let μ denote the Lebesgue measure on $\mathbb{T}^m$. Let $N \in M_{md}(\mathbb{R})$ be a matrix with m rows and d columns. We consider the dynamical system $T = T(y)_{y\in\mathbb{R}^d}$ defined by for all $y \in \mathbb{R}^d$,

$$T(y) : \ \omega \in \mathbb{T}^m \longmapsto \omega + Ny \in \mathbb{T}^m.$$

The Lebesgue measure being invariant under translation, so is μ under T.

At what condition is T ergodic? It depends on how the orbits $y \mapsto T(y)\omega$ fill the torus $\mathbb{T}^m$. Let $f = f(\omega)$ defined on $\Sigma = \mathbb{T}^m$, f invariant under T i.e. for almost every $\omega \in \Sigma$, for all $y \in \mathbb{R}^d$,

$$f(T(y)\omega) = f(\omega + Ny) = f(\omega). \tag{2.4}$$

Fix ω such that (2.4). There are two situations:

(1) either there is $k \in \mathbb{Z}^m$, $k \neq 0$, and there exists $1 \leq \alpha \leq d$ so that

$$(N^T k)_\alpha = \sum_{\beta=1}^{m} N_{\beta\alpha} k_\beta = 0;$$

in that case the orbit $y \mapsto T(y)\omega$ is not dense in $\mathbb{T}^m$ and T is not ergodic;

(2) or for all $k \in \mathbb{Z}^m$, $k \neq 0$, for all $1 \leq \alpha \leq d$,

$$(N^T k)_\alpha = \sum_{\beta=1}^{m} N_{\beta\alpha} k_\beta \neq 0;$$

in that case, the orbit $y \mapsto T(y)\omega$ is dense in $\mathbb{T}^m$ and thus T is ergodic.

Random checkerboard For $k \in \mathbb{Z}^2$, let $\Sigma_{(k)} := \{-1, 1\}$, $\mathcal{F}_{(k)}$ the power set of $\Sigma_{(k)}$ and $\mu_{(k)} := p\delta_{-1} + (1-p)\delta_1$ a probability measure. Let

$$\Sigma = \prod_{k \in \mathbb{Z}^2} \Sigma_{(k)},$$

and $\mathcal{F}$ the σ-algebra on Σ generated by the products

$$F = \prod_{k \in \mathbb{Z}^2} F_{(k)}, \tag{2.5}$$

$F_{(k)} = \{-1, 1\}$ but a finite number $F_{(k)} \in \mathcal{F}_{(k)}$. Then using Kolmogorov's extension theorem [Bil95, Section 36] and [Øks98, Theorem 2.1.5], one can define a unique probability measure μ on Σ such that

$$\mu(F) = \prod_{k \in \mathbb{Z}^2} \mu_{(k)}\left(F_{(k)}\right)$$

for all F of the type (2.5).

We now consider the dynamical system $T = (T(y))_{y \in \mathbb{Z}^d}$ defined by for all $\omega \in \Sigma$, for all $k \in \mathbb{Z}^2$,

$$(T(y)\omega)_k := \omega_{k+y},$$

which is nothing but the shifting by y. Since all the measures $\mu_{(k)}$ are identical, μ is invariant under T. With some more work, it can be showed that T is ergodic (see [Bil78, Chapter 1]).

Notice that in this setting the dynamical system T is defined on $\mathbb{Z}^d \times \Sigma$ rather than $\mathbb{R}^d \times \Sigma$. However, it is possible to make a slightly more involved construction of the underlying probability space and of T, so that T is a random process defined on $\mathbb{R}^d \times \Sigma$.

In the probability space

Hilbert spaces and Weyl's decomposition in the probability space

A function $f \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ is *potential* (or *curl-free*, *vortex-free*), if $\mathrm{curl}(f) = 0$ in the sense of distributions, i.e. for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left(f_\alpha \partial_{x_\beta} \varphi - f_\beta \partial_{x_\alpha} \varphi\right) = 0.$$

A function $f \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ is *solenoidal* (or *incompressible*), if $\mathrm{div}(f) = 0$ in the sense of distributions, i.e. for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f_\alpha \partial_{x_\alpha} \varphi = 0.$$

Using the action of the dynamical system T on Σ, we transpose these notions to functions $f \in L^2(\Sigma)$. We say that $f \in L^2(\Sigma)$ is *potential* (resp. *solenoidal*) if for almost every $\omega \in \Sigma$, $f(T(\cdot)\omega) \in L^2_{loc}(\mathbb{R}^d)$ is potential (resp. solenoidal).

We define subspaces of the Hilbert space $L^2(\Sigma)$. Let

$$\begin{aligned} L^2_{pot}(\Sigma) &:= \left\{ f \in L^2(\Sigma) : \ f \text{ is potential} \right\}, \\ L^2_{sol}(\Sigma) &:= \left\{ f \in L^2(\Sigma) : \ f \text{ is solenoidal} \right\}, \\ L^2_{pot,0}(\Sigma) &:= \left\{ f \in L^2(\Sigma) : \ f \text{ is potential and } \mathbb{E}(f) = 0 \right\}, \\ L^2_{sol,0}(\Sigma) &:= \left\{ f \in L^2(\Sigma) : \ f \text{ is solenoidal and } \mathbb{E}(f) = 0 \right\}. \end{aligned}$$

Since

$$f \in L^2(\Sigma) \longmapsto \mathbb{E}(f) = 0,$$

is continuous, $L^2_{pot,0}(\Sigma)$ (resp. $L^2_{sol,0}(\Sigma)$) are closed subspaces of $L^2_{pot}(\Sigma)$ (resp. $L^2_{sol}(\Sigma)$). The fact that $L^2_{pot}(\Sigma)$ and $L^2_{sol}(\Sigma)$ are closed subspaces of $L^2(\Sigma)$ follows from the following fact. Let $f_n \in L^2(\Sigma)$ so that $f_n \to f$ in $L^2(\Sigma)$. Then for almost every $\omega \in \Sigma$, $f_n(T(\cdot)\omega)$ converges to $f(T(\cdot)\omega)$ in $L^2_{loc}(\mathbb{R}^d)$.

With some more work (see [JKO94, Lemma 7.3]) one can prove that the space $L^2(\Sigma)$ has a type of Helmholtz decomposition.

Theorem 1 (Weyl's decomposition)**.** *The following orthogonal decomposition holds:*

$$L^2(\Sigma) = L^2_{pot,0}(\Sigma) \oplus L^2_{sol}(\Sigma).$$

Correctors

Let $1 \leq \alpha \leq d$. We consider the following corrector problem posed on the probability space Σ:

$$\text{find} \quad \gamma^*_\alpha \in L^2_{pot}(\Sigma), \quad \text{such that} \quad A^*\gamma^*_\alpha \in L^2_{sol}(\Sigma) \quad \text{and} \quad \mathbb{E}(\gamma^*_\alpha) = e_\alpha, \tag{2.6}$$

with e_α the α-th vector of the canonical basis of $\mathbb{R}^d$. Let us stress that we look for a vector field $\gamma^*_\alpha = \gamma^*_\alpha(\omega) \in \mathbb{R}^d$ defined on Σ.

By Weyl's decomposition of the probability space (Theorem 1), we can reformulate the corrector problem (2.6) as:

$$\text{find} \quad \Gamma^*_\alpha \in L^2_{pot,0}(\Sigma), \quad \text{such that for all} \quad \varphi \in L^2_{pot,0}(\Sigma), \quad \mathbb{E}\left\{\varphi \cdot A^*(e_\alpha + \Gamma^*_\alpha)\right\} = 0. \tag{2.7}$$

Notice that $\mathbb{E}(\Gamma^*_\alpha) = 0$ and that $\gamma^*_\alpha(\omega) = e_\alpha + \Gamma^*_\alpha(\omega)$.

The variational formulation (2.7) is clearly suitable for the Lax-Milgram theorem. Since $L^2_{pot,0}(\Sigma)$ is closed in the Hilbert space $L^2(\Sigma)$, and the bilinear form

$$(\varphi, \psi) \in L^2(\Sigma) \longmapsto E\left\{\varphi \cdot A^*\psi\right\},$$

is coercive, there exists a unique solution $\Gamma^*_\alpha \in L^2_{pot,0}$ to (2.7).

Back to physical space

Spatial averages and expectations: the ergodic theorem

Ergodicity is fundamental in order to make a connection between spatial averages of random stationary fields, and expectations. For instance, the limit of $A(x/\varepsilon, \omega) =$

$A(T(x/\varepsilon)\omega)$ for a fixed realization $\omega \in \Sigma$ is a spatial average. How does this limit relate to $\mathbb{E}(A) = \int_\Sigma A(\omega)d\mu(\omega)$?

Let T be a dynamical system as defined above: with the group and measurability properties, and measure preserving. Let $1 \le p < \infty$. Let $f \in L^p_{loc}(\mathbb{R}^d)$. We say that f has a *mean value*, if there exists $\overline{f}$ such that

$$f(x/\varepsilon) \overset{\varepsilon \to 0}{\rightharpoonup} \overline{f} \quad \text{weakly in} \quad L^p_{loc}(\mathbb{R}^d).$$

Theorem 2 (Birkhoff Ergodic Theorem, see [JKO94, Theorem 7.2]). *Let $1 \le p < \infty$. Let $f \in L^p(\Sigma)$. Then:*

(1) For almost every $\omega \in \Sigma$, $f(T(\cdot)\omega) \in L^p_{loc}(\mathbb{R}^d)$ has a mean value, denoted by $\overline{f(T(\cdot)\omega)}$.

(2) The mean value considered as a function of ω,

$$\omega \in \Sigma \longmapsto \overline{f(T(\cdot)\omega)},$$

is invariant and

$$\int_\Sigma f(\omega)d\mu(\omega) = \int_\Sigma \overline{f(T(\cdot)\omega)}d\mu(\omega).$$

(3) Moreover, if T is ergodic, then for all $\omega \in \Sigma$,

$$\overline{f(T(\cdot)\omega)} = \int_\Sigma f(\omega)d\mu(\omega) = \mathbb{E}(f).$$

It follows that in the case when T is ergodic, the mean value of any realization $\overline{f(T(\cdot)\omega)}$ is constant independent of ω and equal to the expectation. In other words, the spatial mean value equals the expectation, i.e. mean value over Σ.

Homogenization

In this part we assume that T is an ergodic dynamical system. We consider the homogenization of

$$\begin{cases} -\nabla \cdot A(T(x/\varepsilon)\omega)\nabla u^\varepsilon(x,\omega) &= f, \quad x \in \Omega, \\ u^\varepsilon(x,\omega) &= 0, \quad x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^d$. We have the classical a priori bounds uniformly in ω. There exists $C = C(d, \Omega, \lambda) > 0$, for all $\omega \in \Sigma$,

$$\left(\int_\Omega \|\nabla u^\varepsilon(x,\omega)\|^2 dx\right)^{1/2} \le C\|f\|_{L^2(\Omega)}.$$

Therefore, the Poincaré inequality implies that $u^\varepsilon(x,\omega)$ is bounded in $H^1_0(\Omega)$ uniformly in ε and $\omega \in \Sigma$. For all $\omega \in \Sigma$ (up to extracting subsequences),

$$\begin{aligned} u^\varepsilon(x,\omega) &\to u^0(x,\omega) \quad \text{strongly in} \quad L^2(\Omega), \\ \nabla u^\varepsilon(x,\omega) &\rightharpoonup \nabla u^0(x,\omega) \quad \text{weakly in} \quad L^2(\Omega), \\ \xi^\varepsilon(x,\omega) := A(T(x/\varepsilon)\omega)\nabla u^\varepsilon(x,\omega) &\rightharpoonup \nabla \xi^0(x,\omega) \quad \text{weakly in} \quad L^2(\Omega). \end{aligned}$$

Question. Do we have $\xi^0(x,\omega) = A^0 \nabla u^0(x,\omega)$ for some constant matrix A^0 and, in particular, is u^0 independent of the realization?

Let $\gamma^*_\alpha \in L^2_{pot}(\Sigma)$ be the unique solution to the corrector problem (2.6). We define the following stationary fields

$$\gamma^{*,\varepsilon}_\alpha(x) := \gamma^*_\alpha(T(x/\varepsilon)\omega),$$
$$\eta^{*,\varepsilon}_\alpha(x) := A^*(T(x/\varepsilon)\omega)\gamma^*_\alpha(T(x/\varepsilon)\omega).$$

By the ergodic theorem, we have for almost every $\omega \in \Sigma$,

$$\gamma^{*,\varepsilon}_\alpha(x) \rightharpoonup \overline{\gamma^*_\alpha(T(\cdot)\omega)} = \mathbb{E}(\gamma^*_\alpha) = e_\alpha \quad \text{weakly in} \quad L^2_{loc}(\mathbb{R}^d),$$
$$\eta^{*,\varepsilon}_\alpha(x) \rightharpoonup \overline{A^*(T(\cdot)\omega)\gamma^*_\alpha(T(\cdot)\omega)} = \mathbb{E}(A^*\gamma^*_\alpha) \quad \text{weakly in} \quad L^2_{loc}(\mathbb{R}^d).$$

Remark 2. Notice that neither $u^\varepsilon(x,\omega)$, $\nabla u^\varepsilon(x,\omega)$, nor $\xi^\varepsilon(x,\omega)$, are stationary.

Now, for all $\omega \in \Sigma$, for all $x \in \Omega$,

$$\xi^\varepsilon(x,\omega) \cdot \gamma^{*,\varepsilon}_\alpha = \nabla u^\varepsilon(x,\omega) \cdot \eta^{*,\varepsilon}_\alpha. \tag{2.8}$$

Notice that for all $\omega \in \Sigma$,

$$\operatorname{div}(\xi^\varepsilon(\cdot,\omega)) = f, \quad \operatorname{curl}(\nabla u^\varepsilon(\cdot,\omega)) = 0,$$

and for almost all $\omega \in \Sigma$,

$$\operatorname{div}(\eta^{*,\varepsilon}_\alpha) = \operatorname{div}(\eta^*_\alpha(T(\cdot)\omega) = 0, \quad \operatorname{curl}(\gamma^{*,\varepsilon}_\alpha) = \operatorname{curl}(\gamma^*_\alpha(T(\cdot)\omega) = 0.$$

Therefore, for almost every ω (fixed), we can apply the div-curl lemma to pass to the limit in (2.9). We get

$$\xi^0(x,\omega) \cdot e_\alpha = \nabla u^0(x,\omega) \cdot \mathbb{E}(A^*\gamma^*_\alpha). \tag{2.9}$$

After simplification

$$\xi^{0,\alpha}(x,\omega) = \Big(\mathbb{E}(A^{*\gamma\alpha}) + \mathbb{E}(A^{*\gamma\beta}\,\{\Gamma^*{}_\alpha\}_\beta)\Big)\,\partial_{x_\gamma} u^0(x,\omega).$$

Finally

$$A^0 := E(A) + \mathbb{E}(A\Gamma),$$

$\Gamma = (\Gamma_\alpha)_{1\le\alpha\le d}$ being the unique solution to the corrector problem (2.6) with A^* replaced by A. Consequently, $u^0 = u^0(x)$ is the unique solution to the deterministic problem

$$\begin{cases} -\nabla \cdot A^0 \nabla u^0 &= f, \quad x \in \Omega, \\ \qquad\quad u^0 &= 0, \quad x \in \partial\Omega. \end{cases}$$

References

[Bil78] Patrick Billingsley. *Ergodic theory and information.* Robert E. Krieger Publishing Co., Huntington, N.Y., 1978. Reprint of the 1965 original.

[Bil95] Patrick Billingsley. *Probability and measure.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, third edition, 1995. A Wiley-Interscience Publication.

[JKO94] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals.* Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].

[Øks98] Bernt Øksendal. *Stochastic differential equations.* Universitext. Springer-Verlag, Berlin, fifth edition, 1998. An introduction with applications.