

Lecture 3: Compensated Compactness

Christophe Prange*

February 18, 2017

One of the essential difficulties of non linear (and even linear!) PDEs is the incompatibility of weak convergence with products. For instance in the first lecture we proved that

$$A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup A^0\nabla u^0 \quad \text{weakly in } L^2(\Omega)$$

where $A^0 = \overline{A} + \overline{A\nabla\chi}$ is in general different from the weak limit \overline{A} of $A(x/\varepsilon)$.

This lecture is devoted to situations where additional information (notably on the boundedness of some linear combination of derivatives) enables to prove that the weak limit of the product is the product of the weak limits. A simple instance of such a phenomenon is the following. Let Ω be as usual a bounded Lipschitz domain of \mathbb{R}^d . Let $v_n = v_n(x) \in \mathbb{R}^d$ and $w_n = w_n(x) \in \mathbb{R}^d$ be two sequences such that

$$\begin{aligned} v_n & \text{ is bounded in } H^1(\Omega), & v_n & \rightharpoonup v \quad \text{weakly in } L^2(\Omega), \\ w_n & \text{ is bounded in } L^2(\Omega), & w_n & \rightharpoonup w \quad \text{weakly in } L^2(\Omega). \end{aligned}$$

Then, by Rellich's compact injection theorem a subsequence of v_n converges strongly to v in $L^2(\Omega)$. Therefore

$$v_n \cdot w_n \rightharpoonup v \cdot w \quad \text{weakly in } L^1(\Omega),$$

(the whole sequence converges). One encounters this situation when passing to the limit in the non linear term $u_n \cdot \nabla u_n$ of the Navier-Stokes equations, for u_n a sequence of smooth approximations.

The core of this lecture is the div-curl lemma. The lecture is mainly based on the paper by Murat [Mur78] and the Chapter 5.B of the book by Evans [Eva90].

1 The div-curl lemma

Let us start with the definition of the divergence div and the rotational curl. For $v \in \mathcal{D}'(\Omega)$, $v = v(x) \in \mathbb{R}^d$, we define $\text{div } v = \text{div } v(x) \in \mathbb{R}$ by

$$\text{div } v := \partial_{x_\alpha} v_\alpha \quad \text{in } \mathcal{D}'(\Omega).$$

For $w \in \mathcal{D}'(\Omega)$, $w = w(x) \in \mathbb{R}^d$, we define $\text{curl } w = \text{curl } w(x) \in \mathbb{R}^{d^2}$ by

$$\{\text{curl } w\}_{\alpha\beta} := \partial_{x_\beta} w^\alpha - \partial_{x_\alpha} w^\beta \quad \text{in } \mathcal{D}'(\Omega),$$

for $1 \leq \alpha, \beta \leq d$.

*Université de Bordeaux, 351 cours de la Libération, 33405 Talence. *E-mail address:* christophe.prange@math.cnrs.fr

Theorem 1 (Div-curl lemma). *Let $1 < p < \infty$ and p' the Hölder conjugate of p , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Let*

$$\begin{aligned} v_n = v_n(x) \in \mathbb{R}^d & \text{ be a sequence in } L^p(\Omega) \text{ such that } \operatorname{div} v_n \in L^p(\Omega), \\ w_n = w_n(x) \in \mathbb{R}^d & \text{ be a sequence in } L^{p'}(\Omega) \text{ such that } \operatorname{curl} w_n \in L^{p'}(\Omega). \end{aligned}$$

Assume that

$$\begin{aligned} \|v_n\|_{L^p(\Omega)} + \|\operatorname{div} v_n\|_{L^p(\Omega)} & \text{ is bounded uniformly and } v_n \rightharpoonup v \text{ weakly in } L^p(\Omega), \\ \|w_n\|_{L^{p'}(\Omega)} + \|\operatorname{curl} w_n\|_{L^{p'}(\Omega)} & \text{ is bounded uniformly and } w_n \rightharpoonup w \text{ weakly in } L^{p'}(\Omega). \end{aligned}$$

Then,

$$v_n \cdot w_n \rightharpoonup v \cdot w$$

in the sense of distributions.

In the div-curl lemma we only control some linear combinations of derivatives, namely the divergence and the curl, whereas the example in the introduction based on Rellich's compactness theorem requires a control of all first-order derivatives of v_n (or w_n). It is often applied to sequences such that $\operatorname{div} v_n = 0$ or $\operatorname{curl} w_n = 0$.

2 A direct proof via the Fourier transform: the case $p = p' = 2$

We give here a proof based on the original ideas of Murat [Mur78] using the Fourier transform in $L^2(\mathbb{R})$ and the Parseval-Plancherel formula. The following proof is direct, in the sense that it only relies on elementary arguments and computations. In particular, we do not use Rellich's compactness theorem. The proof is valid for an arbitrary bounded domain Ω , i.e. without any assumption on $\partial\Omega$. The strategy of the proof (splitting between low and high frequencies) is inspired from the proof of Rellich's injection $H_0^1(\Omega) \Subset L^2(\Omega)$ given in the Introduction.

Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$ a test function. We need to show that

$$\int_{\Omega} v_n \cdot w_n \varphi \longrightarrow \int_{\Omega} v \cdot w \varphi.$$

Step 1: extension to \mathbb{R}^d Before resorting to the Fourier transform, we need to define functions on the whole space \mathbb{R}^d . Let $\psi \in \mathcal{C}_c^\infty(\Omega)$ such that $\psi \equiv 1$ on $\operatorname{Supp} \varphi$. Let

$$\tilde{v}_n := \begin{cases} \varphi v_n & \text{in } \Omega \\ 0 & \text{on } \mathbb{R}^d \setminus \Omega \end{cases} \quad \text{and} \quad \tilde{w}_n := \begin{cases} \psi w_n & \text{in } \Omega \\ 0 & \text{on } \mathbb{R}^d \setminus \Omega \end{cases}.$$

Denoting by $\widehat{\tilde{v}_n}$ (resp. $\widehat{\tilde{w}_n}$) the Fourier transform of \tilde{v}_n (resp. \tilde{w}_n), we have

$$\int_{\Omega} v_n \cdot w_n \varphi = \int_{\mathbb{R}^d} \tilde{v}_n \cdot \tilde{w}_n = \int_{\mathbb{R}^d} \widehat{\tilde{v}_n} \cdot \overline{\widehat{\tilde{w}_n}}. \quad (2.1)$$

It is clear that

$$\begin{aligned} \widehat{v}_n \quad \text{and} \quad \widehat{w}_n & \text{ are bounded in } L^2(\mathbb{R}^d), \\ \widehat{v}_n \rightharpoonup \widehat{v} & \text{ weakly in } L^2(\mathbb{R}^d), \\ \widehat{w}_n \rightharpoonup \widehat{w} & \text{ weakly in } L^2(\mathbb{R}^d), \\ \xi_\alpha \widehat{v}_{n,\alpha} & \text{ is bounded in } L^2(\mathbb{R}^d), \\ \xi_\beta \widehat{w}_{n,\alpha} - \xi_\alpha \widehat{v}_{n,\beta} & \text{ is bounded in } L^2(\mathbb{R}^d). \end{aligned}$$

The proof of the convergence of the last term in (2.1) is based on a splitting between low and high frequencies: for $R > 0$

$$\int_{\mathbb{R}^d} \widehat{v}_n \cdot \widehat{w}_n = \int_{B(0,R)} \widehat{v}_n \cdot \widehat{w}_n + \int_{\mathbb{R}^d \setminus B(0,R)} \widehat{v}_n \cdot \widehat{w}_n.$$

Step 2: low frequencies To handle the low frequencies, we rely on Lebesgue's theorem of dominated convergence. On the one hand,

$$\|\widehat{v}_n\|_{L^\infty(\mathbb{R}^d)} \leq \|\widetilde{v}_n\|_{L^1(\Omega)} \leq C_\Omega \|v_n\|_{L^2(\Omega)}, \quad (2.2)$$

and on the other hand, the weak convergence of \widetilde{v}_n in $L^2(\mathbb{R}^d)$ implies the pointwise convergence

$$\widehat{v}_n(\xi) = \int_{\Omega} v_n(x) \exp(-ix \cdot \xi) d\xi \longrightarrow \int_{\Omega} v(x) \exp(-ix \cdot \xi) d\xi = \widehat{v}(\xi). \quad (2.3)$$

The analogues of (2.2) and (2.3) also hold for \widetilde{w}_n . Therefore,

$$\int_{B(0,R)} \widehat{v}_n \cdot \widehat{w}_n \longrightarrow \int_{B(0,R)} \widehat{v} \cdot \widehat{w},$$

when $n \rightarrow \infty$.

Step 3: high frequencies For the control of the high frequencies, we use the boundedness of the divergence and the curl in $L^2(\Omega)$. We simply have

$$\left| \int_{\mathbb{R}^d \setminus B(0,R)} \widehat{v}_n \cdot \widehat{w}_n \right| = \left| \int_{\mathbb{R}^d \setminus B(0,R)} \frac{\xi_\beta \xi_\beta}{|\xi|^2} \widehat{v}_n \cdot \widehat{w}_n \right| \leq \frac{1}{R} \int_{\mathbb{R}^d \setminus B(0,R)} \left| \xi_\beta \widehat{v}_n \cdot \widehat{w}_n \right|.$$

It is subsequently enough to show that for all $1 \leq \beta \leq d$, $\xi_\beta \widehat{v}_{n,\alpha} \widehat{w}_{n,\alpha}$ is bounded in $L^1(\mathbb{R}^d)$ uniformly in n . This is the place where the assumptions on the divergence and the curl are crucial.

Lemma 2. For any vectors $V, W \in \mathbb{C}^d$, for any $1 \leq \beta \leq d$,

$$\xi_\beta \sum_{\alpha=1}^d V_\alpha \overline{W_\alpha} = \overline{W_\beta} \sum_{\alpha=1}^d \xi_\alpha V_\alpha + \sum_{\alpha=1}^d \{\xi_\beta \overline{W_\alpha} - \xi_\alpha \overline{W_\beta}\} V_\alpha.$$

Applying this identity to $V = \widehat{v_{n,\alpha}}(\xi)$ and $W = \widehat{w_{n,\alpha}}(\xi)$, we get

$$\xi_\beta \widehat{v_{n,\alpha}} \widehat{w_{n,\alpha}} = \widehat{w_{n,\beta}} \underbrace{\xi_\alpha \widehat{v_{n,\alpha}}}_{L^2(\mathbb{R}^d)} + \underbrace{\left\{ \xi_\beta \widehat{w_{n,\alpha}} - \xi_\alpha \widehat{w_{n,\beta}} \right\}}_{L^2(\mathbb{R}^d)} \widehat{v_{n,\alpha}}$$

so that $\xi_\beta \widehat{v_{n,\alpha}} \widehat{w_{n,\alpha}}$ is bounded in $L^1(\mathbb{R}^d)$ uniformly in n . Thus,

$$\int_{\mathbb{R}^d \setminus B(0,R)} \widehat{v_n} \cdot \widehat{w_n} \rightarrow 0,$$

when $R \rightarrow \infty$.

Finally, combining Step 1, Step 2 and Step 3 we get

$$\int_{\Omega} v_n \cdot w_n \varphi = \int_{\mathbb{R}^d} \widehat{v_n} \cdot \widehat{w_n} \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} \widehat{v} \cdot \widehat{w} = \int_{\mathbb{R}^d} \widetilde{v} \cdot \widetilde{w} = \int_{\Omega} \widetilde{v} \cdot \widetilde{w} = \int_{\Omega} v \cdot w \varphi.$$

3 The case $1 < p, p' < \infty$

To address the case $p \neq 2$, Fourier analysis is not suitable any longer and we need to resort to tools from harmonic analysis, except in one case: when there exists a sequence z_n such that $w_n = \nabla z_n$ with

$$\begin{aligned} z_n &\rightarrow z \quad \text{strongly in } L^{p'}(\Omega), \\ \nabla z_n &\rightharpoonup \nabla z \quad \text{weakly in } L^{p'}(\Omega). \end{aligned} \tag{3.1}$$

Easy case

Let us assume (3.1). As we will see below, this is equivalent to $\text{curl } w_n = 0$. Then the div-curl lemma simply follows an integration by parts. Indeed, we have

$$\begin{aligned} \int_{\Omega} v_n \cdot w_n \varphi &= \int_{\Omega} v_n \cdot \nabla z_n \varphi = - \int_{\Omega} \nabla \cdot v_n z_n \varphi - \int_{\Omega} v_n \cdot \nabla \varphi z_n \\ &\xrightarrow{n \rightarrow \infty} - \int_{\Omega} \nabla \cdot v z \varphi - \int_{\Omega} v \cdot \nabla \varphi z = \int_{\Omega} v \cdot w \varphi. \end{aligned}$$

Remark 1. This computation was implicitly done in the first lecture, when proving the homogenization via the oscillating test function (step (2)). Though there we did not invoke the div-curl lemma, the same phenomenon is at work.

A few singular integrals

Let $\Gamma = \Gamma(x)$ denote the fundamental solution of the Laplacian $-\Delta$:

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \log |x|, & d = 2, \\ c_d \frac{1}{|x|^{d-2}}, & d \geq 3. \end{cases}$$

Then, for all $f \in C_0^\infty(\mathbb{R}^d)$, the Newtonian potential u given by

$$u(x) = \Gamma \star f(x) := \int_{\mathbb{R}^d} \Gamma(x - \tilde{x}) f(\tilde{x}) d\tilde{x},$$

is a weak solution to

$$-\Delta u = f, \quad x \in \mathbb{R}^d. \quad (3.2)$$

Our goal is to prove the following proposition.

Proposition 3.

(1) Let $1 < q < d$. There exists $C > 0$ such that for all $f \in L^q(\Omega)$, the Newtonian potential $u = \Gamma \star f$ satisfies

$$\|\nabla u\|_{L^{q^*}(\Omega)} \leq C \|f\|_{L^q(\Omega)} \quad \text{with} \quad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}.$$

(2) Let $1 < q < \infty$. There exists $C > 0$ such that for all $f \in L^q(\Omega)$, the Newtonian potential $u = \Gamma \star f$ satisfies

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C \|f\|_{L^q(\Omega)}.$$

The estimate for ∇u follows from an estimate on singular integrals of the following type: for $0 < \alpha < d$ and $f \in C_c^\infty(\mathbb{R}^d)$, we define

$$I_\alpha f(x) := \frac{1}{|\cdot|^{d-\alpha}} \star f(x) = \int_{\mathbb{R}^d} \frac{1}{|x - \tilde{x}|^{d-\alpha}} f(\tilde{x}) d\tilde{x}.$$

Indeed,

$$\nabla \Gamma(x) = \begin{cases} \frac{1}{2\pi} \frac{x}{|x|^2}, & d = 2, \\ c_d(2-d) \frac{x}{|x|^d}, & d \geq 3. \end{cases}$$

Proposition 4. For all $0 < \alpha < d$, for all $1 < p < d/\alpha$, for all $f \in L^p(\mathbb{R}^d)$,

$$\|I_\alpha f\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for} \quad 1/q = 1/p - \alpha/d \quad (3.3)$$

with $C = C(\alpha, d, p)$. We say that I_α is of (strong) type (p, q) .

For a proof, see [Ste70, Theorem 1.1 p. 119] or [O'N63, section III] (more general theorem on convolution in Lorentz spaces).

Remark 2 (homogeneity). The condition $1/q = 1/p - \alpha/d$ comes from homogeneity arguments. Let $f \in L^p(\mathbb{R}^d)$. Let $1 < p, q < \infty$ and assume an estimate like (3.3) holds, i.e. I_α is of type (p, q) . For all $\delta > 0$, we have

$$\|I_\alpha \{f(\delta \cdot)\}\|_{L^q(\mathbb{R}^d)} \leq C \|f(\delta \cdot)\|_{L^p(\mathbb{R}^d)}.$$

Now, on the one hand

$$\|I_\alpha \{f(\delta \cdot)\}\|_{L^q(\mathbb{R}^d)} = \frac{1}{\delta^\alpha} \|I_\alpha f(\delta x)\|_{L^q(\mathbb{R}^d)} = \frac{1}{\delta^{\alpha+d/q}} \|I_\alpha f\|_{L^q(\mathbb{R}^d)},$$

and on the other hand

$$\|f(\delta \cdot)\|_{L^p(\mathbb{R}^d)} = \frac{1}{\delta^{d/p}} \|f\|_{L^p(\mathbb{R}^d)}.$$

Therefore, $\alpha + d/q = d/p$.

It follows from the Proposition that for $1 < p < d$, $1/p_* = 1/p - 1/d$,

$$\|\nabla u\|_{L^{p_*}(\mathbb{R}^d)} \leq C \|I_1 f\|_{L^{p_*}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (3.4)$$

We now turn to the estimate of $\nabla^2 u$. We have for all $d \geq 2$, for all $1 \leq \alpha, \beta \leq 1$,

$$\partial_{x_\alpha} \partial_{x_\beta} \Gamma(x) = \begin{cases} -d \frac{\tilde{c}_d}{|x|^d} \frac{x_\alpha x_\beta}{|x|^2}, & \alpha \neq \beta, \\ \frac{\tilde{c}_d}{|x|^d} \left\{ 1 - d \frac{x_\alpha^2}{|x|^2} \right\}, & \alpha = \beta, \end{cases}$$

with for $d = 2$, $\tilde{c}_2 := \frac{1}{2\pi}$ and for $d = 3$, $\tilde{c}_d := (2 - d)c_d$.

Operators with a kernel of homogeneity $-d$ such as $\frac{1}{|\cdot|^d}$ or $\partial_{x_\alpha} \partial_{x_\beta} \Gamma(x)$ cannot be handled using Proposition 4. Indeed, the kernel is neither integrable around 0, nor at ∞ . In some cases, subtle cancellations properties make it possible to define an operator and prove its boundedness. One of the purposes of Calderon-Zygmund theory is to address such situations.

Let $\omega = \omega(s)$ be a Lipschitz function on \mathbb{S}^{d-1} such that

$$\int_{\mathbb{S}^{d-1}} \omega(s) d\sigma(s) = 0, \quad (3.5)$$

and denote by $K = K(x)$ the kernel

$$K(x) := \frac{1}{|x|^d} \omega\left(\frac{x}{|x|}\right).$$

Notice that p. v. $K \star f$ is well defined: for all $f \in C_c^\infty(\mathbb{R}^d)$, for $\varepsilon > 0$,

$$\begin{aligned} & \left| \int_{B(x, \varepsilon)} \frac{1}{|x - \tilde{x}|^d} \omega\left(\frac{x - \tilde{x}}{|x - \tilde{x}|}\right) f(\tilde{x}) d\tilde{x} \right| = \left| \int_0^\varepsilon \int_{\mathbb{S}^{d-1}} \frac{1}{r^d} \omega(s) f(x - rs) r^{d-1} d\sigma(s) dr \right| \\ & = \left| \int_0^\varepsilon \frac{1}{r} \int_{\mathbb{S}^{d-1}} \omega(s) \underbrace{[f(x - rs) - f(x)]}_{\leq r} d\sigma(s) dr \right| \leq \int_0^\varepsilon dr \int_{\mathbb{S}^{d-1}} |\omega(s)| d\sigma(s) \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

Proposition 5. *The operator T , defined by for all $f \in C_0^\infty(\mathbb{R}^d)$ by*

$$Tf(x) := \text{p. v. } K \star f(x),$$

is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$.

Proposition 5 is actually a corollary of a general theory due to Calderon and Zygmund. For a proof of Proposition 5, see [Ste70, theorem 3 p. 39].

For $1 \leq \alpha, \beta \leq d$, let

$$\omega_{\alpha\beta}(s) := \begin{cases} s_\alpha s_\beta, & \alpha \neq \beta, \\ 1 - ds_\alpha^2, & \alpha = \beta, \end{cases}$$

for $s \in \mathbb{S}^{d-1}$. If $\alpha \neq \beta$, $\omega_{\alpha\beta}$ is odd with respect to x_α . If $\alpha = \beta$, we have

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \omega_{\alpha\beta}(s) d\sigma(s) &= \int_{\mathbb{S}^{d-1}} d\sigma(s) - d \int_{\mathbb{S}^{d-1}} s_\alpha^2 d\sigma(s) = \int_{\mathbb{S}^{d-1}} d\sigma(s) - \sum_{\gamma=1}^d \int_{\mathbb{S}^{d-1}} s_\gamma^2 d\sigma(s) \\ &= \int_{\mathbb{S}^{d-1}} ds - \int_{\mathbb{S}^{d-1}} \sum_{\gamma=1}^d s_\gamma^2 d\sigma(s) = 0. \end{aligned}$$

Since (3.5) is satisfied in both cases, Proposition 5 applies and implies that for $1 < p < \infty$,

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$

Remark 3. For $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, for all $1 \leq \alpha \leq d$,

$$\widehat{\partial_{x_\alpha} \partial_{x_\beta} u}(\xi) = -\xi_\alpha \xi_\beta \widehat{u}(\xi) = -\frac{\xi_\alpha}{|\xi|} \frac{\xi_\beta}{|\xi|} |\xi|^2 \widehat{u}(\xi) = \widehat{R_\alpha R_\beta} \widehat{\Delta u}(\xi),$$

where the Riesz transforms R_α are defined by: for $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, for $1 \leq \alpha \leq d$,

$$R_\alpha f(x) := \hat{c}_d \text{p.v.} \int_{\mathbb{R}^d} \frac{x_\alpha - \tilde{x}_\alpha}{|x - \tilde{x}|^{d+1}} f(\tilde{x}) d\tilde{x}.$$

Therefore,

$$\partial_{x_\alpha} \partial_{x_\beta} u = -R_\alpha R_\beta \Delta u,$$

and by the L^p boundedness of the Riesz potentials (see Proposition 5), for all $1 < p < \infty$, there exists a constant $C = C(d, p)$ such that

$$\|\partial_{x_\alpha} \partial_{x_\beta} u\|_{L^p(\mathbb{R}^d)} \leq C \|\Delta u\|_{L^p(\mathbb{R}^d)}.$$

Helmholtz's decomposition and the proof of the div-curl lemma

The following proof of the div-curl lemma is inspired from Evans [Eva90, Chapter 5.B] and from [Mur78].

Technical point. The estimate of the singular integral $I_1(f)$ in Proposition 4 and its corollary estimate (4) prompts the need to separate the case $p > \frac{d}{d-1}$ (treated first) and $p \leq \frac{d}{d-1}$.

Let v_n and w_n be two sequences satisfying the assumptions of Theorem 1. The idea of the proof is to decompose w_n as a sum

$$w_n = y_n + \nabla z_n, \tag{3.6}$$

with

$$\begin{aligned} y_n &\rightarrow y \quad \text{strongly in } L^{p'}(\Omega), \\ z_n &\rightharpoonup z \quad \text{weakly in } W^{1,p'}(\Omega). \end{aligned}$$

Identity (3.6) is nothing more than the Helmholtz decomposition of w_n , y_n being the projection of w_n on the solenoidal fields and ∇z_n being the projection of w_n on the curl-free fields.

- Let $p > \frac{d}{d-1}$ and p' the conjugate Hölder exponent of p :

$$\boxed{p > \frac{d}{d-1}, \quad 1 < p' < d \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1.}$$

(1) As in the proof of the case $p = p' = 2$ (see Step 1 above), we extend v_n and w_n on the whole space \mathbb{R}^d in the following way

$$\tilde{v}_n := \begin{cases} \varphi v_n & \text{in } \Omega \\ 0 & \text{on } \mathbb{R}^d \setminus \Omega \end{cases} \quad \text{and} \quad \tilde{w}_n := \begin{cases} \psi w_n & \text{in } \Omega \\ 0 & \text{on } \mathbb{R}^d \setminus \Omega \end{cases}.$$

(2) For each n , we consider the Newtonian potential $u_n = u_n(x) \in \mathbb{R}^d$ defined by $u_n := \Gamma \star w_n$, which is a weak solution to

$$-\Delta u_n = \widetilde{w}_n, \quad x \in \mathbb{R}^d.$$

where $\widetilde{w}_n = \widetilde{w}_n(x) \in \mathbb{R}^d$. Since by assumption \widetilde{w}_n is bounded in $L^{p'}(\mathbb{R}^d)$, Proposition 3 implies

$$\|\nabla u_n\|_{L^{p'_*}(\mathbb{R}^d)} + \|\nabla^2 u_n\|_{L^{p'}(\mathbb{R}^d)} \leq C \|\widetilde{w}_n\|_{L^{p'}(\mathbb{R}^d)},$$

for $1/p'_* = 1/p' - 1/d$.

(3) Define $z_n := -\nabla \cdot u_n$ and $y_n := \widetilde{w}_n - \nabla z_n$. We have for all $\alpha = 1, \dots, d$,

$$y_n^\alpha = \widetilde{w}_n^\alpha - \partial_\alpha z_n = -\partial_\beta \partial_\beta u_n^\alpha + \partial_\alpha \partial_\beta u_n^\beta = \partial_\beta (-\partial_\beta u_n^\alpha + \partial_\alpha u_n^\beta) = -\partial_\beta \{(\operatorname{curl} u_n)_{\alpha\beta}\}.$$

Since $\operatorname{curl} u_n = \Gamma \star \operatorname{curl} \widetilde{w}_n$, we have

$$\|\nabla \operatorname{curl} u_n\|_{L^{p'_*}(\mathbb{R}^d)} + \|\nabla^2 \operatorname{curl} u_n\|_{L^{p'}(\mathbb{R}^d)} \leq C \|\operatorname{curl} \widetilde{w}_n\|_{L^{p'}(\mathbb{R}^d)},$$

from which we infer that

$$\|y_n\|_{L^{p'_*}(\mathbb{R}^d)} + \|\nabla y_n\|_{L^{p'}(\mathbb{R}^d)} \leq C \|\operatorname{curl} \widetilde{w}_n\|_{L^{p'}(\mathbb{R}^d)}.$$

Restricting to Ω using $p'_* > p'$, we get

$$\|y_n\|_{L^{p'}(\Omega)} + \|\nabla y_n\|_{L^{p'}(\Omega)} \leq C_\Omega \|\operatorname{curl} \widetilde{w}_n\|_{L^{p'}(\mathbb{R}^d)}.$$

Therefore, y_n is bounded in $W_0^{1,p'}(\Omega)$ and Rellich's compactness theorem implies that (up to a subsequence)

$$y_n \rightarrow y \quad \text{strongly in } L^{p'}(\Omega).$$

Moreover, by a similar reasoning z_n is bounded in $W_0^{1,p'}(\Omega)$, so up to a subsequence,

$$\begin{aligned} z_n &\rightarrow z \quad \text{strongly in } L^{p'}(\Omega), \\ \nabla z_n &\rightharpoonup \nabla z \quad \text{weakly in } L^{p'}(\Omega). \end{aligned}$$

We thus have:

$$\psi w = y + \nabla z.$$

(4) It remains to pass to the limit in the product, proceeding as above (integration by parts) for the term involving ∇z_n :

$$\begin{aligned} \int_\Omega v_n \cdot w_n \varphi &= \int_{\mathbb{R}^d} \widetilde{v}_n \cdot \widetilde{w}_n = \int_\Omega \widetilde{v}_n \cdot (y_n + \nabla z_n) \\ &\longrightarrow \int_\Omega \widetilde{v} \cdot (y + \nabla z) = \int_\Omega \varphi v \cdot \psi w = \int_\Omega v \cdot w \varphi. \end{aligned}$$

The whole sequence $v_n \cdot w_n$ converges in $\mathcal{D}'(\Omega)$ (and not only a subsequence) because $v \cdot w$ is the only possible limit.

- Let $p \leq \frac{d}{d-1}$ and p' the conjugate Hölder exponent of p :

$$\boxed{p \leq \frac{d}{d-1}, \quad p' \geq d \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1.}$$

Let us outline the changes which have to be made. We concentrate on steps (2) and (3) above. Since \widehat{w}_n and $\text{curl} \widehat{w}_n$ are compactly supported in Ω , we have, for all $q < p'$

$$\|\widehat{w}_n\|_{L^q(\mathbb{R}^d)} \leq C \|\widehat{w}_n\|_{L^{p'}(\mathbb{R}^d)} \quad \text{and} \quad \|\text{curl} \widehat{w}_n\|_{L^q(\mathbb{R}^d)} \leq C \|\text{curl} \widehat{w}_n\|_{L^{p'}(\mathbb{R}^d)}$$

where C only depends on p' , q and the measure of Ω . Now we take advantage of the gain in integrability provided by I_1 by taking $q < d$ close enough to d , so that

$$q_* := \frac{dq}{d-q} = p'.$$

Then,

$$\begin{aligned} \|\nabla u_n\|_{L^{p'}(\mathbb{R}^d)} + \|\nabla^2 u_n\|_{L^{p'}(\mathbb{R}^d)} &\leq C \|\widehat{w}_n\|_{L^{p'}(\mathbb{R}^d)}, \\ \|\nabla \text{curl} u_n\|_{L^{p'}(\mathbb{R}^d)} + \|\nabla^2 \text{curl} u_n\|_{L^{p'}(\mathbb{R}^d)} &\leq C \|\text{curl} \widehat{w}_n\|_{L^{p'}(\mathbb{R}^d)}. \end{aligned}$$

Therefore

$$\|y_n\|_{L^{p'}(\Omega)} + \|\nabla y_n\|_{L^{p'}(\Omega)} \leq C_\Omega \|\text{curl} \widehat{w}_n\|_{L^{p'}(\mathbb{R}^d)},$$

and the rest of the proof is similar.

Remark 4. Notice that

$$\widehat{\partial_\alpha y_n^\alpha} = \xi_\alpha \widehat{w_{n,\alpha}} - \frac{\xi_\alpha \xi_\alpha}{|\xi|^2} \xi_\beta \widehat{w_{n,\beta}} = \widehat{\nabla \cdot w_n} - \widehat{\nabla \cdot w_n} = 0.$$

Therefore,

$$y_n = \mathbb{P}w,$$

where \mathbb{P} is the Leray projector. In Fourier space,

$$\widehat{y}_n = \widehat{w}_n - \xi \frac{\xi \cdot \widehat{w}_n}{|\xi|^2} = \left\{ I_d - \left(\frac{\xi_\alpha \xi_\beta}{|\xi|^2} \right)_{\alpha,\beta} \right\} \widehat{w}_n.$$

4 Applications

Convergence of the energy

From the first lecture we have

$$\begin{aligned} \xi^\varepsilon(x) &:= A(x/\varepsilon) \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } L^2(\Omega), \\ \nabla u^\varepsilon &\rightharpoonup \nabla u^0 \quad \text{weakly in } L^2(\Omega). \end{aligned}$$

Moreover, on the one hand

$$\nabla \cdot \xi^\varepsilon = -f$$

so that

$$\|\xi^\varepsilon\|_{L^2(\Omega)} + \|\nabla \cdot \xi^\varepsilon\|_{L^2(\Omega)}$$

is bounded uniformly in ε , and on the other hand

$$\operatorname{curl}(\nabla u^\varepsilon) = 0.$$

The div-curl lemma now implies the convergence of the energy

$$A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla u^\varepsilon \rightharpoonup A^0\nabla u^0 \cdot \nabla u^0$$

in the sense of distributions.

Weak continuity of the Jacobian determinant

Proposition 6. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}^d$. Assume that*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,d}(\mathbb{R}^d).$$

Then,

$$\det(\nabla u^\varepsilon) \rightharpoonup \det(\nabla u)$$

in the sense of distributions.

This proposition goes back to Ball [Bal77]. A proof of the result stated here can be found in [Mur, p. 56, Lemma 6.20] (see also [Eva90, Chapter 3.E]). Since ∇u^ε is bounded in $L^d(\Omega)$, we have $\det(\nabla u^\varepsilon)$ is bounded in $L^1(\Omega)$. Thus

$$\det(\nabla u^\varepsilon) \xrightarrow{*} \mu$$

to a Radon measure $\mu \in (L^\infty(\Omega))'$.

Developping with respect to the first column we have

$$\det(\nabla u^\varepsilon) = \partial_\beta u_1^\varepsilon m_{1,\beta}(\nabla u^\varepsilon) \tag{4.1}$$

where $m_{\alpha,\beta}$ is the minor in position (α, β) associated to ∇u^ε .

(1) The key point is the following algebraic formula: for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\partial_1 m_{1,1}(\nabla \varphi) + \dots + \partial_d m_{1,d}(\nabla \varphi) = 0. \tag{4.2}$$

(2) We now extend the formula (4.2) to $u \in W^{1,d}(\Omega)$. Let $u \in W^{1,d}(\Omega)$. There exists a sequence $\varphi_k \in \mathcal{C}_c^\infty(\overline{\Omega})$ such that

$$\varphi_k \rightarrow u \quad \text{strongly in } W^{1,d}(\Omega).$$

Therefore,

$$m_{1,\cdot}(\nabla \varphi_k) \rightarrow m_{1,\cdot}(\nabla u) \quad \text{strongly in } L^{\frac{d}{d-1}}(\Omega),$$

and for all $\psi \in \mathcal{C}_c^\infty(\Omega)$

$$\begin{aligned} 0 &= \langle \partial_1 m_{1,1}(\nabla \varphi_k) + \dots + \partial_d m_{1,d}(\nabla \varphi_k), \psi \rangle_{\mathcal{D}', \mathcal{D}} = - \int_{\Omega} m_{1,\cdot}(\nabla \varphi_k) \cdot \nabla \psi \\ &\xrightarrow{\varepsilon \rightarrow 0} - \int_{\Omega} m_{1,\cdot}(\nabla u) \cdot \nabla \psi = \langle \partial_1 m_{1,1}(\nabla u) + \dots + \partial_d m_{1,d}(\nabla u), \psi \rangle_{\mathcal{D}', \mathcal{D}}. \end{aligned}$$

(3) It follows from (4.1) that

$$\det(\nabla u^\varepsilon) = m_{1,\cdot}(\nabla u^\varepsilon) \cdot \nabla u_1^\varepsilon$$

has a div-curl structure. Indeed on the one hand

$$\operatorname{div}(m_{1,\cdot}(\nabla u^\varepsilon)) = 0 \quad \mathcal{D}'(\Omega) \quad \text{and} \quad m_{1,\cdot}(\nabla u^\varepsilon) \quad \text{is bounded uniformly in} \quad L^{\frac{d}{d-1}}(\Omega),$$

and on the other hand

$$\operatorname{curl}(\nabla u_1^\varepsilon) = 0 \quad \text{and} \quad \nabla u_1^\varepsilon \quad \text{is bounded uniformly in} \quad L^d(\Omega),$$

so that the result follows from the div-curl lemma for $p = \frac{d}{d-1}$ and $p' = d$.

References

- [Bal77] John M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/77.
- [Eva90] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.
- [Mur] François Murat. Compacité par compensation et homogénéisation. Notes du cours de DEA rédigées par Isabelle Gruais.
- [Mur78] François Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.
- [O’N63] Richard O’Neil. Convolution operators and $L(p, q)$ spaces. *Duke Math. J.*, 30:129–142, 1963.
- [Ste70] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.