Lecture 2: A Strange Term Coming From Nowhere

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In this lecture, we consider the Poisson equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u^\varepsilon &= f, \quad x \in \Omega^\varepsilon, \\ \quad u^\varepsilon &= 0, \quad x \in \partial\Omega^\varepsilon, \end{cases} \tag{0.1}$$

in a perforated (or porous) domain

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{n_\varepsilon} T_i^\varepsilon \subset \mathbb{R}^d.$$

The source term f belongs to $L^2(\Omega)$. The bounded domain Ω is perforated by n_ε holes T_i^ε. Contrary to the first lecture, the inhomogeneities are not in the coefficients, but in the domain itself. A way to connect the two situations is to consider that in the holes the viscosity is infinite, though we will not use this point of view.

This lecture is based on the paper by Cioranescu and Murat, *A Strange Term Coming from Nowhere* in [CK97].

Several things can happen depending on the size of the holes, their density, the way they are distributed and the distance between adjacent holes:

(1) Assume that for any $K \Subset \Omega$, $K \subset \Omega^\varepsilon$ for all ε sufficiently small. Then we easily have that u^ε converges to the solution u^0 to the Poisson problem in Ω with Dirichlet boundary conditions on $\partial\Omega$.

(2) Assume that the characteristic function χ_{T^ε} of the holes (bounded in $L^\infty(\Omega)$) converges weakly star in $L^\infty(\Omega)$ to a positive function $\chi \in L^\infty$. Then the holes fill the whole of Ω and therefore u^ε goes to zero. Indeed, extending u^ε by zero in the holes, we get for $\varphi \in L^1(\Omega)$,

$$\int_\Omega u^\varepsilon \varphi = \int_{\Omega^\varepsilon} u^\varepsilon \varphi = \int_\Omega \left(1 - \chi_{\cup_{i=1}^{n_\varepsilon} T_i^\varepsilon}\right) u^\varepsilon \varphi,$$

so that in the limit

$$\int_\Omega u^0 \varphi = \int_\Omega (1-\chi) u^0 \varphi,$$

i.e.

$$\int_\Omega \chi u^0 \varphi = 0.$$

Thus $\chi u^0 = 0$ and by positivity of χ, $u^0 = 0$.

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(3) The domain Ω is perforated by an increasing number of *regularly* distributed holes whose diameters go to zero. This is the situation we consider in this lecture. Of course, the two extreme cases u^ε goes to u^0 or u^ε goes to 0 are still possible. However, there is an intermediate situation where something non trivial arises in the limit, hence the title of this lecture.

1 A general framework for the oscillating test function method

In this section, $d \geq 1$.

The variational formulation for (0.1) reads: for all $v \in H_0^1(\Omega^\varepsilon)$,

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla v = \int_{\Omega^\varepsilon} fv. \tag{1.1}$$

We now extend any function $v \in H_0^1(\Omega^\varepsilon)$ by zero in the holes to a function $\tilde{v} \in H_0^1(\Omega)$:

$$\tilde{v}(x) = v(x) \text{ for } x \in \Omega^\varepsilon, \qquad \tilde{v}(x) = 0 \text{ for } x \in T_i^\varepsilon.$$

For the rest of this lecture, we drop the tildes.

Remark 1. The class of test functions in (1.1) cannot be extended to the whole of $H_0^1(\Omega)$. The test function have to vanish on the holes. The point is that the solution u^ε extended by zero in the holes belongs to $H_0^1(\Omega)$, but it does not satisfy the equation in a neighborhood of the holes ∂T_i^ε. On ∂T_i^ε, there is a jump in ∇u^ε. Thus Δu^ε is a dirac mass (in a sense to be made precise below).

Taking $v = u^\varepsilon \in H_0^1(\Omega^\varepsilon)$ in the variational formulation (1.1) and using Poincaré's inequality, we get the a priori estimate

$$\left(\int_\Omega |\nabla u^\varepsilon|^2 \right)^{1/2} \leq C \|f\|_{L^2(\Omega)}.$$

Therefore, $\|u^\varepsilon\|_{H^1(\Omega)}$ is bounded uniformly in ε so that by Rellich's theorem yields (up to a subsequence)

$$\begin{aligned} &\nabla u^{\varepsilon_k} \rightharpoonup \nabla u^0 \text{ weakly in } L^2(\Omega), \\ &u^{\varepsilon_k} \to u^0 \text{ strongly in } L^2(\Omega). \end{aligned}$$

Question. Can we identify the limit u^0?

We have seen in the first lecture that the oscillating test function method is useful in similar situations. Therefore, let us take a test function $v := w^\varepsilon \varphi$, where $\varphi \in \mathcal{C}_c^\infty(\Omega)$ (test function for the limit system, so no requirement to be zero on the holes) and w^ε is a corrector. We adopt a heuristic approach in two steps:

(1) We first try to identify properties the correctors w^ε have to satisfy in order to see something non trivial in the limit. This is the easy step in the sense that we may impose whatever we like on w^ε which gives rise to interesting phenomena in the limit.

(2) Adding some structure on the distribution of the holes (for instance periodicity) and taking specific scalings (size of the holes relative to the distance between adjacent holes), we show that ad hoc correctors exist (see section 2). This is the difficult step.

Again, w^ε should be thought of some function which behaves similarly to u^ε in the sense that

$$w^\varepsilon \quad \text{is almost a solution of the Poisson equation, and it is zero on the holes.} \tag{1.2}$$

The additional requirement is that

$$\text{in the limit} \quad \varphi w^\varepsilon \quad \text{should converge (in some sense) to} \quad \varphi, \tag{1.3}$$

because φ is the test function for the limit system posed in Ω. As underlined in the first lecture, the difficulty of the oscillating test function method lies in the construction of correctors, i.e. of w^ε.

Keeping in mind the heuristic conditions (1.2) and (1.3), we now list a couple a conditions on w^ε.

First of all, we want the test function $w^\varepsilon\varphi$ to belong to $H^1(\Omega)$. Thus, we take

$$w^\varepsilon \in H^1(\Omega). \tag{1.4}$$

Second, the test function $w^\varepsilon\varphi$, in order to be admissible in (1.1), has to vanish on ∂T_i^ε. Therefore, we impose

$$w^\varepsilon = 0 \quad \text{on} \quad T_i^\varepsilon. \tag{1.5}$$

We do not impose $w^\varepsilon = 0$ on $\partial\Omega$, since there φ already vanishes.

Let us now take $v = w^\varepsilon\varphi$ in (1.1). We have

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla(w^\varepsilon\varphi) = \int_\Omega \varphi\nabla u^\varepsilon \cdot \nabla w^\varepsilon + \int_\Omega w^\varepsilon \nabla u^\varepsilon \cdot \nabla\varphi = \int_\Omega f w^\varepsilon \varphi.$$

Provided that w^ε converges strongly in $L^2(\Omega)$ (necessarily to 1 because we want that φw^ε to converge to φ), it is easy to pass to the limit in the terms

$$\int_\Omega w^\varepsilon \nabla u^\varepsilon \cdot \nabla\varphi \longrightarrow \int_\Omega \nabla u^0 \cdot \nabla\varphi,$$
$$\int_\Omega f w^\varepsilon \varphi \longrightarrow \int_\Omega f\varphi.$$

We impose

$$w^\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(\Omega), \tag{1.6}$$

which in particular implies that $\nabla w^\varepsilon \to 0$ in $L^2(\Omega)$ and enables to get rid of one term (see below),.

Remark 2. If moreover $w^\varepsilon \to 1$ in $H^1(\Omega)$, then

$$\int_\Omega \varphi \nabla u^\varepsilon \cdot \nabla w^\varepsilon \longrightarrow 0,$$

so that $u^0 \in H^1_0(\Omega)$ is simply a solution to the Poisson problem

$$\begin{cases} -\Delta u^0 &= f, \quad x \in \Omega, \\ \quad u^0 &= 0, \quad x \in \partial\Omega. \end{cases} \tag{1.7}$$

No interesting phenomenon arises from the limit in that case.

Remark 3. Condition (1.6) prevents the holes from coalescing/merging in the limit, because w^ε has to be zero on the holes. In some sense, it implies that the diameter of the holes go sufficiently fast to zero, and that the holes are sufficiently spaced.

Remark 4 (one-dimensional case). Let $d = 1$ and take $\Omega = (0,1)$ for simplicity. Let

$$\Omega^\varepsilon := (0,1) \setminus \bigcup_{i=1}^{n_\varepsilon} B(x_i^\varepsilon, r_i^\varepsilon)$$

with $x_i^\varepsilon \in (0,1)$. Assume that assumptions (1.4), (1.5) and (1.6) hold. Then, a subsequence of the family of points $\{x_i^\varepsilon\}$ converges to a point $x^0 \in [0,1]$. In dimension $d = 1$, Rellich's theorem implies that $H^1(\Omega)$ is compactly embedded in $\mathcal{C}^{0,\gamma}([0,1])$ for $0 < \gamma < 1/2$. Since w^ε is bounded uniformly in $H^1(0,1)$, up to a subsequence, w^ε converges strongly in $\mathcal{C}^{0,\gamma}([0,1])$ to w^0. From (1.6), we get that $w^0 = 1$. Now,

$$|w^0(x^0)| = |w^0(x^0) - w^\varepsilon(x_i^\varepsilon)| \leq |w^0(x^0) - w^0(x_i^\varepsilon)| + |w^0(x_i^\varepsilon) - w^\varepsilon(x_i^\varepsilon)| \longrightarrow 0.$$

Therefore, $w^0(x^0) = 0$, which is a contradiction. To put it in a nutshell, there does not exist a family of correctors w^ε such that (1.4), (1.5) and (1.6) hold.

Of course, if ∇u^ε is merely weakly convergent, then the term

$$\int_\Omega \varphi \nabla u^\varepsilon \cdot \nabla w^\varepsilon$$

is a product of two merely weakly convergent sequences (to 0), and nothing can be said a priori about the limit. Let us integrate by parts in the previous term seen as a duality product. We get

$$\int_\Omega \varphi \nabla u^\varepsilon \cdot \nabla w^\varepsilon = -\int_\Omega u^\varepsilon \nabla \varphi \cdot \nabla w^\varepsilon - \langle \Delta w^\varepsilon, \varphi u^\varepsilon \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

A priori Δw^ε, as is Δu^ε is no better than a distribution (actually a measure). Using the weak convergence of the gradient, we get

$$-\int_\Omega u^\varepsilon \nabla \varphi \nabla w^\varepsilon \longrightarrow 0.$$

There remains the tricky term

$$\langle \Delta w^\varepsilon, \varphi u^\varepsilon \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$$

for which a priori nothing is known. If

$$\text{there exists} \quad \mu \in W^{-1,\infty}(\Omega) \tag{1.8}$$

such that

$$\langle \Delta w^\varepsilon, \varphi u^\varepsilon \rangle_{H^{-1}(\Omega),H^1(\Omega)} \longrightarrow \langle \mu, \varphi u^0 \rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

then (of course!) we are done. In fact, what we will prove is that $-\Delta w^\varepsilon$ breaks down into the difference of two measures: one μ^ε which converges strongly to μ in $H^{-1}(\Omega)$, and another one γ^ε which converges merely weakly to μ, but whose support does not see the test function φu^ε. In more precise terms, we look for w^ε such that

$$\begin{array}{l} \text{there exists two sequences} \quad \mu^\varepsilon,\ \gamma^\varepsilon \in H^{-1}(\Omega) \quad \text{such that} \quad -\Delta w^\varepsilon = \mu^\varepsilon - \gamma^\varepsilon \\ \mu^\varepsilon \to \mu \quad \text{strongly in} \quad H^{-1}(\Omega) \quad \text{and} \quad \gamma^\varepsilon \rightharpoonup \mu \quad \text{weakly in} \quad H^{-1}(\Omega) \quad \text{and} \\ \langle \gamma^\varepsilon, v^\varepsilon \rangle = 0 \quad \text{for all} \quad v^\varepsilon \in H_0^1(\Omega) \quad \text{such that} \quad v^\varepsilon = 0 \quad \text{on} \quad T_i^\varepsilon. \end{array} \tag{1.9}$$

Notice that (1.8) implies

$$\mu \in W^{-1,\infty}(\Omega) = \left(W_0^{1,1}(\Omega)\right)' \subset H^{-1}(\Omega).$$

We have proved the following conditional theorem.

Theorem 1. *Assume that there exist a sequence of correctors w^ε meeting assumptions* (1.4), (1.5), (1.6), (1.8) *and* (1.9). *Then, the family of solutions u^ε to* (0.1) *converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to the unique solution u^0 of the modified Poisson equation*

$$\begin{cases} -\Delta u^0 + \mu u^0 &= f, \quad x \in \Omega, \\ u^0 &= 0, \quad x \in \partial\Omega. \end{cases} \tag{1.10}$$

Lemma 2. *The distribution $\mu \in W^{-1,\infty}(\Omega)$ satisfying* (1.9) *is positive, i.e. it is a Radon measure.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi \geq 0$. Then, by integrating by parts on Ω^ε first, and second by using the convergence results (1.6) and (1.9) we get

$$0 \leq \int_\Omega \varphi \nabla w^\varepsilon \cdot \nabla w^\varepsilon = \langle -\Delta w^\varepsilon, \varphi w^\varepsilon \rangle_{H^{-1},H_0^1} - \int_\Omega w^\varepsilon \nabla\varphi \cdot \nabla w^\varepsilon \longrightarrow \langle \mu, \varphi \rangle_{W^{-1,\infty},W^{1,1}}.$$

Thus, $\langle \mu, \varphi \rangle_{W^{-1,\infty},W^{1,1}} \geq 0$. □

The strange term, is reminiscent from the holes, and represents some kind of friction or drag due to the holes. It will be computed explicitly in the next section.

Lemma 3. *The solution to* (1.10) *is unique.*

Proof. Call $u = u(x)$ the difference of two solutions of (1.10). Testing against u yields

$$\int_\Omega \nabla u \cdot \nabla u + \langle \mu, u^2 \rangle_{W^{-1,\infty},W^{1,1}} = 0.$$

The positivity of μ (see Lemma 2) implies $u = 0$. □

Remark 5. The convergence proof of a subsequence of u^ε to a solution u^0 of (1.10) holds with μ barely $H^{-1}(\Omega)$. In the uniqueness proof however, condition (1.8) enables to make sense of the duality product

$$\langle \mu, u^2 \rangle_{W^{-1,\infty}, W^{1,1}},$$

since u^2 is merely $W^{1,1}(\Omega)$.

Remark 6 (comparison between lecture 1 and 2). We now compare the ways we pass to the limit in the variational formulation, and how we use the oscillating test functions. We concentrate on the tricky term. In lecture 1, here is what we did for the equation with highly oscillating coefficients: using the oscillating test function $\varphi(x_\gamma + \varepsilon\chi^{*,\gamma}(x/\varepsilon))$,

$$\int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla(x_\gamma + \varepsilon\chi^{*,\gamma}(x/\varepsilon))\varphi = -\langle \nabla \cdot A^*(x/\varepsilon)\nabla(x_\gamma + \varepsilon\chi^{*,\gamma}(x/\varepsilon)), u^\varepsilon\varphi\rangle \\ - \int_\Omega u^\varepsilon A^*(x/\varepsilon)\nabla(x_\gamma + \varepsilon\chi^{*,\gamma}(x/\varepsilon)) \cdot \nabla\varphi. \quad (1.11)$$

Now we use the fact that $x_\gamma + \varepsilon\chi^{*,\gamma}(x/\varepsilon)$ is A^*-linear to get that the first term in the right hand side of (1.11) is zero. It is then easy to pass to the limit in the second term. The role of the oscillating test function for porous media in this lecture is in some sense opposite. We use φw^ε where $w^\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$ in

$$\int_\Omega \varphi\nabla u^\varepsilon \cdot \nabla w^\varepsilon = -\langle \Delta w^\varepsilon, \varphi u^\varepsilon\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} - \int_\Omega u^\varepsilon \nabla\varphi \cdot \nabla w^\varepsilon, \quad (1.12)$$

to see that the second term in the right hand side of (1.12) goes to zero.

2 Construction of correctors in the periodic case

The case $d = 1$ is very particular because of Remark 4, so we discard it. We will stick to $d \geq 3$ in order to allow for a unified presentation. However, ideas and techniques are the same for $d = 2$.

We consider the Poisson problem (0.1) in the domain

$$\Omega^\varepsilon := \Omega \setminus \bigcup_{\xi \in 2\varepsilon\mathbb{Z}^d} B(\xi, r^\varepsilon) = \Omega \setminus \bigcup_{i=1}^{n_\varepsilon} T_i^\varepsilon,$$

where T_i^ε is a ball centered at a point ξ_i of the lattice $2\varepsilon\mathbb{Z}^d$ and of radius r^ε (see Figure 1). We take $0 < r^\varepsilon < \varepsilon$.

Theorem 4. *Let $d \geq 3$ and $C_0 > 0$. Let $a^\varepsilon := C_0\varepsilon^{\frac{d}{d-2}}$. Then,*

1. *either $r^\varepsilon \ll a^\varepsilon$, then u^ε converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to u^0 the unique solution to the Poisson problem (1.7),*
2. *or $r^\varepsilon \gg a^\varepsilon$, then u^ε converges strongly in $H^1(\Omega)$ to 0,*

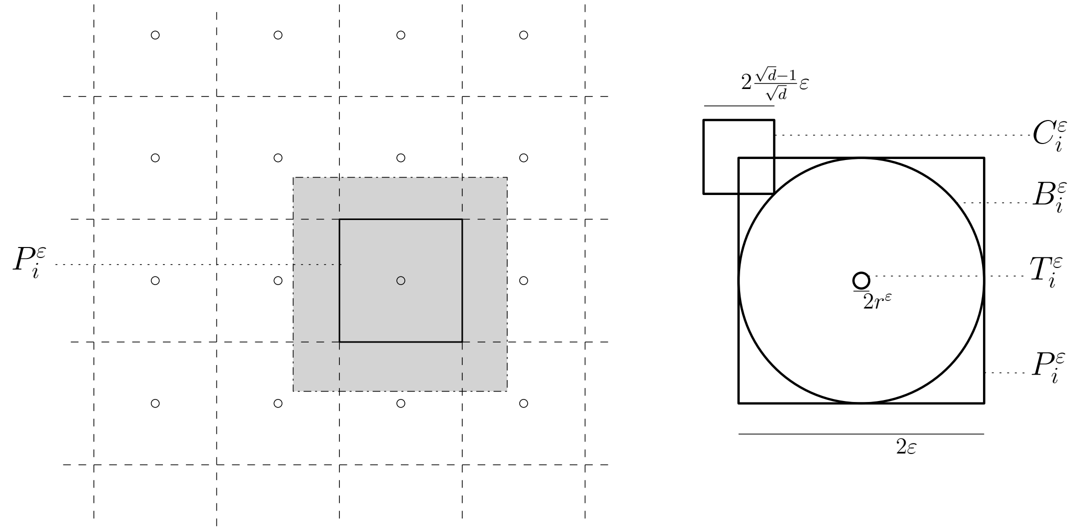

Figure 1: The porous medium: zoom on the unit cell P_i^ε

3. or $r^\varepsilon = a^\varepsilon$, then u^ε converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to the solution u^0 to (1.10)*, where the measure μ is an explicit constant*

$$\mu := \frac{S_d(d-2)}{2^d} C_0^{d-2}$$

and S_d is the surface of the sphere in $\mathbb{R}^d$.

The critical scale $\varepsilon^{\frac{d}{d-2}}$ is related to the decay of the fundamental solution of $-\Delta$ in $\mathbb{R}^d$ for $d \geq 3$.

We define a family of $2\varepsilon\mathbb{Z}^d$-periodic correctors w^ε in the following way. On the microscopic cell $[-\varepsilon, \varepsilon]^d$, $w^\varepsilon = w^\varepsilon(x) \in H^1([-\varepsilon, \varepsilon]^d)$ is a solution to

$$\begin{cases} w^\varepsilon(x) &= 0, \quad x \in B(0, r^\varepsilon), \\ -\Delta w^\varepsilon &= 0, \quad x \in B(0, \varepsilon) \setminus B(0, r^\varepsilon), \\ w^\varepsilon &= 1, \quad x \in [-\varepsilon, \varepsilon]^d \setminus B(0, \varepsilon). \end{cases} \tag{2.1}$$

The condition $w^\varepsilon \in H^1([-\varepsilon, \varepsilon]^d)$ contains the fact that there are no jumps at the interfaces $\partial B(0, \varepsilon)$ and $\partial B(0, r^\varepsilon)$. By the maximum principle, $0 \leq w^\varepsilon \leq 1$. The corrector w^ε is then extended to the whole of $\mathbb{R}^d$ by periodicity and is still denoted w^ε. Thus $w^\varepsilon \in H^1(\mathbb{R}^d)$ and vanishes on the holes T_i^ε, so (1.4) and (1.5) are satisfied.

Remark 7. The solution to (2.1) can be computed explicitely by using the rotational invariance (polar coordinates). We have

$$w^\varepsilon(x) = \frac{(r^\varepsilon)^{-d+2} - |x|^{-d+2}}{(r^\varepsilon)^{-d+2} - \varepsilon^{-d+2}}, \quad \text{for all} \quad x \in B(0, \varepsilon) \setminus B(0, r^\varepsilon). \tag{2.2}$$

Every explicit computation relies on this formula.

Lemma 5. *We have*

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 \simeq \frac{S_d(d-2)|\Omega|}{(2\varepsilon)^d (r^\varepsilon)^{-d+2}}. \tag{2.3}$$

Proof. On the one hand, a simple count gives asymptotically

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \simeq \frac{|\Omega|}{(2\varepsilon)^d} \int_{[-\varepsilon,\varepsilon]^d} |\nabla w^\varepsilon|^2,$$

and on the other hand, an explicit computation based on (2.2) gives

$$\int_{[-\varepsilon,\varepsilon]^d} |\nabla w^\varepsilon|^2 = \frac{S_d(d-2)}{(r^\varepsilon)^{-d+2} - \varepsilon^{-d+2}}.$$

Since $\varepsilon^{-d+2} = o\big((r^\varepsilon)^{-d+2}\big)$, we have the result. □

Case $r^\varepsilon \ll a^\varepsilon$: small holes

Now, if $r^\varepsilon \ll \varepsilon^{\frac{d}{d-2}}$, then by (2.3)

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \longrightarrow 0,$$

so that we are in the setting of Remark 2 and thus u^ε goes to the solution u^0 of (1.7). In other words, the strange term μ is zero.

Case $r^\varepsilon \gg a^\varepsilon$: large holes

Lemma 6 (Poincaré inequality in Ω^ε)**.** *There exists a constant $C > 0$ independent of ε such that for all $u \in H^1_0(\Omega^\varepsilon)$,*

$$\|u\|_{L^2(\Omega^\varepsilon)} \le C \left(\frac{\varepsilon^d}{(r^\varepsilon)^{d-2}}\right)^{1/2} \|\nabla u\|_{L^2(\Omega^\varepsilon)}. \tag{2.4}$$

This Poincaré inequality can be found in the paper by Allaire [All90, Lemma 3.4.1] dealing with the homogenization of the Stokes equation. It is true of course in every regime, but in the case when $r^\varepsilon \gg a^\varepsilon$, it directly gives some information about the limit of u^ε. Notice that

$$r^\varepsilon \gg \varepsilon^{\frac{d}{d-2}} \quad \text{implies} \quad \frac{\varepsilon^d}{(r^\varepsilon)^{d-2}} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Remark 8. The fact that we "gain" a factor $\varepsilon^{d/2}$ comes from the fact that the holes are close to each other. The "loss" of the factor $\frac{1}{r^{\varepsilon\frac{d-2}{2}}}$ is a consequence of the fact that the part where u is zero, that is on the holes T_i^ε, is relatively small.

As a corollary of Lemma 6, we get:

$$\int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^2 \le \|f\|_{L^2(\Omega)} \|u^\varepsilon\|_{L^2(\Omega)} \le C \left(\frac{\varepsilon^d}{(r^\varepsilon)^{d-2}}\right)^{1/2} \|f\|_{L^2(\Omega)} \|\nabla u^\varepsilon\|_{L^2(\Omega)},$$

thus

$$\left(\int_\Omega |\nabla u^\varepsilon|^2\right)^{1/2} = \left(\int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^2\right)^{1/2} \le C \left(\frac{\varepsilon^d}{(r^\varepsilon)^{d-2}}\right)^{1/2} \|f\|_{L^2(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

By Poincaré's inequality in Ω

$$\left(\int_{\Omega} |u^\varepsilon|^2\right)^{1/2} \leq C \left(\int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^2\right)^{1/2} \xrightarrow{\varepsilon\to 0} 0,$$

so that u^ε converges strongly to 0 in $H^1(\Omega)$.

Proof of Lemma 6. (1) For a point $\xi \in 2\varepsilon\mathbb{Z}^d$, we notice that the ball $B(\xi, \sqrt{d}\varepsilon)$ is circumscribed (touching at the vertices) to the cell $P_\xi^\varepsilon = [\xi - \varepsilon, \xi + \varepsilon]^d$ centered in ξ. One given point $x \in \mathbb{R}^d$ belongs to exactly one cell P_i^ε, and maybe to some of the $2d$ circumscribed balls to adjacent cells. Therefore, we have the following estimate of the $L^2(\Omega^\varepsilon)$ norm:

$$\|u\|^2_{L^2(\Omega^\varepsilon)} \leq \sum_{i=1}^{n_\varepsilon} \|u\|^2_{L^2(B(\xi_i,\sqrt{d}\varepsilon))} \leq (2d+1)\|u\|^2_{L^2(\Omega^\varepsilon)}.$$

This estimate is rather crude, but all we care about is to have equivalence of then norms. We aim at proving Poincaré's inequality for a function $u \in H^1(B(\xi_i, \sqrt{d}\varepsilon))$, zero on $T_i^\varepsilon = B(\xi_i, r^\varepsilon)$.

(2) For simplicity (everything is translation invariant) let us consider the point $\xi = 0$. Let r denote the radial variable. Let $u \in \mathcal{C}^0(B(0, \sqrt{d}\varepsilon)) \cap H^1(B(0, \sqrt{d}\varepsilon))$ such that $u(x) = 0$ on $r = r^\varepsilon$. For $x \in B(0, \sqrt{d}\varepsilon)$, the radial vector e_r is equal to $x/|x|$. Writing the fundamental theorem of calculus by taking x as the reference point (rather than 0), for the function

$$t \mapsto u(x + (t-r)e_r)$$

yields

$$u(x) = \int_{r^\varepsilon}^{r} \nabla u(x + (t-r)e_r) \cdot e_r dt.$$

Then, integrating over $x = r\omega \in B(0, \sqrt{d}\varepsilon)$,

$$\|u\|^2_{L^2(B(0,\sqrt{d}\varepsilon))} \leq C \int_{\mathbb{S}^{d-1}} \int_{r^\varepsilon}^{\sqrt{d}\varepsilon} \left(\int_{r^\varepsilon}^{r} \nabla u(r\omega + (t-r)\omega) \cdot \omega dt\right)^2 r^{d-1} dr d\omega.$$

Cauchy-Schwarz's inequality yields

$$\left(\int_{r^\varepsilon}^{r} \nabla u(r\omega + (t-r)\omega) \cdot \omega dt\right)^2 \leq \left(\int_{r^\varepsilon}^{r} |\nabla u(r\omega + (t-r)\omega) \cdot \omega|^2 t^{d-1} dt\right) \left(\int_{r^\varepsilon}^{r} \frac{dt}{t^{d-1}}\right)$$

$$\leq \left(\int_{r^\varepsilon}^{\sqrt{d}\varepsilon} [\nabla u(r\omega + (t-r)\omega) \cdot \omega]^2 t^{d-1} dt\right) \left(\int_{r^\varepsilon}^{\sqrt{d}\varepsilon} \frac{dt}{t^{d-1}}\right).$$

Therefore,

$$\|u\|^2_{L^2(B(0,\sqrt{d}\varepsilon))} \leq \int_{\mathbb{S}^{d-1}} \int_{r^\varepsilon}^{\sqrt{d}\varepsilon} \int_{r^\varepsilon}^{\sqrt{d}\varepsilon} |\nabla u(r\omega + (t-r)\omega)|^2 t^{d-1} dt r^{d-1} dr d\omega \left(\int_{r^\varepsilon}^{\sqrt{d}\varepsilon} \frac{dt}{t^{d-1}}\right)$$

$$\leq C\varepsilon^d \|\nabla u\|^2_{L^2(B(0,\sqrt{d}\varepsilon))} \left(\int_{r^\varepsilon}^{\sqrt{d}\varepsilon} \frac{dt}{t^{d-1}}\right) \leq C \frac{\varepsilon^d}{(r^\varepsilon)^{d-2}},$$

since $r^\varepsilon \ll \varepsilon$. □

Case $r^\varepsilon = a^\varepsilon$: critical size

We focus now on the case when $r^\varepsilon \simeq \varepsilon^{\frac{d}{d-2}}$. More precisely, we assume that $r^\varepsilon = C_0\varepsilon^{\frac{d}{d-2}}$. In this regime, it follows from (2.3) that

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} \frac{S_d(d-2)|\Omega|}{2^d} C_0^{d-2}.$$

This fact combined with the bound $0 \le w^\varepsilon \le 1$ infered from (2.2) yields that w^ε is bounded in $H^1(\Omega)$, so that

$$\begin{aligned} w^\varepsilon &\rightharpoonup w \quad \text{weakly in} \quad H^1(\Omega),\\ w^\varepsilon &\to w \quad \text{strongly in} \quad L^2(\Omega). \end{aligned}$$

We aim at showing that $w = 1$. In that perspective, we use the fact that

$$w^\varepsilon \equiv 1 \quad \text{on} \quad C^\varepsilon := \bigcup_{i=1}^{n_\varepsilon} C_i^\varepsilon.$$

We denote the characteristic function of C^ε by χ_{C^ε}. Let E be a measurable set in Ω and χ_E its characteristic function. We have

$$\int_\Omega \chi_{C^\varepsilon}\chi_E = |C^\varepsilon \cap E| \overset{\varepsilon\to 0}{\sim} \frac{|E|}{(2\varepsilon)^d}|C_i^\varepsilon| = \frac{|E|}{(2\varepsilon)^d}\left(\frac{\sqrt{d}-1}{\sqrt{d}}\right)^d (2\varepsilon)^d = |E|\left(\frac{\sqrt{d}-1}{\sqrt{d}}\right)^d.$$

Therefore

$$\chi_{C^\varepsilon} \overset{\star}{\rightharpoonup} \left(\frac{\sqrt{d}-1}{\sqrt{d}}\right)^d \quad \text{weakly star in} \quad L^\infty(\Omega).$$

We argue now as in point (2) of the introduction (see above): by definition of w^ε

$$w^\varepsilon \chi_{C^\varepsilon} = \chi_{C^\varepsilon},$$

so that passing to the weak limit (using the strong convergence of w^ε in L^2) we get

$$\left(\frac{\sqrt{d}-1}{\sqrt{d}}\right)^d (w-1) = 0,$$

i.e. $w = 1$ which proves (1.6).

Our next (and final goal) is to compute $-\Delta w^\varepsilon$. Since the gradient of w^ε has jumps at the interfaces $\partial B_i^\varepsilon \cup T_i^\varepsilon$, $-\Delta w^\varepsilon$ is a distribution (actually a measure) supported on $\partial B_i^\varepsilon \cup \partial T_i^\varepsilon$. We have, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \langle -\Delta w^\varepsilon, \varphi\rangle_{\mathcal{D}',\mathcal{D}} &= \langle \nabla w^\varepsilon, \nabla\varphi\rangle_{\mathcal{D}',\mathcal{D}}\\ &= \int_{\mathbb{R}^d} \nabla w^\varepsilon \cdot \nabla\varphi\\ &= \sum_{i=1}^{n_\varepsilon} \int_{B_i^\varepsilon\setminus T_i^\varepsilon} \nabla w^\varepsilon \cdot \nabla\varphi\\ &= \sum_{i=1}^{n_\varepsilon} \int_{\partial B_i^\varepsilon} \nabla w^\varepsilon \cdot n_{ext}\varphi + \sum_{i=1}^{n_\varepsilon} \int_{\partial T_i^\varepsilon} \nabla w^\varepsilon \cdot n_{ext}\varphi, \end{aligned}$$

the last line following by integration by parts, and the fact that

$$\int_{B_i^\varepsilon \setminus T_i^\varepsilon} \Delta w^\varepsilon \varphi = 0.$$

To summarize

$$-\Delta w^\varepsilon = \mu^\varepsilon - \gamma^\varepsilon,$$

with

$$\begin{aligned}\langle \mu^\varepsilon, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \sum_{i=1}^{n_\varepsilon} \int_{\partial B_i^\varepsilon} \nabla w^\varepsilon \cdot n_{ext} \varphi \\ &= \frac{(d-2)C_0^{d-2}}{1 - C_0^{d-2}\varepsilon^2} \sum_{i=1}^{n_\varepsilon} \varepsilon \int_{\partial B_i^\varepsilon} \varphi(s)\sigma(ds)\end{aligned}$$

(the explicit formula is computed thanks to (2.2)), and

$$\langle \gamma^\varepsilon, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = -\sum_{i=1}^{n_\varepsilon} \int_{\partial T_i^\varepsilon} \nabla w^\varepsilon \cdot n_{ext} \varphi.$$

We clearly have that for all $v^\varepsilon \in H_0^1(\Omega^\varepsilon)$ (i.e. v^ε is zero on the holes),

$$\langle \gamma^\varepsilon, v^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0.$$

It remains to check that μ^ε converges strongly in $H^{-1}(\Omega)$ (even in $W^{-1,\infty}(\Omega)$) to $\mu \in W^{-1,\infty}(\Omega)$. Let $\delta_{\partial B_i^\varepsilon}$ be the distribution defined by

$$\langle \delta_i^\varepsilon, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \int_{\partial B_i^\varepsilon} \varphi(s)\sigma(ds) \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Lemma 7. *We have*

$$\sum_{i=1}^{n_\varepsilon} \varepsilon \delta_i^\varepsilon \to \frac{S_d}{2^d} \quad \textit{strongly in} \quad W_{loc}^{-1,\infty}(\mathbb{R}^d).$$

It follows from the lemma that

$$\mu^\varepsilon \to \mu := \frac{(d-2)S_d}{2^d} C_0^{d-2} \quad \text{strongly in} \quad W_{loc}^{-1,\infty}(\mathbb{R}^d)$$

Proof of Lemma 7. (1) We first show that

$$\sum_{i=1}^{n_\varepsilon} \varepsilon \delta_i^\varepsilon = -\Delta q^\varepsilon - d\chi_{B^\varepsilon}, \tag{2.5}$$

where $B^\varepsilon = \bigcup_{i=1}^{n_\varepsilon} B_i^\varepsilon$. Here q^ε is the unique solution to the Neumann problem

$$\left\{ \begin{array}{rll} -\Delta q^\varepsilon &= d, & x \in \bigcup_{i=1}^{n_\varepsilon} B_i^\varepsilon, \\ \nabla q^\varepsilon \cdot n_{ext} &= \varepsilon, & x \in \bigcup_{i=1}^{n_\varepsilon} \partial B_i^\varepsilon, \end{array} \right. \tag{2.6}$$

such that $q^\varepsilon = 0$ on ∂B_i^ε. We extend q^ε by zero on $\mathbb{R}^d \setminus B^\varepsilon$. Notice that the compatibility condition

$$\int_{B_i^\varepsilon} -\delta q^\varepsilon = \int_{\partial B_i^\varepsilon} \nabla q^\varepsilon \cdot n_{ext}$$

is satisfied. Let us prove (2.5): integrating by parts against $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we get

$$\begin{aligned}
\langle -\Delta q^\varepsilon, \varphi \rangle_{\mathcal{D}',\mathcal{D}} &= \langle \nabla q^\varepsilon, \nabla \varphi \rangle_{\mathcal{D}',\mathcal{D}} \\
&= \int_{\mathbb{R}^d} \nabla q^\varepsilon \cdot \nabla \varphi \\
&= \sum_{i=1}^{n_\varepsilon} \int_{\partial B_i^\varepsilon} \nabla q^\varepsilon \cdot n_{ext} \varphi + d \int_{B^\varepsilon} \varphi \\
&= \sum_{i=1}^{n_\varepsilon} \varepsilon \langle \delta_i^\varepsilon, \varphi \rangle - \langle d\chi_{B^\varepsilon}, \varphi \rangle.
\end{aligned}$$

(2) The solution q^ε of (2.6) is radially symmetric, thus

$$-\Delta q^\varepsilon = \frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r q^\varepsilon(r) \right) = d \quad \text{implies} \quad \partial_r q^\varepsilon(r) = r,$$

r being the distance from a point in B_i^ε to the center ξ_i. Eventually,

$$\|\nabla q^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon,$$

so that

$$q^\varepsilon \to 0 \quad \text{strongly in} \quad W^{1,\infty}(\mathbb{R}^d).$$

This convergence directly implies

$$-\Delta q^\varepsilon \to 0 \quad \text{strongly in} \quad W^{-1,\infty}(\mathbb{R}^d).$$

Moreover,

$$\langle \chi_{B^\varepsilon}, \varphi \rangle_{L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d)} \longrightarrow \langle \overline{\chi}, \varphi \rangle_{L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d)}, \tag{2.7}$$

where

$$\overline{\chi} = \int_{\mathbb{T}^d} \chi_B(y) dy = |B_d(0, 1/2)| = \frac{|B_d(0,1)|}{2^d} = \frac{|S_d(0,1)|}{2^d d}.$$

From (2.7) we get that (up to a subsequence)

$$\chi_{B^\varepsilon} \to \overline{\chi} \quad \text{strongly in} \quad W^{-1,\infty}(\Omega),$$

hence the lemma. □

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