Lecture 1: The Oscillating Test Function Method

Christophe Prange*

February 15, 2016

In this lecture, we consider the divergence form elliptic equation (or system)

$$\begin{cases} -\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon &= f, \quad x \in \Omega, \\ u^\varepsilon &= 0, \quad x \in \partial\Omega, \end{cases} \tag{0.1}$$

with highly oscillating periodic coefficients. For ease of notation, we will only consider equations, although everything straightforwardly extends to systems.

The domain Ω is a compact subset of $\mathbb{R}^d$, $u^\varepsilon = u^\varepsilon(x) \in \mathbb{R}$, $A = (A^{\alpha\beta}(y))$ and $f \in L^2(\Omega)$. Let us assume that:

(A1) A is elliptic, i.e. there exists $\lambda > 0$, such that for all $\xi \in \mathbb{R}^d$, for all $y \in \mathbb{R}^d$,

$$\lambda|\xi|^2 \leq A(y)\xi \cdot \xi \leq \lambda^{-1}|\xi|^2,$$

(A2) A is $\mathbb{Z}^d$-periodic, i.e. $A = A(y)$ with $y \in \mathbb{T}^d$.

1 A priori bounds and one dimensional case

A priori bounds

The weak formulation of the Dirichlet problem (0.1) reads: for all $v \in H^1_0(\Omega)$,

$$\int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla v = \int_\Omega fv. \tag{1.1}$$

Taking $v = u^\varepsilon$ in the previous weak formulation, Poincaré's inequality implies the a priori bound

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

with $C = C(d, \Omega, \lambda)$.

Therefore, (using Poincaré's inequality one more time) u^ε is bounded in $H^1(\Omega)$ uniformly in ε. The compact injection $H^1(\Omega)$ into $L^2(\Omega)$ (Rellich's theorem) now implies

$$\begin{aligned} &\nabla u^{\varepsilon_k} \rightharpoonup \nabla u^0 \text{ weakly in } L^2(\Omega), \\ &u^{\varepsilon_k} \to u^0 \text{ strongly in } L^2(\Omega), \\ &\xi^{\varepsilon_k} := A(x/\varepsilon_k)\nabla u^{\varepsilon_k} \rightharpoonup \xi^0 \text{ weakly in } L^2(\Omega). \end{aligned}$$

*Université de Bordeaux, 351 cours de la Libération, 33405 Talence. *E-mail address:* `christophe.prange@math.cnrs.fr`

Passing to the limit in (1.1) we get for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \xi^0 \cdot \nabla v = \int_\Omega f v. \tag{1.2}$$

Question. Is ξ^0 of the form $\xi^0 = A^0 \nabla u^0$ for some matrix A^0?

The answer is non trivial. Indeed, since A is periodic

$$A(x/\varepsilon_k) \rightharpoonup \overline{A} := \fint_{\mathbb{T}^d} A(y) dy \text{ weakly in } L^2(\Omega).$$

This convergence cannot be improved to strong convergence. Moreover, there are no obvious a priori bounds uniform in ε making it possible to improve the weak convergence of ∇u^ε into strong convergence. Therefore, the product

$$\xi^{\varepsilon_k} = A(x/\varepsilon_k) \nabla u^{\varepsilon_k}$$

is a product of weakly converging sequences. A priori

$$\xi^0 \neq \overline{A} \nabla u^0.$$

Particular case of dimension one

In dimension one, the equation gives additional controls, which make it possible to improve the convergence of ξ^{ε_k} from weak to strong convergence in $L^2(\Omega)$. Our equation reads

$$\partial_x(\xi^{\varepsilon_k}) = \partial_x \left(A(x/\varepsilon_k) \partial_x u^{\varepsilon_k} \right) = f, \tag{1.3}$$

for $\Omega = (0,1)$. This equation (contrary to the case $d \geq 2$) gives that the whole gradient of ξ^{ε_k} is controlled, not only the divergence: $\partial_x(\xi^{\varepsilon_k})$ is bounded in $L^2(\Omega)$, which together with the a priori bound above implies that ξ^{ε_k} is bounded in $H^1(\Omega)$. By Rellich's theorem, we then infer that (up to extracting a subsequence still denoted the same)

$$\xi^{\varepsilon_k} \to \xi^0 \text{ strongly in } L^2(\Omega).$$

Now,

$$\partial_x u^0 \leftharpoonup \partial_x u^{\varepsilon_k} = \frac{\xi^{\varepsilon_k}}{A(x/\varepsilon_k)} \rightharpoonup \xi^0 \overline{A^{-1}},$$

the last convergence being seen as a product of a strongly times a weakly convergent sequence. Therefore, identifying the limit on the left and on the right, we get

$$\xi^0 = \frac{1}{\overline{A^{-1}}} \partial_x u^0.$$

We notice that:

(1) the limit equation has the same form as (1.3), i.e. it is divergence form, elliptic,

$$-\partial_x(A^0 \partial_x u^0) = f, \; x \in (0,1)$$

with A^0 a constant matrix,

$$A^0 := \frac{1}{\overline{A^{-1}}}, \tag{1.4}$$

(2) of course, the whole sequence converges (there is just one possible limit), not only a subsequence,

(3) even in this (simple) case, the limit of ξ^{ε_k} is not equal in general to the product of the weak limits!

2 Multiscale expansions and existence of correctors

There are two scales in the problem: the slow scale x and the fast scale $y = x/\varepsilon$. Let us assume that u^ε retains this structure and therefore we speculate the following Ansatz:

$$u^\varepsilon(x) \simeq u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon^2 u^2(x, x/\varepsilon) + \dots$$

with $u^i = u^i(x, y)$ $\mathbb{Z}^d$-periodic in y. Let us simply plug the Ansatz in the equation (0.1) and identify the different powers of ε. This leaves us with a cascade of equations:

$$\begin{aligned}
&-\nabla_y \cdot A(y)\nabla_y u^0 = 0,\\
&-\nabla_y \cdot A(y)\nabla_y u^1 = \nabla_y \cdot A(y)\nabla_x u^0 + \nabla_x \cdot A(y)\nabla_y u^0,\\
&-\nabla_y \cdot A(y)\nabla_y u^2 = f + \nabla_y \cdot A(y)\nabla_x u^1 + \nabla_x \cdot A(y)\nabla_y u^1 + \nabla_x \cdot A(y)\nabla_x u^0.
\end{aligned}$$

Each equation is posed on the torus $\mathbb{T}^d$, where x is but a parameter. The first equation implies that u^0 is a constant, i.e. $u^0 = u^0(x)$. This is somehow expected since u^0 is meant to represent an averaged behavior of u^ε. The second equation becomes

$$-\nabla_y \cdot A(y)\nabla_y u^1 = \nabla_y \cdot A(y)\nabla_x u^0 = \partial_{y_\alpha} A^{\alpha\gamma}(y)\partial_{x_\gamma} u^0.$$

By linearity, we can look for u^1 of separated form

$$u^1(x, y) = \chi^\gamma(y)\partial_{x_\gamma} u^0(x),$$

where $\chi = \chi^\gamma(y) \in \mathbb{R}$, $1 \leq \gamma \leq d$ is a solution to the cell (or interior) corrector problem

$$-\nabla \cdot A(y)\nabla\chi^\gamma = \partial_\alpha A^{\alpha\gamma},\ y \in \mathbb{T}^d \qquad \text{and} \qquad \fint_{\mathbb{T}^d} \chi^\gamma(y)dy = 0. \tag{2.1}$$

This problem has a unique weak solution, uniqueness coming from the Poincaré-Wirtinger inequality: for all $\varphi \in H^1(\mathbb{T}^d)$,

$$\left\| \varphi - \fint_{\mathbb{T}^d} \varphi \right\|_{L^2(\mathbb{T}^d)} \leq C\|\nabla\varphi\|_{L^2(\mathbb{T}^d)}.$$

Notice that the function $y_\gamma + \chi^\gamma(y)$ is a solution of

$$-\nabla \cdot A(y)\nabla\left(y_\gamma + \chi^\gamma\right) = 0,\ y \in \mathbb{R}^d.$$

We say that $y_\gamma + \chi^\gamma$ is an A-linear function. Rescaling, we get

$$-\nabla \cdot A(x/\varepsilon)\nabla\left(x_\gamma + \varepsilon\chi^\gamma(x/\varepsilon)\right) = 0,\ x \in \mathbb{R}^d. \tag{2.2}$$

It remains to look at the last equation in our hierarchy (of course, one could go on forever). On the left hand side, $-\nabla_y \cdot A(y)\nabla_y u^2(x,y)$ is a periodic function in y, therefore,

$$\begin{aligned}
0 &= -\int_{\mathbb{T}^d} \left\{\nabla_y \cdot A(y)\nabla_y u^2\right\} dy \\
&= \int_{\mathbb{T}^d} \left\{f + \nabla_y \cdot A(y)\nabla_x u^1 + \nabla_x \cdot A(y)\nabla_y u^1 + \nabla_x \cdot A(y)\nabla_x u^0\right\} \\
&= f(x) + \int_{\mathbb{T}^d} \left\{A^{\alpha\beta}(y)\partial_{y_\beta}\chi^\gamma(y)\partial_{x_\alpha}\partial_{x_\gamma}u^0(x) + A^{\alpha\beta}(y)\partial_{x_\alpha}\partial_{x_\beta}u^0(x)\right\} dy.
\end{aligned}$$

Eventually we get

$$-\int_{\mathbb{T}^d} \left\{A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y) + A^{\alpha\beta}(y)\right\} \partial_{x_\alpha}\partial_{x_\beta}u^0(x) = f(x) \quad \text{for} \quad x \in \Omega,$$

or in a more compact form

$$-\nabla \cdot A^0 \nabla u^0 = f(x) \quad \text{for} \quad x \in \Omega,$$

where

$$A^0 := \int_{\mathbb{T}^d} \left\{A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y) + A^{\alpha\beta}(y)\right\} dy$$

is the constant homogenized matrix.

Remark 1 (correctors for $d = 1$). In dimension $d = 1$, we clearly have

$$\partial_y \chi(y) = \frac{C_1}{A(y)} - 1.$$

It remains to compute the constant C_1. Using that χ is periodic, we have

$$0 = \int_{\mathbb{T}^d} \partial_y \chi(y) dy = C_1 \overline{A^{-1}} - 1,$$

so that

$$C_1 = \frac{1}{\overline{A^{-1}}}.$$

Plugging the expression of $A(y)\partial_y \chi$, we recover the formula for A^0 discovered in (1.4).

3 Oscillating test function method

As underlined above, the product $\xi^{\varepsilon_k} = A(x/\varepsilon_k)\nabla u^{\varepsilon_k}$ is a product of two weakly convergent sequences. Apart from dimension one, there is, it seems, no easy way to improve these convergences from weak to strong. *We now drop the subscript k and just write ε.*

A fundamental idea to untangle this situation (and many others) has been suggested and developpped by Tartar in the late 70's (Tartar 1977 Cours Peccot, see

Murat and Tartar, *H-Convergence* in [CK97]). We take in the weak formulation 1.1 particular test functions, namely *oscillating test functions*, whose behavior mimic the behavior of u^ε.

Two (among many!) references on the oscillating test function method, also called energy method, are the book by Cioranescu and Donato [CD99] in particular Chapter 8, and the book by Evans [Eva90] in particular Chapter 5.A.2.

(1) Let χ^* be the cell corrector for the adjoint equation

$$-\nabla \cdot A^*(y)\nabla\chi^{*,\gamma} = \partial_\alpha A^{\gamma\alpha},\ y \in \mathbb{T}^d \qquad \text{and} \qquad \fint_{\mathbb{T}^d} \chi^{*,\gamma}(y)dy = 0. \tag{3.1}$$

Let $v^\varepsilon = v^\varepsilon(x) := x_\gamma + \varepsilon\chi^{*,\gamma}(x/\varepsilon)$. For $\varphi \in \mathcal{C}_c^\infty(\Omega)$, we take $v = \varphi v^\varepsilon$. By (2.2),

$$-\nabla \cdot A^*(x/\varepsilon)\nabla v^\varepsilon = 0 \quad \text{in} \quad \Omega.$$

One can think in the case when A is symmetric that the oscillations of $v^\varepsilon(x)$ imitate the oscillations of u^ε (at least away from boundaries). Notice that

$$v^\varepsilon \longrightarrow x^\gamma \text{ strongly in } L^2(\Omega),$$

and

$$\begin{aligned}(A^*(x/\varepsilon)\nabla v^\varepsilon)_\alpha &= A^{*\alpha\gamma}(x/\varepsilon) + A^{*\alpha\beta}(x/\varepsilon)\partial_{y_\beta}\chi^{*,\gamma}(x/\varepsilon)\\ &\rightharpoonup \overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}} = \fint_\Omega A^{*\alpha\gamma}(y) + A^{*\alpha\beta}(y)\partial_{y_\beta}\chi^{*,\gamma}(y)dy.\end{aligned}$$

(2) Taking the test function $v = \varphi v^\varepsilon$ as above, we get

$$\int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla\varphi v^\varepsilon + \int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla v^\varepsilon\varphi = \int_\Omega f\varphi v^\varepsilon. \tag{3.2}$$

It is easy to pass to the limit in the right hand side of the previous equality:

$$\int_\Omega f\varphi v^\varepsilon \longrightarrow \int_\Omega f\varphi x^\gamma.$$

By the strong convergence of v^ε, we have

$$\int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla\varphi v^\varepsilon \longrightarrow \int_\Omega \xi^0 \cdot \nabla\varphi x^\gamma.$$

It remains to pass to the limit in the second term of the left hand side of (3.2). Integrating by parts, and using that v^ε is A^*-linear, we get

$$\begin{aligned}\int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla v^\varepsilon\varphi &= -\int_\Omega u^\varepsilon A^*(x/\varepsilon)\nabla v^\varepsilon \cdot \nabla\varphi\\ &\longrightarrow -\int_\Omega u^0\left(\overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}}\right)\partial_{x_\alpha}\varphi = \int_\Omega \left(\overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}}\right)\partial_{x_\alpha}u^0\varphi.\end{aligned}$$

Consequently,

$$\int_\Omega \xi^0 \cdot \nabla\varphi x^\gamma + \int_\Omega \left(\overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}}\right)\partial_{x_\alpha}u^0\varphi = \int_\Omega f\varphi x^\gamma.$$

(3) Now, taking $v(x) = x^\gamma \varphi$ in the weak formulation (1.2), we get

$$\int_\Omega f\varphi x^\gamma = \int_\Omega \xi^0 \cdot \nabla \varphi x^\gamma + \int_\Omega \xi^{0,\gamma}\varphi.$$

(4) We end up with

$$\int_\Omega \left(\overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}}\right)\partial_{x_\alpha} u^0 \varphi = \int_\Omega \xi^{0,\gamma}\varphi,$$

i.e.

$$\xi^{0,\gamma} = \left(\overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}}\right)\partial_{x_\alpha} u^0 = A^{0,\gamma\alpha}\partial_{x_\alpha} u^0,$$

which concludes the proof.

One can show that

$$\left(\overline{A^{*\alpha\gamma}} + \overline{A^{*\alpha\beta}\partial_{y_\beta}\chi^{*,\gamma}}\right) = \left(\overline{A^{\gamma\alpha}} + \overline{A^{\gamma\beta}\partial_{y_\beta}\chi^{\alpha}}\right),$$

so that eventually

$$A^0 := \overline{A} + \overline{A\nabla\chi}.$$

Moreover, since

$$-\int_{\mathbb{T}^d} \nabla \cdot A(y)\nabla\chi^\beta \chi^\alpha dy = -\int_{\mathbb{T}^d} A^{\gamma\beta}(y)\partial_{y_\gamma}\chi^\alpha,$$

we have

$$\begin{aligned}
&\int_{\mathbb{T}^d} A(y)\nabla(y_\beta + \chi^\beta)\cdot\nabla(y_\alpha + \chi^\alpha)dy \\
&= -\int_{\mathbb{T}^d} \nabla\cdot A(y)\nabla\chi^\beta\chi^\alpha dy + \int_{\mathbb{T}^d} A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta dy + \int_{\mathbb{T}^d} A^{\gamma\beta}(y)\partial_{y_\gamma}\chi^\alpha dy + \int_{\mathbb{T}^d} A^{\alpha\beta}(y)dy \\
&= A^{0,\alpha\beta}.
\end{aligned}$$

Therefore, A^0 is elliptic: for all $\eta,\ \hat\eta \in \mathbb{R}^d$

$$A\eta\cdot\hat\eta + A\hat\eta\cdot\eta \geq \frac{2}{\lambda}\eta\cdot\hat\eta$$

so that

$$\begin{aligned}
&A^{0,\alpha\beta}\xi_\alpha\xi_\beta = \int_{\mathbb{T}^d} A(y)\nabla(y_\beta + \chi^\beta)\xi_\beta\cdot\nabla(y_\alpha+\chi^\alpha)\xi_\alpha dy \\
&= \sum_{\alpha\leq\beta}\int_{\mathbb{T}^d}\left\{A(y)\nabla(y_\beta+\chi^\beta)\xi_\beta\cdot\nabla(y_\alpha+\chi^\alpha)\xi_\alpha + A(y)\nabla(y_\alpha+\chi^\alpha)\xi_\alpha\cdot\nabla(y_\beta+\chi^\beta)\xi_\beta\right\}dy \\
&\geq \frac{1}{\lambda}\sum_{\alpha,\beta}\int_{\mathbb{T}^d}\nabla(y_\beta+\chi^\beta)\xi_\beta\cdot\nabla(y_\alpha+\chi^\alpha)\xi_\alpha dy = \frac{1}{\lambda}\int_{\mathbb{T}^d}|\nabla(y_\alpha+\chi^\alpha)\xi_\alpha|^2 \\
&\geq \frac{1}{\lambda}\left|\int_{\mathbb{T}^d}\nabla(y_\alpha+\chi^\alpha)\xi_\alpha\right|^2 = \frac{1}{\lambda}|\xi|^2,
\end{aligned}$$

the last line following from Jensen's inequality.

We have proved the following theorem:

Theorem 1. *The family of solutions u^ε of* (0.1) *converges weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$ to the solution u^0 of*

$$\begin{cases} -\nabla \cdot A^0 \nabla u^0 &= f, \quad x \in \Omega, \\ \qquad\quad u^0 &= 0, \quad x \in \partial\Omega. \end{cases} \tag{3.3}$$

A few comments:

(1) Remark: Oscillating test function method works extremely well (and easily).

(2) Question of convergence becomes: How to construct an ad hoc oscillating test function (in particular oscillating part v^ε) in more general situations (A not periodic, but quasi-, almostperiodic, or random; porous media)?

(3) The crucial point in this proof is $v^\varepsilon \to x_\gamma$ strongly in $L^2(\Omega)$. This is a consequence of the fact that χ^* is bounded (sublinear would be enough).

4 Error estimates

Our goal is to estimate the remainder

$$r^\varepsilon(x) := u^\varepsilon(x) - u^0(x) - \varepsilon\chi(x/\varepsilon) \cdot \nabla u^0(x).$$

We have

$$\begin{aligned} \nabla r^\varepsilon(x) &= \nabla u^\varepsilon(x) - \nabla u^0(x) - \nabla_y \chi(x/\varepsilon) \cdot \nabla u^0(x) - \varepsilon\chi(x/\varepsilon) \cdot \nabla^2 u^0(x), \\ A(x/\varepsilon)\nabla r^\varepsilon(x) &= A(x/\varepsilon)\nabla u^\varepsilon(x) - A(x/\varepsilon)\nabla u^0(x) \\ &\quad - A(x/\varepsilon)\nabla_y \chi(x/\varepsilon) \cdot \nabla u^0(x) - A(x/\varepsilon)\chi(x/\varepsilon) \cdot \nabla^2 u^0(x), \end{aligned}$$

and

$$\begin{aligned} &- \nabla \cdot A(x/\varepsilon)\nabla r^\varepsilon(x) = f(x) + \frac{1}{\varepsilon}\big(\nabla_y \cdot A(x/\varepsilon)\big)\nabla u^0(x) + A(x/\varepsilon) \cdot \nabla^2 u^0(x) \\ &+ \frac{1}{\varepsilon}\left\{\nabla \cdot (A(y)\nabla\chi)\right\}(x/\varepsilon)\nabla u^0(x) + A(x/\varepsilon)\nabla\chi(x/\varepsilon) \cdot \nabla^2 u^0(x) + \varepsilon\nabla \cdot \{A(x/\varepsilon)\chi(x/\varepsilon) \cdot \nabla^2 u^0(x)\}. \end{aligned}$$

Now, the terms of order ε^{-1} vanish because χ is a solution of the cell corrector equation (2.1). Therefore,

$$\begin{aligned} - \nabla \cdot A(x/\varepsilon)\nabla r^\varepsilon(x) = f(x) + \left\{A^{\alpha\beta}(x/\varepsilon) + A^{\alpha\gamma}(x/\varepsilon)\partial_{y_\gamma}\chi^\beta(x/\varepsilon)\right\}\partial_{x_\alpha}\partial_{x_\beta} u^0(x) \\ + \varepsilon\nabla \cdot \{A(x/\varepsilon)\chi(x/\varepsilon) \cdot \nabla^2 u^0(x)\}. \end{aligned}$$

For $1 \leq \alpha, \beta \leq d$, let

$$\psi_{\alpha\beta}(y) := A^{\alpha\beta}(y) + A^{\alpha\gamma}(y)\partial_{y_\gamma}\chi^\beta(y).$$

Notice that,

$$\int_{\mathbb{T}^d} \psi_{\alpha\beta}\partial_{x_\alpha}\partial_{x_\beta} u^0(x) = \nabla \cdot A^0 \nabla u^0(x) = -f(x),$$

so that

$$\psi_{\alpha\beta}(x/\varepsilon)\partial_{x_\alpha}\partial_{x_\beta}u^0(x) = \left\{\psi_{\alpha\beta}(x/\varepsilon) - \int_{\mathbb{T}^d}\psi_{\alpha\beta}\right\}\partial_{x_\alpha}\partial_{x_\beta}u^0(x) - f(x).$$

Eventually

$$-\nabla\cdot A(x/\varepsilon)\nabla r^\varepsilon(x) = \left\{\psi_{\alpha\beta}(x/\varepsilon) - \int_{\mathbb{T}^d}\psi_{\alpha\beta}\right\}\partial_{x_\alpha}\partial_{x_\beta}u^0(x) + \varepsilon\nabla\cdot\{A(x/\varepsilon)\chi(x/\varepsilon)\cdot\nabla^2u^0(x)\}. \tag{4.1}$$

We need to work on the first term in the right hand side of the previous equation, to show it is actually of order ε.

Lemma 2. *Let $F = F(y) \in \mathbb{R}^d$ be an $L^2(\mathbb{T}^d)$ function such that*

$$\int_{\mathbb{T}^d} F = 0 \quad \textit{and} \quad \nabla\cdot F = 0.$$

Then, there exists $W_{\alpha\beta} \in H^1(\mathbb{T}^d)$ such that

$$W_{\alpha\beta} = -W_{\beta\alpha} \quad \textit{and} \quad F_\beta = \partial_{y_\alpha}W_{\alpha\beta}.$$

Proof. For all $1 \le \alpha \le d$, let $f_\alpha \in H^2(\mathbb{T}^d)$ be the solution to

$$\Delta f_\alpha = F_\alpha \quad y \in \mathbb{T}^d \quad \text{and} \quad \int_{\mathbb{T}^d} f_\alpha = 0.$$

Notice that $\nabla\cdot F = 0$ implies $\nabla\cdot f$ is constant. Now let

$$W = \operatorname{curl} f \quad \text{i.e. for all} \quad 1 \le \alpha,\beta \le d, W_{\alpha\beta} := \partial_{y_\alpha}f_\beta - \partial_{y_\beta}f_\alpha.$$

Clearly, $W_{\alpha\beta} = -W_{\beta\alpha}$ and

$$\partial_{y_\alpha}W_{\alpha\beta} = \partial_{y_\alpha}\partial_{y_\alpha}f_\beta - \partial_{y_\beta}\partial_{y_\alpha}f_\alpha = \Delta f_\beta = F_\beta,$$

which concludes the proof. □

The cell corrector equation (2.1) implies that

$$\partial_{y_\alpha}\left(\psi_{\alpha\beta} - \int_{\mathbb{T}^d}\psi_{\alpha\beta}\right) = 0,$$

and we have

$$\int_{\mathbb{T}^d}\left(\psi_{\alpha\beta} - \int_{\mathbb{T}^d}\psi_{\alpha\beta}\right) = 0.$$

Therefore, for fixed β, we can apply the lemma to

$$F = \psi_{\cdot\beta} - \int_{\mathbb{T}^d}\psi_{\cdot\beta},$$

and get the existence of $\Psi_{\gamma\alpha\beta}(y)$ such that

$$\psi_{\alpha\beta} - \int_{\mathbb{T}^d}\psi_{\alpha\beta} = \partial_{y_\gamma}\Psi_{\gamma\alpha\beta} \quad \text{and} \quad \Psi_{\gamma\alpha\beta} = -\Psi_{\alpha\gamma\beta}. \tag{4.2}$$

We finally use $\Psi_{\gamma\alpha\beta}$ to rewrite the first term in the right hand side of (4.1):

$$\left\{\psi_{\alpha\beta}(x/\varepsilon) - \int_{\mathbb{T}^d} \psi_{\alpha\beta}\right\} \partial_{x_\alpha}\partial_{x_\beta} u^0(x) = \partial_{y_\gamma}\Psi_{\gamma\alpha\beta}(x/\varepsilon)\partial_{x_\alpha}\partial_{x_\beta}u^0(x)$$
$$= \varepsilon\partial_{x_\gamma}\left(\Psi_{\gamma\alpha\beta}(x/\varepsilon)\partial_{x_\alpha}\partial_{x_\beta}u^0(x)\right) - \varepsilon\Psi_{\gamma\alpha\beta}(x/\varepsilon)\partial_{x_\alpha}\partial_{x_\beta}\partial_{x_\gamma}u^0(x).$$

It happens that the last term involving three derivatives of u^0 is zero:

$$\Psi_{\gamma\alpha\beta}(x/\varepsilon)\partial_{x_\alpha}\partial_{x_\beta}\partial_{x_\gamma}u^0(x) = \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\sum_{\gamma=1}^{d}\Psi_{\gamma\alpha\beta}(x/\varepsilon)\partial_{x_\alpha}\partial_{x_\beta}\partial_{x_\gamma}u^0(x)$$
$$= \sum_{\beta=1}^{d}\left\{\sum_{\alpha<\gamma} + \sum_{\gamma<\alpha} + \sum_{\alpha=\gamma}\right\} = \sum_{\beta=1}^{d}\left\{\sum_{\alpha<\gamma} - \sum_{\alpha<\gamma}\right\} = 0,$$

because of the second property in (4.2).

To put it in a nutshell

$$-\nabla\cdot A(x/\varepsilon)\nabla r^\varepsilon(x) = \varepsilon\nabla\cdot\left(\Psi(x/\varepsilon)\cdot\nabla^2u^0(x)\right) + \varepsilon\nabla\cdot\{A(x/\varepsilon)\chi(x/\varepsilon)\cdot\nabla^2u^0(x)\}.$$

These two terms are divergence form, so one can integrate by parts. By linearity, r^ε is the sum of r^ε_{int} solving

$$\begin{cases} -\nabla\cdot A(x/\varepsilon)\nabla r^\varepsilon_{int} &= \varepsilon\nabla\cdot(\Psi(x/\varepsilon)\cdot\nabla^2u^0(x)) + \varepsilon\nabla\cdot\{A(x/\varepsilon)\chi(x/\varepsilon)\cdot\nabla^2u^0(x)\}, \quad x\in\Omega,\\ r^\varepsilon_{int} &= 0, \qquad x\in\partial\Omega,\end{cases}$$

and the boundary layer term u^ε_{bl} solving

$$\begin{cases} -\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon_{bl} &= 0, \qquad x\in\Omega,\\ u^\varepsilon_{bl} &= -\chi(x/\varepsilon)\cdot\nabla u^0(x), \quad x\in\partial\Omega.\end{cases}$$

Theorem 3. *Assume that $u^0 \in H^2(\Omega)$ and $A \in L^\infty(\mathbb{R}^d)$ (for an equation), $A \in \mathcal{C}^{0,\mu}(\mathbb{R}^d)$, $\mu > 0$ (for a system). Then*

$$\left\|u^\varepsilon - u^0 - \varepsilon\chi(x/\varepsilon)\cdot\nabla u^0 - \varepsilon u^{1,\varepsilon}_{bl}\right\|_{H^1(\Omega)} \lesssim \varepsilon\|u^0\|_{H^2(\Omega)}.$$

Remark 2. The more higher-order correctors, the more regularity is needed on u^0.

Proposition 4. *Assume that:*

(1) Ω is of class $\mathcal{C}^2$, so that $u^0 \in H^2(\Omega)$,

(2) A is $\mathcal{C}^{0,\mu}$, $\mu > 0$, so that $\chi \in W^{1,\infty}(\mathbb{R}^d)$.

Then,

$$\left\|u^\varepsilon - u^0 - \varepsilon\chi(x/\varepsilon)\cdot\nabla u^0\right\|_{H^1(\Omega)} \lesssim \varepsilon^{1/2}\|f\|_{L^2(\Omega)}.$$

Notice that if $u \in W^{1,\infty}(\Omega)$ and $v \in H^1(\Omega)$, then $uv \in H^1(\Omega)$. Thus by assumption of the previous proposition $\chi(\cdot/\varepsilon)\cdot\nabla u^0 \in H^1(\Omega)$. Moreover,

$$\|u^{1,\varepsilon}_{bl}\|_{H^1(\Omega)} \lesssim \frac{1}{\varepsilon^{1/2}}\|\nabla u^0\|_{H^{3/2}(\partial\Omega)}.$$

In particular,

$$\nabla u^\varepsilon(x) - \nabla u^0(x) - \nabla_y \chi(x/\varepsilon) \cdot \nabla u^0(x) \to 0 \quad \text{strongly in} \quad L^2(\Omega).$$

This shows that one has to add the corrector term $\nabla_y \chi(x/\varepsilon) \cdot \nabla u^0(x)$ to upgrade the weak convergence of the gradient ∇u^ε into a strong convergence in $L^2(\Omega)$.

Remark 3. It is a fundamental (and generally difficult) problem to construct bounded correctors (see the fourth lecture), or to have an information on how the corrector grows at space infinity. Such information directly translates into error estimates. Actually, the construction and estimation of correctors is the main problem of the proof of error estimates. This problem is trivial in the periodic setting considered in this lecture.

References

[CD99] Doina Cioranescu and Patrizia Donato. *An introduction to homogenization*, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1999.

[CK97] Andrej Cherkaev and Robert Kohn, editors. *Topics in the mathematical modelling of composite materials*. Progress in Nonlinear Differential Equations and their Applications, 31. Birkhäuser Boston, Inc., Boston, MA, 1997.

[Eva90] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.