Introduction

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February 9, 2016

This set of lectures is motivated by the following kind of phenomena:

$$\sin(x/\varepsilon) \rightharpoonup 0, \quad \text{while} \quad \sin^2(x/\varepsilon) \rightharpoonup 1/2.$$

Therefore the weak limit of the product is in general different from the product of the weak limits. We will analyze this incompatibility of weak convergence with products in the context of linear and nonlinear Partial Differential Equations.

The goal of the lectures is to introduce some methods, either using soft analysis i.e. weak convergence methods (for instance the structure of the nonlinear term) or hard analysis i.e. strong convergence methods (construction of correctors). The lectures will cover subjects which are familiar to the author, particularly homogenization theory. One of the intention is to show that hard analysis requires to understand some underlying dynamical problems. The plan of the course goes as follows:

(1) information on weak limits by oscillating test functions (lectures 1 and 2) or subtle cancellations (lecture 3),

(2) refined information via the construction of interior and boundary layer correctors correctors (lectures 4 and 5),

(3) regularity theory for elliptic equations via compactness methods (lecture 6).

This is only a very small part of all the methods and phenomena that one could talk about in such a course. Many important subjects will be ignored (young measures, defect measures, vanishing viscosity for Hamilton Jacobi equations...). To get a glimpse into some of these topics, one should read Evans' inspiring book [Eva90].

A few classical results

Most of the following material can be found in [Bre83] or [Eva90].

Proposition 1 (weak convergence in a Banach space)**.** *Let E be a Banach space, E' its dual, and let $\langle \cdot, \cdot \rangle_{E',E}$ denote the duality bracket. Let (x_n) be a sequence in E. The following statements hold:*

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(1) $x_n \rightharpoonup x$ *weakly in* E *(alternative notation* $\sigma(E, E')$*) is equivalent to for all* $f \in E'$*,* $\langle f, x_n\rangle_{E',E} \to \langle f, x\rangle_{E',E}$.

(2) If $x_n \rightharpoonup x$ *weakly in* E*, then* $\|x_n\|_E$ *is bounded and* $\|x\|_E \leq \liminf \|x_n\|_E$.

(3) If $x_n \rightharpoonup x$ *weakly in* E *and* $f_n \to f$ *strongly in* E'*, then*

$$\langle f_n, x_n\rangle_{E',E} \to \langle f, x\rangle_{E',E}.$$

The second point is a consequence of Banach-Steinhaus's theorem.

Proposition 2 (weak star convergence)**.** *Let* E *be a Banach space,* E' *its dual, and let* $\langle \cdot, \cdot\rangle_{E',E}$ *denote the duality bracket. Let* (f_n) *be a sequence in* E'*. The following statements hold:*

(1) $f_n \overset{\star}{\rightharpoonup} f$ *weakly star in* E' *(alternative notation* $\sigma(E', E)$*) is equivalent to for all* $x \in E$*,* $\langle f_n, x\rangle_{E',E} \to \langle f, x\rangle_{E',E}$.

(2) If $f_n \overset{\star}{\rightharpoonup} f$ *weakly star in* E'*, then* $\|f_n\|_{E'}$ *is bounded and* $\|f\|_{E'} \leq \liminf \|f_n\|_{E'}$.

(3) If $f_n \overset{\star}{\rightharpoonup} f$ *weakly star in* E' *and* $x_n \to x$ *strongly in* E*, then*

$$\langle f_n, x_n\rangle_{E',E} \to \langle f, x\rangle_{E',E}.$$

Proposition 3 (Banach Alaoglu theorem)**.** *The unit ball* $B_{E'}(0, 1)$ *of the dual space* E' *is compact for the weak star topology* $\sigma(E', E)$.

Proposition 4 (Kakutani)**.** *A Banach space* E *is reflexive if and only if the unit ball* $B_E(0, 1)$ *of* E *is compact for the weak topology* $\sigma(E, E')$.

Let $f_n \in L^1(\mathbb{R})$ be defined by

$$f_n(x) = \begin{cases} n, & x \in \left(0, \frac{1}{n}\right), \\ 0, & x \text{ elsewhere.} \end{cases}$$

Then $\|f_n\|_{L^1(\mathbb{R})} = 1$ and by the fundamental theorem of calculus, for all $\varphi \in \mathcal{C}^0_c(\mathbb{R})$,

$$\int_{(0,1)} f_n(x)\varphi(x) = n \int_0^{1/n} \varphi(x)dx \longrightarrow \varphi(0).$$

Therefore, $f_n \overset{\star}{\rightharpoonup} \delta_0$ weakly star in the space of Radon measures.

Proposition 5 (weak convergence in a Hilbert space)**.** *Let* H *be a Hilbert space and let* $\langle \cdot, \cdot\rangle$ *denote its scalar product. Assume that* $x_n \rightharpoonup x$ *weakly in* H *and that* $\|x_n\|_H \to \|x\|_H$*. Then,* $x_n \to x$ *strongly in* H.

This property comes from the polarization formula:

$$\|x - x_n\|_H^2 = \|x\|_H^2 + \|x_n\|_H^2 - 2\langle x_n, x\rangle.$$

It does not hold in a general Banach space. However, we have the following refinement for $L^p(\Omega)$, $1 < p < \infty$ and Ω an arbitrary open subset of $\mathbb{R}^d$, and more generally for any uniformly convex Banach space E.

Proposition 6. *Let $1 < p < \infty$ and Ω be a open subset of $\mathbb{R}^d$. Let (f_n) be a sequence in $L^p(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ and*

$$\limsup \|f_n\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

Then $f_n \to f$ strongly in $L^p(\Omega)$.

We refer to [Bre83, Proposition III.30] for the proof of the proposition for a general uniformly convex Banach space, and to [Bre83, Theorem IV.10] for the proof of the uniform convexity of $L^p(0,1)$.

How about $L^1(0,1)$? Take $f_n \in L^2(0,1)$ defined by

$$f_n(x) = \begin{cases} 1, & x \in \left(\frac{k}{n}, \frac{2k+1}{2n}\right), \\ 0, & x \in \left(\frac{2k+1}{2n}, \frac{k+1}{n}\right). \end{cases}$$

Then $f_n \rightharpoonup 1/2$ weakly in $L^2(0,1)$ (and weakly in $L^1(0,1)$). Moreover,

$$\|f_n\|_{L^1(0,1)} = 1/2 = \|1/2\|_{L^1(0,1)},$$

but

$$\|f_n\|_{L^2(0,1)} = 1/\sqrt{2} \neq 1/2 = \|1/2\|_{L^1(0,1)}.$$

It is thus clear that f_n does not converge strongly in $L^2(0,1)$, therefore neither in $L^1(0,1)$.

Proposition 7. *Let $1 \leq p \leq \infty$ and $f = f(y)$ be a $\mathbb{Z}^d$-periodic function. Assume that $f \in L^p(\mathbb{T}^d)$. Let*

$$\bar{f} := \fint_{\mathbb{T}^d} f.$$

Then, for any $K \Subset \mathbb{R}^d$,

(1) for $1 \leq p < \infty$

$$f(x/\varepsilon) \rightharpoonup \bar{f} \quad \text{weakly in} \quad L^p(K),$$

(2) for $p = \infty$,

$$f(x/\varepsilon) \overset{*}{\rightharpoonup} \bar{f} \quad \text{weakly star in} \quad L^\infty(K).$$

Remark 1 (weak convergence and nonlinear function). Let $F = F(u) \in \mathbb{R}$ be a nonlinear function. Take $a < b$ and $0 < \lambda < 1$ so that

$$F(\lambda a + (1-\lambda)b) \neq \lambda F(a) + (1-\lambda)F(b).$$

We define

$$u_n(x) = \begin{cases} a, & x \in \left(\frac{j}{n}, \frac{j+\lambda}{n}\right), \\ b, & x \text{ elsewhere.} \end{cases}$$

Then, by the previous proposition

$$u_n \overset{\star}{\rightharpoonup} \overline{u} = \lambda a + (1-\lambda)b \quad \text{weakly star in} \quad L^\infty(0,1),$$
$$F(u_n) \overset{\star}{\rightharpoonup} \lambda F(a) + (1-\lambda)F(b) \neq F(\lambda a + (1-\lambda)b) \quad \text{weakly star in} \quad L^\infty(0,1).$$

Theorem 8 (Rellich's compactness theorem)**.** *Let Ω be a bounded domain of $\mathbb{R}^d$. Let $1 \leq p \leq d$ and let $p^* := \frac{pd}{d-p}$.*

(1) If the sequence u_n is bounded in $W_0^{1,p}(\Omega)$, then u_n is precompact in $L^q(\Omega)$ for all $1 \leq q < p^*$.

(2) Assume in addition that Ω is Lipschitz. If the sequence u_n is bounded in $W^{1,p}(\Omega)$, then u_n is precompact in $L^q(\Omega)$ for all $1 \leq q < p^*$.

In other words, if Ω is Lipschitz, then $W^{1,p}(\Omega) \Subset L^q(\Omega)$ for all $1 \leq q < p^*$. For a proof, see Chapter 7 of the book by Gilbarg and Trudinger [GT01].

A simple proof of Rellich's theorem

We give a simple proof of Rellich's injection theorem (due to Hörmander) based on the use of the Fourier transform: for an arbitrary bounded domain Ω, $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.

The proof goes as follows. Let u_n be a bounded sequence in $H_0^1(\Omega)$. First we extend u_n by 0 outside Ω. The extension belongs to $H^1(\mathbb{R}^d)$ and is still denoted by u_n. Since u_n is bounded in $L^2(\mathbb{R}^d)$, there is a weakly converging subsequence (still denoted u_n):

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^2(\mathbb{R}^d).$$

If we show that

$$\|u_n\|_{L^2(\mathbb{R}^d)} \longrightarrow \|u\|_{L^2(\mathbb{R}^d)},$$

then we have that u_n converges strongly to u in $L^2(\mathbb{R}^d)$ (thus in $L^2(\Omega)$).

Now, for $R > 0$,

$$\int_{\mathbb{R}^d} u_n^2 = \int_{\mathbb{R}^d} \widehat{u_n}\overline{\widehat{u_n}} = \int_{B(0,R)} \widehat{u_n}\overline{\widehat{u_n}} + \int_{\mathbb{R}^d \setminus B(0,R)} \widehat{u_n}\overline{\widehat{u_n}}.$$

Low frequencies The low frequencies are handled by Lebesgue's theorem, using:

(1) the fact that

$$\|\widehat{u_n}\|_{L^\infty(\mathbb{R}^d)} \leq \|u_n\|_{L^1(\mathbb{R}^d)} \leq C_\Omega \|u_n\|_{L^2(\Omega)}$$

is bounded uniformly in n,

(2) and that the weak convergence of u_n in $L^2(\Omega)$ implies the pointwise convergence of $\widehat{u_n}(\xi)$, i.e. for all $\xi \in \mathbb{R}^d$,

$$\widehat{u_n}(\xi) = \int_\Omega u_n(x)\exp(-ix\cdot\xi)d\xi \longrightarrow \int_\Omega u(x)\exp(-ix\cdot\xi)d\xi = \widehat{u}(\xi).$$

Therefore,

$$\int_{B(0,R)} \widehat{u_n}\overline{\widehat{u_n}} \longrightarrow \int_{B(0,R)} \widehat{u}\overline{\widehat{u}}.$$

High frequencies The high frequencies are addressed thanks to the bound on the $H^1(\mathbb{R}^d)$ norm:

$$\int_{\mathbb{R}^d\setminus B(0,R)} \widehat{u_n}\overline{\widehat{u_n}} = \int_{\mathbb{R}^d\setminus B(0,R)} \frac{1}{|\xi|^2}|\xi|^2\widehat{u_n}\overline{\widehat{u_n}} \leq \frac{1}{R^2}\int_{\mathbb{R}^d} |\xi|^2\widehat{u_n}\overline{\widehat{u_n}} \longrightarrow 0,$$

when $R \to \infty$.

Remark 2. For the injection $W^{1,2}(\Omega) \Subset L^2(\Omega)$, we need to extend functions to the whole of $\mathbb{R}^d$. This restricts the class of domains to the class of Lipschitz Ω. Let $\widetilde{\Omega}$ be a fixed bounded domain such that $\Omega \subset \widetilde{\Omega}$. We extend a given sequence of functions $u_n \in W^{1,2}(\Omega)$ to functions $\widetilde{u_n} \in W^{1,2}(\mathbb{R}^d)$ such that

$$\widetilde{u_n} \equiv u_n \quad \text{on} \quad \Omega, \qquad \text{and} \quad \operatorname{Supp}\widetilde{u_n} \subset \widetilde{\Omega}.$$

The rest of the proof is similar.

Remark 3. The scheme of this proof will be implemented to prove the div-curl lemma in the third lecture.

Some notations

We use Einstein's convention for repeated indices: a repeated index stands for a summation on the index, as for instance for $x,\ \hat{x} \in \mathbb{R}^d$,

$$x \cdot \hat{x} = \sum_{\alpha=1}^{d} x_\alpha \hat{x}_\alpha =: x_\alpha \hat{x}_\alpha.$$

In the whole series of lectures, we will from now on always consider

a bounded and Lipschitz domain $\quad \Omega$

unless explicitly stated otherwise.

References

[Bre83] Haïm Brezis. *Analyse fonctionnelle: théorie et applications.* 1983.

[Eva90] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.